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LOCALIZATION OF ARNOLD TONGUES OF DISCRETE DYNAMICAL SYSTEMS

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Abstract. The work is devoted to the exposition of the method of localizing the Arnold tongues for finite-dimensional dynamical systems with a discrete time which are the sets corresponding to rationally synchronized relations between the system's parameters. Such sets correspond to regions of parameter values for which the system has cycles of certain periods. The method allows us to obtain an approximate representation of the Arnold tongues, to study their properties in the major and minor resonances.

Keywords: bifurcation, dynamical systems, Arnold tongues, operator equations, functionalization of parameter.

Mathematics Subject Classification: 37G10, 37G15.

1. Introduction

One of notions interesting and important from theoretical and practical point of view is that of the Arnold tongue [1]–[3] which are the sets corresponding to rationally synchronized relations of a system's parameters. Such sets correspond to the values of the parameters for which the system can have cycles of certain periods. Various questions related to the properties and applications of Arnold tongues in nonlinear dynamics were discussed in a series of works (see, for instance, [1]–[8] and the references therein).

The system of Arnold tongue can be observed in many works of nonlinear dynamics. For instance, in the problem on the stability loss for a cycle of period T for an autonomous system under the passage of the multiplier via the unit circle, as the parameters, one can treat the modulus and argument of the multiplier. In this case, on the parameters plane there form narrow rhamphoid sets (tongues) having their cusps at the points $e^{2\pi\theta i}$ of the unit circle, where $\theta = \dfrac{p}{q}$ is rational; such sets correspond to the domains of the existence of qT-periodic solutions to the system emerging in a vicinity of the initial cycle of period T.

Another example, where Arnold tongues appear in a natural way, is the problem on synchronization of self-oscillating system with eigenfrequency ν_0 by an external signal of frequency ν. As the parameters, here one can employ the fraction of the frequencies $\varkappa = \dfrac{\nu}{\nu_0}$ and the amplitude a of the external signal. On the parameters plane $(\varkappa, a)$, there appears a character structure of the regime domains which are the domains of synchronization with different relation between the frequencies ν and ν_0. These domains look like tongues appearing from each rational point on the axis $\varkappa$; the domains of periodic (as a rule, long-periodic) regimes of the system are associated with the corresponding values of the parameters $\varkappa$ and a. Between

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the tongues, there exist the domain of quasi-periodic regimes with an irrational fraction of the frequencies.

In the present paper we give basic concepts of a new operator method for localizing the Arnold tongues of a finite-dimensional system with a discrete time. The method allows one to obtain an approximate representation for these sets and to study their properties in major and minor resonances. In the justification of the method, we obtain new asymptotic formulae for the solutions to the problems on bifurcation of the cycles of dynamical systems allowing one to study the bifurcations in details.

1.1. Bifurcation of q-cycles. We consider a finite-dimensional dynamical system with a discrete time,

$$x_{n+1} = F(x_n, \mu), \quad n = 0, 1, 2, ..., \quad x_k \in R^N \tag{1}$$

depending on a parameter $\mu \in R^m$. The function $F(x, \mu)$ is supposed to be defined and continuously differentiable w.r.t. all the variables on the set

$$\Omega = \{(x, \mu) : \ \|x\| \leqslant \delta_1 , \ \ \|\mu - \mu_0\| \leqslant \delta_2\} ,$$

where δ_1 and δ_2 are some positive numbers; hereinafter the symbol $\|\cdot\|$ is employed to denote Euclidean norm in the spaces R^N and R^m.

As usually, by m-cycle or m-periodic solution of system (1) we call a set of different vectors x_0^*, x_1^*, $x_2^*, \ldots, x_{m-1}^*$, such that

$$x_1^* = F(x_0^*, \mu), \ x_2^* = F(x_1^*, \mu), \ \ldots, x_{m-1}^* = F(x_{m-2}^*, \mu), \ x_0^* = F(x_{m-1}^*, \mu) ;$$

For $m = 1$, the given definition becomes the notion of the equilibrium point of system (1); a vector x_0^* is an equilibrium point if $x_0^* = F(x_0^*, \mu)$.

Suppose system (1) has the equilibrium point $x^* = 0$ for all values of μ, i.e., $F(0, \mu) \equiv 0$. Denote by $A(\mu) = F'_x(0, \mu)$ the Jacobi matrix of the function $F(x, \mu)$ calculated at the point $x = 0$. Our main assumption is

S1) the matrix $A(\mu_0)$ has a pair of simple eigenvalues $e^{\pm 2\pi\theta_0 i}$, where $0 < \theta_0 \leqslant \dfrac{1}{2}$ and θ_0 is rational: $\theta_0 = \dfrac{p}{q}$ is a irreducible fraction.

At that, it is assumed that the moduli of all other eigenvalues of the matrix $A(\mu_0)$ are equal to one.

Under the above assumptions, as $\mu = \mu_0$, the equilibrium point $x^* = 0$ of system (1) is non-hyperbolic (see, for instance, [4]), and the value $\mu = \mu_0$ is bifurcational. The codimension of the associated bifurcation equals two. This is why it is natural to assume that the parameter μ is two-dimensional, i.e., $\mu = (\alpha, \beta)$, where α and β are scalar parameters. We also let $\mu_0 = (\alpha_0, \beta_0)$.

Let P be the plane of the parameters $\mu = (\alpha, \beta)$ of system (1). The bifurcation scenarios in a vicinity of the equilibrium point $x^* = 0$ of system (1) are determined by the character of a passage of the parameter $\mu \in P$ through the point μ_0. This passage can be realized in infinitely many various directions, along straight lines or curves passing the point μ_0. Here periodic solutions of various periods can emerge or disappear.

One of basic scenarios (but not the unique one) is the bifurcation of q-cycles of system (1), when for the parameters μ close to μ_0, cycles of period q of system (1) emerge and the amplitudes of these cycles tends to zero as μ does to μ_0. In other words, the value μ_0 is a *bifurcation point of q-cycles for system (1)* if there exists a sequence $\mu_k \to \mu_0$ such that as $\mu = \mu_k$, system (1) has the q-cycle $x_0^k, x_1^k, x_2^k, ..., x_{q-1}^k$ and $\max\limits_{0\leqslant j\leqslant q-1} \|x_j^k\| \to 0$ as $k \to \infty$.

1.2. Arnold tongues. To describe possible scenarios of bifurcation for system (1) in a vicinity on the equilibrium point $x^* = 0$, we denote by $\mathcal{K}$ the set of the points in the plane P of the parameters (α, β), for which the matrix $A(\alpha, \beta)$ has the eigenvalue λ, $|\lambda| = 1$. The set $\mathcal{K}$ is usually a smooth curve in the plane P.

In the plane P, there forms a character structure of the domains of nonlinear system (1); this structure is the domains of synchronization with different fraction of the parameters α and β. These domains has a rhamphoid form or tongue-like $\Psi(\alpha^*, \beta^*)$ with the cusps in the points (α^*, β^*) of the curve $\mathcal{K}$ at which the matrix $A(\alpha^*, \beta^*)$ has eigenvalues $e^{\pm 2\pi\theta^* i}$ with rational θ^*: $\theta^* = \dfrac{l}{m}$ (see fig. 1).

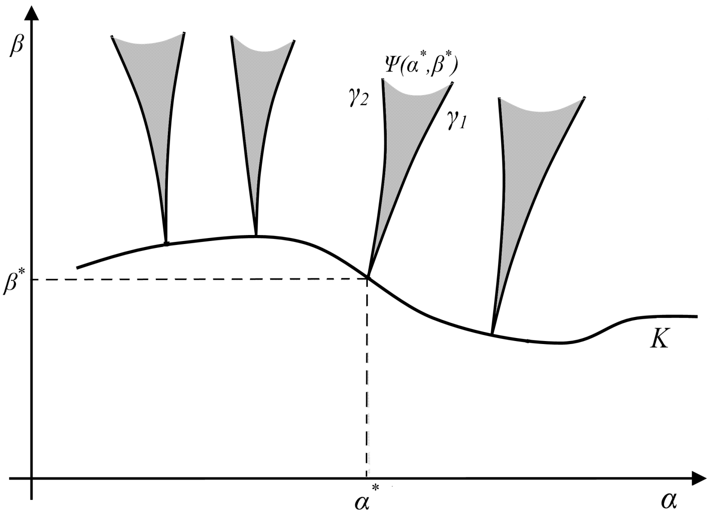

Figure 1. Arnold tongues on parameters plane

Such tongues correspond to the domains of the parameters (α, β) for which system (1) has periodic regimes of period m with the amplitudes tending to zero as (α, β) tends to (α^*, β^*). In other words, the set $\Psi(\alpha^*, \beta^*)$ contains the sequences $(\alpha_k, \beta_k) \to (\alpha^*, \beta^*)$ for which the bifurcation scenario of m-cycles of systems (1) realizes.

For instance, by Condition S1), the inclusion $(\alpha_0, \beta_0) \in \mathcal{K}$ holds true, since the matrix $A(\alpha_0, \beta_0)$ has the eigenvalues $e^{\pm 2\pi\theta_0 i}$, where $\theta_0 = \dfrac{p}{q}$. The corresponding tongue $\Psi(\alpha_0, \beta_0)$ is the set of the values of the parameters (α, β) for which system (1) has q-cycles with the amplitudes tending to zero as (α, β) tends to (α_0, β_0).

Thus, the aforementioned tongues $\Psi(\alpha^*, \beta^*)$ correspond to rationally synchronized (in the natural sense) fractions of the parameters α and β. Between these tongues, there exist the domain of quasi-periodic regimes with irrational fraction of the parameters. The main features of this picture were found out by Russian mathematician V.I. Arnold [1], so the system of synchronization tongues corresponding to rationally synchronized fractions of the parameters was named as Arnold tongues [2], [3].

The mentioned structure of the regimes domains has a local character. As the parameters α and β go far from (α^*, β^*), the domains of periodic regimes supplant quasi-periodic ones and the tongues start overlapping. A chaos becomes possible. The system of Arnold tongues can be observed in self-oscillating systems excited by a periodic signal, in the problems on mutual synchronization of two self-oscillating system, and others (see, for instance, [5], [6]).

In literature the notion of Arnold tongues can be introduced by other interpretations as well. Quite often (see, for instance, [3], [7]) this notion is introduced in terms of spectral characteristics of the matrix $A(\alpha, \beta)$. This interpretation is employed in the present paper; let us introduce it.

Let $\mathbb{C}$ be the complex plane and $S = \{z : |z| = 1\}$ be the unit circle in this plane. Suppose the value $\mu = (\alpha^*, \beta^*)$ of system (1) is a bifurcation point of m-cycles. To realize such scenario for system (1), the matrix $A(\mu)$ should have the eigenvalues in a set $U(m) \in \mathbb{C}$ being a union of narrow rhamphoid sets $\Psi(l,m)$, where $0 < \frac{l}{m} \leqslant \frac{1}{2}$ and the number $\frac{l}{m}$ is an irreducible fraction.

The sets $\Psi(l,m)$ are called the Arnold tongues in the complex plane $\mathbb{C}$. By its cusp, each set $\Psi(l,m)$ abuts the point $e^{2\pi\theta^* i}$ in the circle S, where $\theta^* = \frac{l}{m}$. A typical Arnold tongue $\Psi(l,m)$ is located between two smooth curves γ_1 and γ_2, as it is shown in Fig. 2.

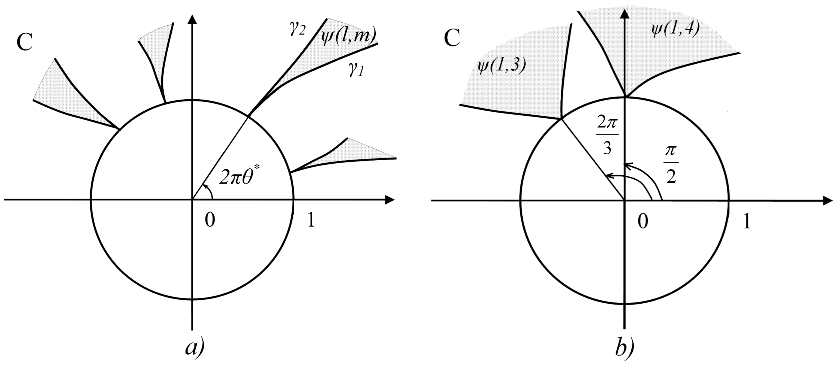

Figure 2. Arnold tongues in complex plane

As $m \geqslant 5$, these curves touch at the point $e^{2\pi\theta^* i}$; in this case the Arnold tongue $\Psi(l,m)$ degenerates to a curve in a small neighborhood of the point $e^{2\pi\theta^* i}$ (Fig. 2 a)). As $m \leqslant 4$, the Arnold tongue $\Psi(l,m)$ is a substantially wider set (Fig. 2 b)).

Such organization of Arnold tongues is due to the structure of so-called resonance term in the Taylor expansion at zero for the mapping $F(x,\mu)$. Main resonance terms are responsible for the existence of cycles of small periods $m \leqslant 4$. Respectively, the cycles of small periods of system (1) appear quite often, while long-periodic (as $m \geqslant 5$) are non-typical and appear rather seldom.

In the unit circle S of the complex plane $\mathbb{C}$, there exists a countable set of the points $e^{2\pi\theta i}$ with rational θ densely located on the circle. Each such point is associated with an Arnold tongue. In particular, it means that in the one-parametric case (i.e., as the parameter μ is scalar), as μ passes μ_0, in general, long-periodic cycles of system (1) emerge and disappear in a vicinity of the point $x = 0$. This phenomenon (subfurcation of periodic oscillations) was first observed by V.S. Kozyakin [9].

2. Formulation of problem

In the present work we provide a scheme allowing us to localize the Arnold tongues $\Psi(p,q)$ of system (1). In order to do it, we make several simplifying assumptions.

First, for the sake of simplicity, we assume that system (1) is two-dimensional, i.e., $N = 2$. The case $N \geqslant 3$ can be reduced to the two-dimensional one by, for instance, theorems on central manifold (see, for instance, [4]).

Second, it is convenient to assume that the parameters α and β of the system (1) are related by a simple expression with the eigenvalues of the matrix $A(\mu)$. We employ the fact that by the perturbation theory of linear operators [10], for each value of the two-dimensional parameter μ close to μ_0, the matrix $A(\mu)$ has the unique eigenvalue $\lambda = \rho(\mu)e^{2\pi\theta(\mu)i}$ close to $e^{2\pi\theta_0 i}$; at that, the functions $\rho(\mu)$ and $\theta(\mu)$ are smooth and the identities $\rho(\mu_0) = 1$ and $\theta(\mu_0) = \theta_0$ hold true. We define the functions $\alpha = \alpha(\mu) \equiv \rho(\mu) - 1$ and $\beta = \beta(\mu) \equiv \theta(\mu) - \theta_0$.

Without loss of generality, we assume that the parameters of system (1) are exactly these α and β. Namely, we suppose that matrix $A(\alpha, \beta)$ reads as follows,

$$A(\alpha, \beta) = (1 + \alpha)Q(\beta)\,, \tag{2}$$

where

$$Q(\beta) = \begin{bmatrix} \cos 2\pi(\theta_0 + \beta) & -\sin 2\pi(\theta_0 + \beta) \\ \sin 2\pi(\theta_0 + \beta) & \cos 2\pi(\theta_0 + \beta) \end{bmatrix}. \tag{3}$$

The matrix $A(\alpha, \beta)$ has a pair of simple eigenvalues

$$\lambda(\alpha, \beta) = (1 + \alpha)e^{\pm 2\pi(\theta_0 + \beta)i}\,. \tag{4}$$

At that, the matrix $A(\alpha, \beta)$ satisfies Condition S1) for $\alpha = 0$ and $\beta = 0$.

Hence, we consider the two-dimensional dynamical system with the discrete time

$$x_{n+1} = A(\alpha, \beta)x_n + a(x_n, \alpha, \beta), \quad n = 0, 1, 2, ..., \quad x_n \in R^2\,, \tag{5}$$

where $A(\alpha, \beta)$ is matrix (2), the nonlinearity $a(x, \alpha, \beta)$ obeys the relation $\|a(x, \alpha, \beta)\| = O(\|x\|^2)$ as $\|x\| \to 0$ uniformly in α and β. We shall assume that the nonlinearity $a(x, \alpha, \beta)$ can be represented as

$$a(x, \alpha, \beta) = a_2(x, \alpha, \beta) + a_3(x, \alpha, \beta) + \tilde{a}_4(x, \alpha, \beta)\,, \tag{6}$$

where $a_2(x, \alpha, \beta)$ and $a_3(x, \alpha, \beta)$ involve quadratic and cubic in x terms, respectively, and $\tilde{a}_4(x, \alpha, \beta)$ is smooth w.r.t. x, and $\tilde{a}_4(x, \alpha, \beta) = O(\|x\|^4)$, $x \to 0$, uniformly in α and β.

The main problem considered in the present work is to localize the Arnold tongues $\Psi(p, q)$ of system (5).

3. Passage to operator equations

The basis for the subsequent construction is the following theorem.

Theorem 1. *The value $\mu_0 = (0, 0)$ is a bifurcation point of q-cycles for system (5).*

The proof of this and other main statements of the work are adduced in Section 7.

For a more detailed study of the bifurcation of q-cycles for system (5) we shall make use of some auxiliary constructions.

Periodic solutions of period q for system (1) are determined by the solutions to the operator equation

$$x = F^{(q)}(x, \mu)\,, \tag{7}$$

where

$$F^{(q)}(x, \mu) = \underbrace{F(F(\cdots(F}_{q}(x, \mu), \mu)\cdots))\,.$$

Namely, the next obvious statement holds true.

Lemma 1. *A vector x^* solves equation (7) if and only if x^* is either an equilibrium point of system (1) or it determines the cycle $x_0 = x^*$, $x_1 = F(x_0, \mu)$, $x_2 = F(x_1, \mu)$, $\ldots$, $x_{r-1} = F(x_{r-2}, \mu)$ of period r for this system, where r is a divisor of the number q.*

For instance, if $q = 6$, the solutions to equation (7) can either be the equilibrium points of system (1) or define cycles of period 2, 3, or 6.

In particular, for system (5), equation (7) becomes

$$x = B(\mu)x + b(x, \mu)\,, \tag{8}$$

where $\mu = (\alpha, \beta)$, the matrix $B(\mu)$ is defined by the identity

$$B(\mu) = A^q(\mu)\,, \tag{9}$$

and the nonlinearity $b(x,\mu)$ satisfies the representation

$$b(x,\mu) = b_2(x,\mu) + b_3(x,\mu) + \tilde{b}_4(x,\mu) \tag{10}$$

similar to (6).

At that, the next lemma holds.

Lemma 2. *The quadratic nonlinearity $b_2(x,\mu)$ in (10) can be represented as*

$$\begin{gathered} b_2(x,\mu) = A^{q-1}a_2(x,\mu) + A^{q-2}a_2(Ax,\mu) + \\ + \cdots + Aa_2(A^{q-2}x,\mu) + a_2(A^{q-1}x,\mu)\,, \end{gathered} \tag{11}$$

and the cubic nonlinearity $b_3(x,\mu)$ satisfies

$$\begin{gathered} b_3(x,\mu) = A^{q-1}a_3(x,\mu) + A^{q-2}a_3(Ax,\mu) + \\ + \cdots + Aa_3(A^{q-2}x,\mu) + a_3(A^{q-1}x,\mu) + g_3(x,\mu)\,, \end{gathered} \tag{12}$$

where

$$\begin{gathered} g_3(x,\mu) = A^{q-2}a'_{2x}(Ax,\mu)a_2(x,\mu) + A^{q-3}a'_{2x}(A^2x,\mu)[Aa_2(x,\mu) + a_2(Ax,\mu)] + \\ + \cdots + a'_{2x}(A^{q-1}x,\mu)[A^{q-2}a_2(x,\mu) + A^{q-3}a_2(Ax,\mu) + \cdots + a_2(A^{q-2}x,\mu)]\,. \end{gathered}$$

Here we have employed the notations: $A = A(\mu)$, $a'_{2x}(x,\mu)$ is the Jacobi matrix of the vector-function $a_2(x,\mu)$.

It is easy to check formula (9)–(12) by straightforward calculations.

4. Well-defined bifurcations

One of important properties of the bifurcation of q-cycles for system (5) is the property of its orientation; let us give an appropriate definition. Let $e \in R^2$ be a non-zero vector. We call the value $\mu_0 = (0,0)$ of the parameter $\mu = (\alpha,\beta)$ the *well-defined bifurcation point of q-cycles for system (5) in the direction of the vector e*, if there exist $\varepsilon_0 > 0$ and continuous functions $\alpha = \alpha(\varepsilon)$, $\beta = \beta(\varepsilon)$, and $x = x(\varepsilon)$ defined for $\varepsilon \in [0,\varepsilon_0)$ such that

1) $\alpha(0) = 0\,,\ \ \beta(0) = 0\,,\ \ x(0) = 0$;
2) $\|x(\varepsilon) - \varepsilon e\| = o(\varepsilon)$ as $\varepsilon \to 0$;
3) for each $\varepsilon > 0$ the vector $x(\varepsilon)$ is a point of a q-cycles for system (5) as $\alpha = \alpha(\varepsilon)$ and $\beta = \beta(\varepsilon)$.

We call $x(\varepsilon)$ and functions $\alpha = \alpha(\varepsilon)$ and $\beta = \beta(\varepsilon)$ the *bifurcating solutions* of system (5).

Well-defined bifurcation points correspond to the situation when system (5), as $\alpha = \alpha(\varepsilon)$ and $\beta = \beta(\varepsilon)$, has a q-cycle starting from the point $x(\varepsilon)$ and the curve $x = x(\varepsilon)$ in the space R^2 tends asymptotically to the line $x = \varepsilon e$ as $\varepsilon \to 0$.

Theorem 2. *The value $\mu_0 = (0,0)$ of the parameter $\mu = (\alpha,\beta)$ is a well-defined bifurcation point of q-cycles of system (5) in the direction of each non-zero vector e.*

Let us give the formulae allowing us to study in greater details the property of well-definiteness for q-cycles of system (5). In order to do it, we define the vectors

$$e = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}, \quad g = \begin{bmatrix} -c_2 \\ c_1 \end{bmatrix}, \tag{13}$$

whose components c_1 and c_2 are so that $c_1^2 + c_2^2 = 1$. In other words, e is an arbitrary unit vector: $\|e\| = 1$, and the unit vector g is orthogonal to the vector e. Letting $x = e$ and $\mu = \mu_0 = (0,0)$ in (11) and (12), we define the vectors

$$b_2 = b_2(e,\mu_0)\,, \qquad b_3 = b_3(e,\mu_0)\,; \tag{14}$$

in particular, by (11) and (2) we get

$$b_2 = Q^{q-1}a_2(e,\mu_0) + Q^{q-2}a_2(Qe,\mu_0) + \tag{15}$$

$$+\cdots+Qa_2(Q^{q-2}e,\mu_0)+a_2(Q^{q-1}e,\mu_0)\,,$$

where $Q=Q(0)$, and $Q(\beta)$ is matrix (3). By (2) and (12) one can get a similar formula for the vector b_3.

We define the numbers

$$\alpha_1=-\frac{1}{q}(b_2,e)\,,\quad \beta_1=-\frac{1}{2\pi q}(b_2,g) \tag{16}$$

and the vectors

$$e_1=\alpha_1 e+\beta_1 g\,, \tag{17}$$

$$\chi=q\left[\frac{1}{2}\alpha_1^2(1+q)-2\pi\beta_1^2(1+\pi q)\right]e+ \tag{18}$$

$$+\alpha_1\beta_1 q(1+2\pi+2\pi q)g+b'_{2x}\cdot(\alpha_1 e+\beta_1 g)+\alpha_1 b'_{2\alpha}+\beta_1 b'_{2\beta}\,.$$

Here $b'_{2x}=b'_{2x}(e,\mu_0)$ is the Jacobi matrix of nonlinearity (11) taken at the point $x=e$ as $\mu=\mu_0=(0,0)$, $b'_{2\alpha}$ and $b'_{2\beta}$ are the derivatives of nonlinearity (11) w.r.t. the parameters α and β, respectively, taken at the point $x=e$ as $\mu=\mu_0=(0,0)$.

Finally, let

$$\alpha_2=-\frac{1}{q}(\chi+b_3,e)\,,\quad \beta_2=-\frac{1}{2\pi q}(\chi+b_3,g)\,, \tag{19}$$

$$e_2=\alpha_2 e+\beta_2 g\,. \tag{20}$$

4.1. Bifurcations for odd q. The bifurcation properties of q-cycles of system (5) depends essentially on the parity of q. We consider first the case of odd q.

Theorem 3. *Suppose the number q is odd. Let e and g be unit vectors (13). Then the bifurcating solutions $x(\varepsilon)$, $\alpha(\varepsilon)$, and $\beta(\varepsilon)$ of system (5) existing in accordance with Theorem 2 can be represented as*

$$\alpha(\varepsilon)=\varepsilon\alpha_1+\varepsilon^2\alpha_2+\alpha_3(\varepsilon)\,,\quad \beta(\varepsilon)=\varepsilon\beta_1+\varepsilon^2\beta_2+\alpha_3(\varepsilon)\,, \tag{21}$$

$$x(\varepsilon)=\varepsilon e+\varepsilon^2 e_1+\varepsilon^3 e_2+e_3(\varepsilon)\,; \tag{22}$$

in these formulae, $\alpha_3(\varepsilon)$, $\beta_3(\varepsilon)$, and $e_3(\varepsilon)$ are continuous functions satisfying the relations

$$\alpha_3(\varepsilon)=o(\varepsilon^2)\,,\quad \beta_3(\varepsilon)=o(\varepsilon^2)\,,\quad \|e_3(\varepsilon)\|=o(\varepsilon^3)\quad \textit{as } \varepsilon\to 0\,. \tag{23}$$

In the plane P of the parameters (α,β) of system (5), functions (21) define a continuous curve $\omega(\varepsilon)$ beginning (as $\varepsilon=0$) at the origin (Fig. 3 a)).

These functions depend on the numbers p and q, and also on the vector e; this is why the curve $\omega(\varepsilon)$ differs for different p, q, and e. Similarly, function (22) in the phase space R^2 of system (5) defines a continuous curve $x(\varepsilon)$ touching the vector e at the origin as $\varepsilon=0$ (Fig. 3 b)).

Theorems 1 and 3 imply that in the case of odd q, for each unit vector $e\in R^2$, the family of emerging q-cycles of system (5) contains continuous branches of the cycles starting from the points of the curve $x(\varepsilon)$ determined by identity (22) for the values of the parameters belonging to the curve $\omega(\varepsilon)$. In other words, the value $\mu=\mu_0=(0,0)$ is a well-defined bifurcation point of q-cycles of system (5) in the direction of the vector e. We shall call formulae (21) and (22) the *asymptotic formulae* for emerging bifurcating solutions of system (5).

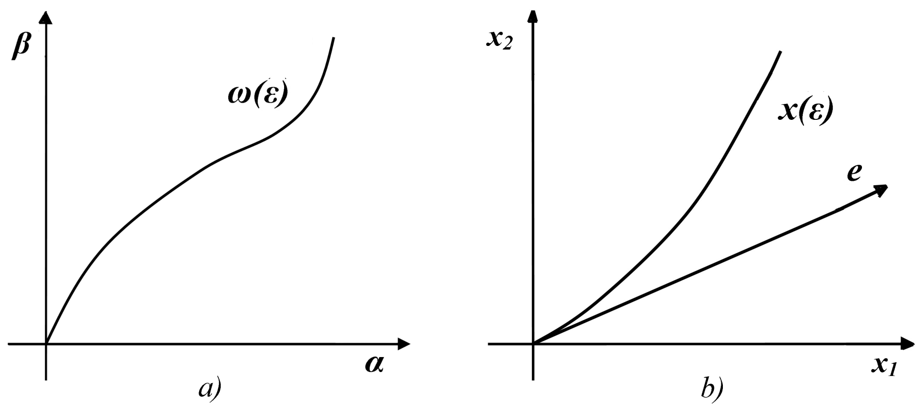

Figure 3. Curves of bifurcating solutions

4.2. Bifurcations for even q. Consider now the case of even q.

Lemma 3. *Suppose q is even. Then the vector b_2 determined by identity (15) vanishes, $b_2 = 0$.*

Corollary 1. *Suppose q is even. Then numbers (16) and vectors (17) and (18) vanish,*

$$\alpha_1 = 0\,, \quad \beta_1 = 0\,, \quad e_1 = 0\,, \quad \chi = 0\,, \tag{24}$$

and numbers (19) and vectors (20) are

$$\alpha_2 = -\frac{1}{q}(b_3, e)\,, \quad \beta_2 = -\frac{1}{2\pi q}(b_3, g)\,, \tag{25}$$

$$e_2 = \alpha_2 e + \beta_2 g\,. \tag{26}$$

Theorem 4. *Suppose the number q is even. Let e and g be unit vectors (13). Then the bifurcating solutions $x(\varepsilon)$, $\alpha(\varepsilon)$, and $\beta(\varepsilon)$ of system (5) existing in accordance with Theorem 2 are represented as*

$$\alpha(\varepsilon) = \varepsilon^2\alpha_2 + \alpha_3(\varepsilon)\,, \quad \beta(\varepsilon) = \varepsilon^2\beta_2 + \beta_3(\varepsilon)\,, \tag{27}$$

$$x(\varepsilon) = \varepsilon e + \varepsilon^3 e_2 + e_3(\varepsilon)\,; \tag{28}$$

in these formulae, the numbers α_2 and β_2 and the vector e_2 are determined by identities (25) and (26), and $\alpha_3(\varepsilon)$, $\beta_3(\varepsilon)$, and $e_3(\varepsilon)$ are some continuous functions satisfying relations (23).

Thus, the main distinction in the bifurcation scenario of q-cycles of system (5) in the direction of the vector e for even and odd q is the form of the asymptotic formulae for the bifurcating solutions. In particular, for even q, the leading terms in formulae (27) and (28) are independent of quadratic terms. The following corollaries hold true as well.

Corollary 2. *Suppose that under the assumption of Theorem 3, numbers (16) are non-zero. For the definiteness let $\alpha_1 > 0$ and $\beta_1 > 0$. Then the value $\mu = \mu_0 = (0,0)$ is a well-defined bifurcation point of q-cycles of system (5) in the direction of the vectors e and $-e$. Two appearing continuous branches of q-cycles are so that one of them exists for $\alpha > 0$ and $\beta > 0$, while the other does for $\alpha < 0$ and $\beta < 0$.*

Corollary 3. *Suppose that under the assumption of Theorem 4 numbers (25)are non-zero. For the definiteness let $\alpha_2 > 0$ and $\beta_2 > 0$. Then the value $\mu = \mu_0 = (0,0)$ is a well-defined bifurcation point of q-cycles of system (5) in the direction of the vectors e and $-e$. Two appearing continuous branches of q-cycles are so that both exist for $\alpha > 0$ and $\beta > 0$.*

In other words, in a natural sense, for odd q, the bifurcation of q-cycles of system (5) is transcritical, while for even q it is a pitchfork bifurcation (see, for instance, [4]).

Theorems 3 and 4 show asymptotic formulae allowing us to obtain an approximate representation in a vicinity of the equilibrium zero state of q-cycles of system (5) as well as in a vicinity of the associated values of the parameters α and β. In what follows, these formulae are employed to localize the Arnold tongues $\Psi(p, q)$ of system (5).

5. Localization of Arnold tongues

5.1. Auxiliary constructions. Let e and g be unit vectors (13). In the complex plane $\mathbb{C}$, we define the curve $\Upsilon(p, q, e)$ described by the equation

$$z = \rho(\varepsilon)e^{\varphi(\varepsilon)i}\,, \quad 0 \leqslant \varepsilon \leqslant 1\,, \tag{29}$$

where

$$\rho(\varepsilon) = 1 + \alpha(\varepsilon)\,, \quad \varphi(\varepsilon) = 2\pi(\theta_0 + \beta(\varepsilon))\,.$$

Here $\alpha(\varepsilon)$ and $\beta(\varepsilon)$ are functions (21) or (27) (subject to the parity of the number q). As $\varepsilon = 0$, the point of the curve $\Upsilon(p, q, e)$ coincides with the point $e^{2\pi\theta_0 i}$ (Fig. 4).

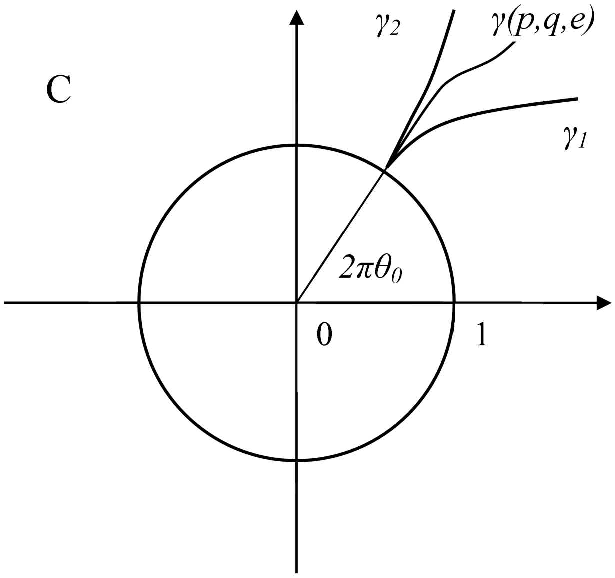

Figure 4. Synchronization curve

The eigenvalues of the matrix $A(\alpha, \beta)$ defined by identity (2) are numbers (4). On the other hand, by Theorems 3 and 4, the bifurcation scenario of q-cycles in the direction of the vector e realizes for system (5) if the eigenvalues of the matrix $A(\alpha, \beta)$ are points of the curve $\Upsilon(p, q, e)$ (for small $\varepsilon \geqslant 0$). This is why the curve $\Upsilon(p, q, e)$ can be treated as one of continuous branches of the eigenvalues of the matrix $A(\alpha, \beta)$ along which the bifurcation scenario of q-cycles of system (5) realizes. We shall call this curve the *synchronization curve* associated with the bifurcation of q-cycles in the direction of the vector e. For small $\varepsilon \geqslant 0$, the synchronization curve is located in the Arnold tongue $\Psi(p, q)$ of system (5).

For fixed p and q, the curve $\Upsilon(p, q, e)$ depends on the vector e: for different e we get different curves $\Upsilon(p, q, e)$, at that, the curve $\Upsilon(p, q, e)$ depends continuously on the vector e in a natural sense (for instance, in the Hausdorff metrics). It allows us to define the Arnold tongue $\Psi(p, q)$ of system (5) as the set of all synchronization curves (for different vectors $e \in R^2$).

Namely, in what follows, the Arnold tongue $\Psi(p, q)$ of system (5) will be determined by the following scheme. We define the continual family of the vectors $(0 \leqslant t \leqslant 2\pi)$

$$e(t) = \begin{bmatrix} \cos t \\ \sin t \end{bmatrix}, \quad g(t) = \begin{bmatrix} -\sin t \\ \cos t \end{bmatrix}. \tag{30}$$

For each t, the vectors $e(t)$ and $g(t)$ are the unit vectors of the form (13). For each $t \in [0, 2\pi]$ we determine the curve $\Upsilon(p, q, e(t))$ (as $\varepsilon = 0$, the point of each of these synchronization curves coincides with $e^{2\pi\theta_0 i}$).

We shall call the set

$$\Psi(p, q) = \bigcup_{t\in[0,2\pi]} \Upsilon(p, q, e(t)) \tag{31}$$

the *Arnold tongue* $\Psi(p, q)$ *of system (5).*

In following subsection we solve the main problem on the localization of sets (31).

5.2. Families of well-defined bifurcations. For each fixed pair of the vectors $e(t)$ and $g(t)$, analogues of Theorems 3 and 4 hold true. To obtain such statements, one should let $x = e(t)$ and $\mu = \mu_0 = (0, 0)$ in (11) and (12) and define vectors

$$b_2(t) = b_2(e(t), \mu_0)\,, \qquad b_3(t) = b_3(e(t), \mu_0) \tag{32}$$

depending on the $t \in [0, 2\pi]$. In particular, by (2) and (11) we get

$$b_2(t) = Q^{q-1}a_2(e(t), \mu_0) + Q^{q-2}a_2(Qe(t), \mu_0) + \cdots + \tag{33}$$

$$+Qa_2(Q^{q-2}e(t), \mu_0) + a_2(Q^{q-1}e(t), \mu_0)\,,$$

where $Q = Q(0)$. Similarly, by (2) and (12) one can obtain a representation for the vector $b_3(t)$. Next, we define the functions

$$\alpha_1(t) = -\frac{1}{q}(b_2(t), e(t))\,, \quad \beta_1(t) = -\frac{1}{2\pi q}(b_2(t), g(t)) \tag{34}$$

and

$$\chi(t) = q\left[\frac{1}{2}\alpha_1^2(t)(1 + q) - 2\pi\beta_1^2(t)(1 + \pi q)\right] e(t)+ \tag{35}$$

$$+\alpha_1(t)\beta_1(t)q(1 + 2\pi + 2\pi q)g(t)+$$

$$+b'_{2x}(t) \cdot (\alpha_1(t)e(t) + \beta_1(t)g(t)) + \alpha_1(t)b'_{2\alpha}(t) + \beta_1(t)b'_{2\beta}(t)\,.$$

Here $b'_{2x}(t) = b'_{2x}(e(t), \mu_0)$ is the Jacobi matrix of nonlinearity (11) taken at the point $x = e(t)$ as $\mu_0 = (0, 0)$, $b'_{2\alpha}(t)$ and $b'_{2\beta}(t)$ are the derivatives of nonlinearity (11) w.r.t. the parameters α and β, respectively, taken at the point $x = e(t)$ as $\mu = \mu_0 = (0, 0)$.

Finally, we let

$$\alpha_2(t) = -\frac{1}{q}(\chi(t) + b_3(t), e(t))\,, \quad \beta_2(t) = -\frac{1}{2\pi q}(\chi(t) + b_3(t), g(t))\,. \tag{36}$$

For each fixed t, the given formulae lead us to analogues of Theorems 3 and 4. Namely, the following statements hold true.

Theorem 5. *Suppose the numberq is odd. Let $e = e(t)$ and $g = g(t)$ be vectors (30) for a fixed $t \in [0, 2\pi]$. Then the bifurcating solutions $x(\varepsilon, t)$, $\alpha(\varepsilon, t)$, and $\beta(\varepsilon, t)$ of system (5) existing in accordance with Theorem 2 can be represented as*

$$x(\varepsilon, t) = \varepsilon e(t) + e_1(\varepsilon, t)\,, \tag{37}$$

$$\alpha(\varepsilon, t) = \varepsilon\alpha_1(t) + \varepsilon^2\alpha_2(t) + \alpha_3(\varepsilon, t),\ \beta(\varepsilon, t) = \varepsilon\beta_1(t) + \varepsilon^2\beta_2(t) + \beta_3(\varepsilon, t)\,, \tag{38}$$

where $e_1(\varepsilon, t)$, $\alpha_3(\varepsilon, t)$ and $\beta_3(\varepsilon, t)$ are continuous w.r.t. all the variables and 2π-periodic w.r.t. t functions satisfying the relations

$$e_1(\varepsilon, t) = o(\varepsilon)\,, \quad \alpha_3(\varepsilon, t) = o(\varepsilon^2)\,, \quad \beta_3(\varepsilon, t) = o(\varepsilon^2) \quad \text{as } \varepsilon \to 0 \tag{39}$$

uniformly in $t \in [0, 2\pi]$.

Theorem 6. *Suppose the number q is even. Let $e = e(t)$ and $g = g(t)$ be vectors (30) for a fixed $t \in [0, 2\pi]$. Then the bifurcating solutions $x(\varepsilon, t)$, $\alpha(\varepsilon, t)$, and $\beta(\varepsilon, t)$ of system (5) existing in accordance with Theorem 2 can be represented as*

$$x(\varepsilon, t) = \varepsilon e(t) + e_1(\varepsilon, t)\,, \tag{40}$$

$$\alpha(\varepsilon, t) = \varepsilon^2 \alpha_2(t) + \alpha_3(\varepsilon, t)\,, \quad \beta(\varepsilon, t) = \varepsilon^2 \beta_2(t) + \beta_3(\varepsilon, t)\,, \tag{41}$$

where

$$\alpha_2(t) = -\frac{1}{q}(b_3(t), e(t))\,, \quad \beta_2(t) = -\frac{1}{2\pi q}(b_3(t), g(t))\,, \tag{42}$$

$e_1(\varepsilon, t)$, $\alpha_3(\varepsilon, t)$ and $\beta_3(\varepsilon, t)$ are continuous w.r.t. all the variables and 2π-periodic w.r.t. t functions satisfying relations (39).

Formulas (37) and (40) can be specified by analogues of asymptotic formulae (22) and (28) provided in Theorems 3 and 4. However, in what follows, we shall be interested only in asymptotics formulae (38) and (41).

5.3. Main statements: weakly-resonance case. We provide now the main statements of the work allowing us to localize the Arnold tongues $\Psi(p, q)$ of system (5) defined by identity (31). Here the cases $q \geqslant 5$ and $q \leqslant 4$ are essentially different. The former is called the weakly-resonance case, while the latter is the strongly-resonance case. We begin with the weakly-resonance case.

Lemma 4. *Let $q \geqslant 5$. Then the functions defined by identity (34) vanish for each t: $\alpha_1(t) \equiv 0$, $\beta_1(t) \equiv 0$.*

Corollary 4. *Let $q \geqslant 5$. Then the function $\chi(t)$ defined by identity (35) vanishes, $\chi(t) \equiv 0$, and functions (36) coincide with functions (42).*

Lemma 5. *Let $q \geqslant 5$. Then functions (36) and (42) are constants equal to corresponding numbers (25).*

Theorem 7. *Let $q \geqslant 5$. Then the Arnold tongue $\Psi(p, q)$ of system (5) is determined by the identity (31), where, for a fixed t, $\Upsilon(p, q, e(t))$ is the curve described by the equation*

$$z = (1 + \alpha(\varepsilon, t))e^{2\pi(\theta_0 + \beta(\varepsilon, t))i}\,, \ \ 0 \leqslant \varepsilon \leqslant 1\,. \tag{43}$$

Here

$$\alpha(\varepsilon, t) = \alpha_2 \varepsilon^2 + \varepsilon^3 \alpha_3(\varepsilon, t)\,, \tag{44}$$

$$\beta(\varepsilon, t) = \beta_2 \varepsilon^2 + \varepsilon^3 \beta_3(\varepsilon, t)\,; \tag{45}$$

α_2 and β_2 are numbers (25), and the functions $\alpha_3(\varepsilon, t)$ and $\beta_3(\varepsilon, t)$ are continuous and 2π-periodic w.r.t. t.

It follows from identities (44) and (45) that for $q \geqslant 5$ and small $\varepsilon \geqslant 0$, the Arnold tongues $\Psi(p, q)$ of system (5) are very narrow. Namely, if numbers (25) are non-zero, the set $\Psi(p, q)$ can be locally identified with the curve $\tilde{\Psi}(p, q)$ described by the equation

$$z = (1 + \alpha_2 \xi)e^{2\pi(\theta_0 + \beta_2 \xi)i}\,, \ \ 0 \leqslant \xi \leqslant 1\,, \tag{46}$$

and beginning (as $\xi = 0$) at the point $e^{\varphi_0 i}$ on the unit circle $S \in \mathbb{C}$; here $\varphi_0 = 2\pi p/q$.

Identities (44) and (45) imply also the following fact. Let number (25) are non-zero, and for definiteness, $\alpha_2 > 0$ and $\beta_2 > 0$. Then the Arnold tongue $\Psi(p, q)$ of system (5) corresponds to the value of the parameters α and β obeying the inequalities $\alpha > 0$ and $\beta > 0$.

5.4. Main statements: strongly-resonance case. Consider now the strongly-resonance case, i.e., let $2 \leqslant q \leqslant 4$. In this case the Arnold tongues $\Psi(p,q)$ of system (5) are in a natural sense essentially wider than as $q \geqslant 5$. Suppose first that q is even.

Theorem 8. *Let $q = 2$ or $q = 4$. Then the Arnold tongue $\Psi(1,q)$ of system (5) is determined by identity (31) (where $p = 1$ and $q = 2$ or $q = 4$), where, for a fixed t, $\Upsilon(p,q,e(t))$ is the curve described by equation (43) as*

$$\alpha(\varepsilon,t) = \alpha_2(t)\varepsilon^2 + \varepsilon^3\alpha_3(\varepsilon,t)\,, \tag{47}$$

$$\beta(\varepsilon,t) = \beta_2(t)\varepsilon^2 + \varepsilon^3\beta_3(\varepsilon,t)\,. \tag{48}$$

Here $\alpha_2(t)$ and $\beta_2(t)$ are functions (42) (as $q = 2$ or $q = 4$), and the functions $\alpha_3(\varepsilon,t)$ and $\beta_3(\varepsilon,t)$ are continuous and 2π-periodic w.r.t. t.

It follows from identities (47) and (48) that for $q = 2$ or $q = 4$, the Arnold tongues $\Psi(p,q)$ of system (5) can be locally identifies with the set (w.r.t. $t \in [0,2\pi]$) of the curves described by the equations

$$z = (1 + \alpha_2(t)\xi)e^{2\pi(\theta_0+\beta_2(t)\xi)i}\,, \;\; 0 \leqslant \xi \leqslant 1\,. \tag{49}$$

Suppose now q is odd, namely, let $q = 3$.

Theorem 9. *The Arnold tongue $\Psi(1,3)$ of system (5) is determined by identity (31) (for $p = 1$ and $q = 3$), where, for a fixed t, $\Upsilon(p,q,e(t))$ is the curve described by equation (43) as*

$$\alpha(\varepsilon,t) = \alpha_1(t)\varepsilon + \alpha_2(t)\varepsilon^2 + \varepsilon^3\alpha_3(\varepsilon,t)\,, \tag{50}$$

$$\beta(\varepsilon,t) = \beta_1(t)\varepsilon + \beta_2(t)\varepsilon^2 + \varepsilon^3\beta_3(\varepsilon,t)\,. \tag{51}$$

Here $\alpha_1(t)$ and $\beta_1(t)$ are functions (34) (as $q = 3$), $\alpha_2(t)$ and $\beta_2(t)$ are functions (36)(as $q = 3$), and the functions $\alpha_3(\varepsilon,t)$ and $\beta_3(\varepsilon,t)$ are continuous and 2π-periodic w.r.t. t.

It follows from identities (50) and (51) that the Arnold tongue $\Psi(1,3)$ of system (5) can be locally identifies with the set (w.r.t. $t \in [0,2\pi]$) of the curves described by the equations

$$z = (1 + \alpha_1(t)\xi)e^{2\pi(\theta_0+\beta_1(t)\xi)i}\,, \;\; 0 \leqslant \xi \leqslant 1\,. \tag{52}$$

The curves γ_1 and γ_2 being in the natural sense the end ones in the set of curves (49) or (52) can be treated as the curves limiting locally , the Arnold tongue $\Psi(p,q)$ of system (5).

6. Examples

6.1. Example 1. Consider the discrete system

$$x_{n+1} = A(\alpha,\beta)x_n + a_3(x_n), \quad n = 0,1,2,..., \quad x_n \in R^2 \tag{53}$$

with $A(\alpha,\beta) = (1+\alpha)Q(\beta)$, where

$$Q(\beta) = \begin{bmatrix} \cos 2\pi(0,25+\beta) & -\sin 2\pi(0,25+\beta) \\ \sin 2\pi(0,25+\beta) & \cos 2\pi(0,25+\beta) \end{bmatrix},$$

and the nonlinearity $a_3(x)$ reads as

$$a_3(x) = \begin{bmatrix} x_1^3 + 2x_2^3 \\ 2x_1x_2^2 \end{bmatrix}.$$

Since

$$Q(0) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$

in this example Condition S1) holds true as $\theta_0 = 1/4$ To localize the Arnold tongue $\Psi(1,4)$ of system (53), we employ Theorem 8. This theorem implies that the set $\Psi(1,4)$ can be locally identified with a set of curves (49), where $\alpha_2(t)$ and $\beta_2(t)$ are functions (42) (as $q = 4$).

Let us calculate the functions $\alpha_2(t)$ and $\beta_2(t)$. Since considered system (53) involves only the cubic nonlinearity $a_3(x)$, formulae (30), (32), (11), and (12) lead us to the identities $b_2(t) \equiv 0$ and

$$b_3(t) = Q^3 a_3(e(t)) + Q^2 a_3(Qe(t)) + Q a_3(Q^2 e(t)) + a_3(Q^3 e(t)),$$

where $Q = Q(0)$. Simple calculations give rise to the identity

$$b_3(t) = 2\begin{bmatrix} 2\sin^2 t\cos t + \sin^3 t - 2\cos^3 t \\ 2\sin t\cos^2 t - 2\sin^3 t - \cos^3 t \end{bmatrix}.$$

Then by (42) we get

$$\alpha_2(t) = -\frac{1}{4}(b_3(t), e(t)) = \frac{1}{4}\cos 2t(4\cos 2t + \sin 2t),$$

$$\beta_2(t) = -\frac{1}{8\pi}(b_3(t), g(t)) = \frac{1}{8\pi}(1 + \cos^2 2t - 2\sin 4t).$$

Substituting these formulae into (49) and analyzing the obtained identity, we get that locally the Arnold tongue $\Psi(1,4)$ of system (53) is located between the curves γ_1 and γ_2 described respectively by the equations

$$z = (1 + \alpha_1\xi)e^{2\pi(0{,}25+\beta_1\xi)i}, \qquad z = (1 + \alpha_2\xi)e^{2\pi(0{,}25+\beta_2\xi)i} \quad (0 \leqslant \xi \leqslant 1).$$

Here

$$\alpha_1 = \frac{1}{2}, \quad \beta_1 = \frac{3-\sqrt{17}}{16\pi}, \quad \alpha_2 = \frac{4-\sqrt{17}}{8}, \quad \beta_2 = \frac{3}{16\pi}.$$

The obtained result is confirmed also by direct numerical calculation of the synchronization curves localizing the Arnold tongue $\Psi(1,4)$ of system (53) in accordance with the formulae in Theorem 8 (Fig. 5).

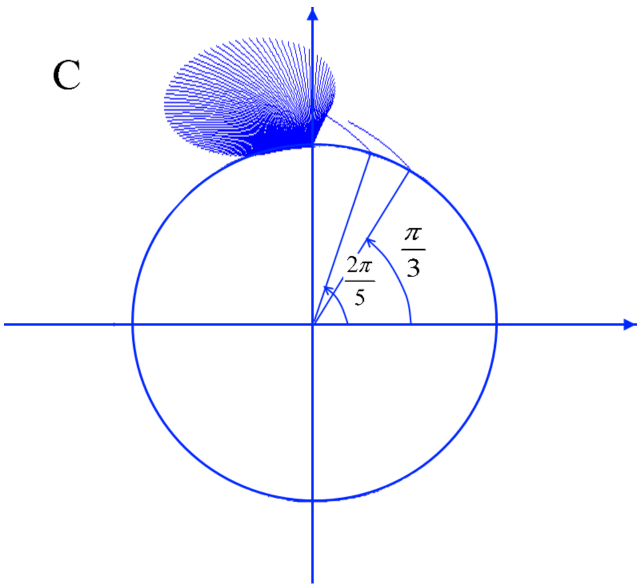

Figure 5. Arnold tongues of system (53)

In Figure 5, we give the synchronization curves localizing the Arnold tongues $\Psi(1,5)$ and $\Psi(1,6)$ of system (53) calculated in accordance with the formulae of Theorem 7. The calculations confirm that the set of these curves (both for the tongues $\Psi(1,5)$ and the tongues $\Psi(1,6)$) forms in fact one curve. In other words, the tongues $\Psi(1,5)$ and $\Psi(1,6)$ are locally very narrow sets coinciding with the synchronization curves starting from the corresponding rational point on the unit circle.

6.2. Example 2. Consider now a non-autonomous dynamical system depending on real variables α and β and described by the differential equation

$$x' = A(\alpha, \beta)x + a(x, t, \alpha, \beta), \quad x \in R^2, \tag{54}$$

where

$$A(\alpha, \beta) = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix},$$

and the nonlinearity $a(x, t, \alpha, \beta)$ is smooth w.r.t. all the variables, 2π-periodic w.r.t. t and can be represented as

$$a(x, t, \alpha, \beta) = a_2(x, t, \alpha, \beta) + a_3(x, t, \alpha, \beta) + \tilde{a}_4(x, t, \alpha, \beta),$$

where $a_2(x, t, \alpha, \beta)$ and $a_3(x, t, \alpha, \beta)$ involves the terms quadratic and cubic w.r.t. x, while $\tilde{a}_4(x, t, \alpha, \beta)$ satisfies the relation $\tilde{a}_4(x, t, \alpha, \beta) = O(\|x\|^4)$, $x \to 0$, uniformly in t, α, and β. For all values of the parameters α and β, system (54) has the equilibrium $x = 0$.

We let $\mu = (\alpha, \beta)$ and $\mu_0 = (0, \beta_0)$, where β_0 is a positive number. As $\mu = \mu_0$, the equilibrium $x = 0$ of system (54) is non-hyperbolic; as the parameter μ passes through μ_0, various bifurcation scenarios are possible. In a particular case, when the nonlinearity $a(x, t, \alpha, \beta)$ is independent of t, the main scenario is the Andronov-Hopf bifurcation: as μ passes through μ_0, in a vicinity of the equilibrium $x = 0$ of system (54), there appear nonstationary periodic solutions of small amplitude and with period close to $T_0 = \dfrac{2\pi}{\beta_0}$.

The presence of a nonstationary periodic nonlinearity $a(x, t, \alpha, \beta)$ follows a change in the mentioned bifurcation scenario. Namely, various scenarios of emergence of subharmonic solutions (i.e., periodic solutions with the period being multiple of T) and quasi-periodic solutions become possible in a vicinity of the equilibrium $x = 0$.

To study such scenarios and, in particular, the localization of the Arnold tongues of system (54), it is possible to employ the scheme suggested in the present work. On the first step of such studying system (54) we pass to the discrete dynamical system described by the equation

$$x_{n+1} = V(\mu)x_n + v(x_n, \mu), \quad n = 0, 1, 2, ..., \tag{55}$$

where $x_n \in R^2$, $V(\mu) = e^{2\pi A(\mu)}$, and the nonlinear operator $v(\cdot, \mu) : R^2 \to R^2$ can be represented as

$$v(x, \mu) = \int_0^{2\pi} e^{(2\pi - \tau)A(\mu)} a(x(\tau), \tau, \mu)\, d\tau,$$

where $x(t)$ is the solution to system (54) satisfying the initial condition $x(0) = x$. The fixed points of system (55) determines the initial values of 2π-periodic solutions of system (54), and the cycles of period q determine the initial values $2\pi q$-periodic solutions of this system.

It is easy to show that the matrix $V(\mu)$ is as follows,

$$V(\mu) = e^{2\pi\alpha} \begin{bmatrix} \cos 2\pi\beta & -\sin 2\pi\beta \\ \sin 2\pi\beta & \cos 2\pi\beta \end{bmatrix}.$$

We replace α and β by new parameters

$$\alpha^* = e^{2\pi\alpha} - 1, \quad \beta^* = \beta - \beta_0,$$

and represent system (55) as

$$x_{n+1} = A(\alpha^*, \beta^*)x_n + b(x_n, \alpha^*, \beta^*), \quad n = 0, 1, 2, ..., , \tag{56}$$

where

$$b(x, \alpha^*, \beta^*) = v(x, \ln(1 + \alpha^*)/(2\pi), \beta^* + \beta_0),$$
$$A(\alpha^*, \beta^*) = (1 + \alpha^*)Q(\beta^*).$$

Here

$$Q(\beta^*) = \begin{bmatrix} \cos 2\pi(\beta_0 + \beta^*) & -\sin 2\pi(\beta_0 + \beta^*) \\ \sin 2\pi(\beta_0 + \beta^*) & \cos 2\pi(\beta_0 + \beta^*) \end{bmatrix}.$$

The problem on the local bifurcations of system (54) is equivalent in a natural sense to a similar problem for system (56). Since this system is similar to system (5), on the next step we can employ the scheme suggested in the previous sections. In particular, in accordance with with this scheme we get that if the number β_0 is rational, $\beta_0 = \dfrac{p}{q}$, as the two-dimensional parameter $\mu = (\alpha, \beta)$ passes through the point $\mu_0 = (0, \beta_0)$, for system (54), the scenario of emergence of subharmonic solutions of period $2\pi q$ becomes possible in a vicinity of the equilibrium $x = 0$.

At that, on the plane of the parameters (α, β), there forms a system of the Arnold tongues with cusps at the points $(0, \beta_0)$ with rational β_0. Such tongues correspond to the domains of the parameters values for which system (54) has periodic regimes of period multiple of 2π with amplitudes tending to zero as the point (α, β) tends to $(0, \beta_0)$. The mentioned tongues can be localized in accordance with the scheme stated in the previous sections.

7. Proof of main statements

7.1. Operator method. The proofs of main statements of the present work are based on the operator method for studying problems on multi-parametric local bifurcations developed in [11] and [12]. We give briefly the main concepts of this method. Here it is sufficient to restrict ourselves by considering two-parametric problems for operator equations on the plane.

Consider the operator equation

$$x = B(\mu)x + b(x, \mu), \quad x \in R^2 \tag{57}$$

depending on the two-dimensional parameter $\mu = (\alpha, \beta) \in R^2$, where the second order square matrix $B(\mu)$ is continuously differentiable w.r.t. μ and the nonlinearity $b(x, \mu)$ depends smoothly on μ as well and is represented as

$$b(x, \mu) = b_2(x, \mu) + b_3(x, \mu) + \tilde{b}_4(x, \mu),$$

where $b_2(x, \mu)$ and $b_3(x, \mu)$ involve the quadratic and cubic w.r.t. x terms, respectively, and $\tilde{b}_4(x, \mu)$ is smooth w.r.t. x and $\tilde{b}_4(x, \mu) = O(\|x\|^4)$, $x \to 0$ uniformly in μ.

For all values of μ, equation (57) has the zero solution $x = 0$. We shall say that a value μ_0 is the *bifurcation point of nonzero solutions to equation (57)* if there exists a sequence $\mu_k \to \mu_0$ such that as $\mu = \mu_k$, equation (57) has a nonzero solution $x = x_k$ and $\|x_k\| \to 0$ as $k \to \infty$.

As a rule, the bifurcations of nonzero solutions to equation (57) has a oriented character; let us provide the corresponding direction. Let $e \in R^2$ be a nonzero vector. We shall call a value μ_0 of the parameter μ the *well-defined bifurcation point of equation (57) in the direction of the vector* e if there exists $\varepsilon_0 > 0$ and continuous functions $\mu = \mu(\varepsilon)$ and $x = x(\varepsilon)$ defined for $\varepsilon \in [0, \varepsilon_0)$ such that

1) $\mu(0) = \mu_0$, $x(0) = 0$;
2) $\|x(\varepsilon) - \varepsilon e\| = o(\varepsilon)$ as $\varepsilon \to 0$;
3) for each $\varepsilon \geqslant 0$, the vector $x(\varepsilon)$ solves equation (57) as $\mu = \mu(\varepsilon)$.

We shall call $x(\varepsilon)$ and values $\mu(\varepsilon)$ the *bifurcating solutions* of equation (57).

Lemma 6. *Suppose the value μ_0 of the parameter μ is a well-defined bifurcation point of equation (57) in the direction of the vector e. Then the vector e is an eigenvector for the matrix $B(\mu_0)$ associated with the eigenvalue.*

In what follows we assume that the matrix $B(\mu_0)$ has a semi-simple eigenvalue 1 of multiplicity 2; in other words, let $B(\mu_0) = I$, where I is the unit matrix of second order. We denote $\mu_0 = (\alpha_0, \beta_0)$ and $B_0 = B(\mu_0)$.

Let e, g and e^*, g^* be two pair of linearly independent vectors chosen by the relations

$$(e, e^*) = (g, g^*) = 1, \ (e, g^*) = (g, e^*) = 0. \tag{58}$$

We let

$$S = \begin{bmatrix} (B'_\alpha(\alpha_0, \beta_0)e, e^*) & (B'_\beta(\alpha_0, \beta_0)e, e^*) \\ (B'_\alpha(\alpha_0, \beta_0)e, g^*) & (B'_\beta(\alpha_0, \beta_0)e, g^*) \end{bmatrix}. \tag{59}$$

Here B'_α and B'_β are the matrices obtained by the differentiation of the matrix $B(\alpha, \beta)$ w.r.t. α and β, respectively.

Theorem 10. *Suppose*

$$\det S \neq 0. \tag{60}$$

Then μ_0 is a well-defined bifurcation point of equation (57) in the direction of the vector e.

In what follows we shall make use of the following notations,

$$b_2 = b_2(e, \alpha_0, \beta_0), \ \ b_3 = b_3(e, \alpha_0, \beta_0), \tag{61}$$

$$b'_{2x} = b'_{2x}(e, \alpha_0, \beta_0), \ \ b'_{2\alpha} = b'_{2\alpha}(e, \alpha_0, \beta_0), \ \ b'_{2\beta} = b'_{2\beta}(e, \alpha_0, \beta_0). \tag{62}$$

We let

$$Fh = -\left[(h, e^*)B'_\alpha e + (h, g^*)B'_\beta e\right], \quad h \in R^2, \tag{63}$$

where it is denoted $B'_\alpha = B'_\alpha(\alpha_0, \beta_0)$ and $B'_\beta = B'_\beta(\alpha_0, \beta_0)$. By condition (60), the linear operator $F : R^2 \to R^2$ is invertible. We let

$$\Gamma_0 = F^{-1} : \ R^2 \to R^2. \tag{64}$$

Lemma 7. *The operator $\Gamma_0 = F^{-1}$ is determined by the formula*

$$\Gamma_0 y = J_\alpha(y)e + J_\beta(y)g\,.$$

Here the functionals $J_\alpha(y)$ and $J_\beta(y)$ are the components of the vector

$$J(y) = \begin{bmatrix} J_\alpha(y) \\ J_\beta(y) \end{bmatrix}$$

given by the formula $J(y) = -S^{-1}\gamma(y)$, where S is matrix (59) and

$$\gamma(y) = \begin{bmatrix} (y, e^*) \\ (y, g^*) \end{bmatrix}.$$

We then let

$$e_1 = \Gamma_0 b_2, \ \ \alpha_1 = J_\alpha(b_2), \ \ \beta_1 = J_\beta(b_2)\,, \tag{65}$$

$$e_2 = \Gamma_0(\varphi + b_3)\,, \quad \alpha_2 = J_\alpha(\varphi + b_3)\,, \quad \beta_2 = J_\beta(\varphi + b_3). \tag{66}$$

Here

$$\varphi = \alpha_1 B'_\alpha \Gamma_0 b_2 + \beta_1 B'_\beta \Gamma_0 b_2 + \frac{\alpha_1^2}{2} B''_{\alpha\alpha} e + \alpha_1 \beta_1 B''_{\alpha\beta} e + \tag{67}$$

$$+ \frac{\beta_1^2}{2} B''_{\beta\beta} e + b'_{2x} \Gamma_0 b_2 + \alpha_1 b'_{2\alpha} + \beta_1 b'_{2\beta},$$

Γ_0 is operator (64), B'_α, B'_β, $B''_{\alpha\alpha}$, $B''_{\alpha\beta}$, $B''_{\beta\beta}$ are the matrices obtained by the differentiation of the matrix $B(\alpha, \beta)$ w.r.t. α and (or) β at the point (α_0, β_0); we also employ notations (61) and (62).

Theorem 11. *The bifurcating solutions $x(\varepsilon)$, $\alpha(\varepsilon)$, and $\beta(\varepsilon)$ of equation (57) existing under the assumption of Theorem 10 can be represented as*

$$x(\varepsilon) = \varepsilon e + \varepsilon^2 e_1 + \varepsilon^3 e_2 + o(\varepsilon^3)\,, \tag{68}$$

$$\alpha(\varepsilon) = \alpha_0 + \varepsilon\alpha_1 + \varepsilon^2\alpha_2 + o(\varepsilon^2)\,, \quad \beta(\varepsilon) = \beta_0 + \varepsilon\beta_1 + \varepsilon^2\beta_2 + o(\varepsilon^2)\,. \tag{69}$$

7.2. Auxiliary statements. In order to prove the main statements of the work, we shall make use of auxiliary statements.

Lemma 8. *A point μ_0 is a bifurcation point for q-cycles of system (5) if and only if μ_0 a bifurcation point of nonzero solutions to equation (8).*

Proof. The necessity. Suppose $\mu = \mu_0$ is a bifurcation point for q-cycles of system (5), i.e., there exists a sequence $\mu_k \to \mu_0$ such that as $\mu = \mu_k$, system (5) has a q-cycle $x_0^k, x_1^k, x_2^k, \dots, x_{q-1}^k$, and $\max\limits_{0\leqslant j\leqslant q-1} \|x_j^k\| \to 0$ as $k \to \infty$. At that, $x_j^k \neq 0$ that follows from the definition of the q-cycle. By Lemma 1 we get that as $\mu = \mu_k$, each of nonzero vectors x_j^k solves equation (8). Hence, the value μ_0 is a bifurcation point for nonzero solutions of equation (8).

The sufficiency. Suppose $\mu = \mu_0$ is a bifurcation point for nonzero solutions of equation (8), i.e., there exists a sequence $\mu_k \to \mu_0$ such that as $\mu = \mu_k$, equation (8) has the nonzero solution $x = x_k$ and $\|x_k\| \to 0$ as $k \to \infty$. Let us show that in this case, as $\mu = \mu_k$, system (5) has a q-cycle and one of its points is the vector x_k; it would mean that $\mu = \mu_0$ is a bifurcation point for q-cycles of system (5).

Indeed, Lemma 1 implies that either $x = x_k$ is a fixed point of system (5) as $\mu = \mu_k$ or it is one of the points of a cycle with period r for this system as $\mu = \mu_k$, where r is a divisor of the number q. The vector x_k can be neither a fixed point of system (5) no a point of a r-cycle of this system once $r \neq q$. We restrict ourselves by proving the first fact. Assuming the opposite, we obtain the identities

$$x_k = A(\mu_k)x_k + a(x_k, \mu_k)\,.$$

Dividing both sides of this identity by a nonzero number $\|x_k\|$ and letting $y_k = x_k/\|x_k\|$, we obtain

$$y_k = A(\mu_k)y_k + \frac{a(x_k, \mu_k)}{\|x_k\|}\,. \tag{70}$$

Since $\|y_k\| = 1$, we can assume that the sequence y_k converges: $y_k \to y^*$, where $\|y^*\| = 1$. Passing in (70) to the limit as $k \to \infty$, we obtain the identity $y^* = A(\mu_0)y^*$, i.e., the matrix $A(\mu_0)$ has an eigenvalue 1. This fact contradicts to formulae (2) and (3) determining the matrix $A(\mu_0) = A(0, 0)$. The proof is complete. □

In the same way one can prove

Lemma 9. *A point μ_0 is a bifurcation point for q-cycles of system (5) in the direction of a vector e if and only if μ_0 a bifurcation point of nonzero solutions to equation (8) in the direction of a vector e.*

7.3. Proof of Theorem 1. Equation (8) is that of the form (57). This is why if we prove that for equation (8) relation (60) holds true for some choice of the vectors e, e^*, g, and g^* (satisfying conditions (58)), by Theorem 10 it will mean that μ_0 is a bifurcation point for nonzero solutions of equation (8). Then Lemma 10 will imply that μ_0 is a bifurcation point for q-cycles of system (5), i.e., then Theorem 1 holds true.

The matrix $B(\alpha, \beta)$ in equation (8) is determined by identity (9),

$$B(\alpha, \beta) = (1 + \alpha)^q Q^q(\beta)\,. \tag{71}$$

Here $Q(\beta)$ is matrix (3). We have

$$Q^q(\beta) = \begin{bmatrix} \cos 2\pi q(\theta_0 + \beta) & -\sin 2\pi q(\theta_0 + \beta) \\ \sin 2\pi q(\theta_0 + \beta) & \cos 2\pi q(\theta_0 + \beta) \end{bmatrix}. \tag{72}$$

Since $\theta_0 = \dfrac{p}{q}$, then $B(0, 0) = I$; the matrix $B(0, 0)$ thus has the semi-simple eigenvalue 1 of multiplicity 2.

Let

$$e(t) = e^*(t) = \begin{bmatrix} \cos t \\ \sin t \end{bmatrix}, \quad g(t) = g^*(t) = \begin{bmatrix} -\sin t \\ \cos t \end{bmatrix}. \qquad (73)$$

For each $t \in [0, 2\pi]$, these vectors satisfy conditions (58). We fix $t \in [0, 2\pi]$ and calculate the matrix S defined by identity (59) S. By (71) and (72) we obtain

$$B'_\alpha(0,0) = qI\,, \quad B'_\beta(0,0) = 2\pi q \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}. \qquad (74)$$

By (59), (73) and simple calculations we get the identity

$$S = \begin{bmatrix} q & 0 \\ 0 & 2\pi q \end{bmatrix}. \qquad (75)$$

Therefore, $\det S = 2\pi q^2 \neq 0$, i.e., relation (60) holds true. The proof is complete.

Remark 1. *In the proof of Theorem 1 it was shown that the matrix S defined by identity (75) is independent of t, i.e., it is the case for each set of vectors (73).*

7.4. Proof of Theorem 2. Due to Lemma 9, Theorem 2 will be proven if we show that the value $\mu_0 = (0,0)$ of the parameter $\mu = (\alpha, \beta)$ is a well-defined bifurcation point for equation (8) in the direction of each nonzero vector e. In fact, it was established in the proof of Theorem 1. Indeed, as an arbitrary nonzero vector e, one can choose the vector $e(t)$ defined by the first identity in (73). For each such vector condition (60) of Theorem 10 holds true, since the corresponding matrix (75) is non-degenerate.

7.5. Proof of Theorem 3. Lemma 9 and Theorem 2 yield that statement of Theorem 11 holds true for equation (8) (with appropriate modifications). This is why Theorems 3 and 4 will be proven if we show that formulae (68) and (69) applied to equation (8) lead us to the formulae given in these theorems. In order to do it, we need to show that the numbers and vectors defined by identities (65)-(66) lead us to the corresponding numbers and vectors in (16)-(20).

We consider first the numbers α_1 and β_1 defined by identities (65). We have

$$\begin{bmatrix} \alpha_1 \\ \beta_1 \end{bmatrix} = \begin{bmatrix} J_\alpha(b_2) \\ J_\beta(b_2) \end{bmatrix} = J(b_2) = -S^{-1}\gamma(b_2) = -S^{-1} \begin{bmatrix} (b_2, e^*) \\ (b_2, g^*) \end{bmatrix}.$$

We convert these identities with equation (8) taken into account. In order to do it, as e, e^*, g, and g^*, we treat vectors (73) for a fixed $t \in [0, 2\pi]$ (for each t, they are vectors of the form (13)), as b_2, we take vector (15), and as S we take matrix (75). Since

$$S^{-1} = \frac{1}{2\pi q} \begin{bmatrix} 2\pi & 0 \\ 0 & 1 \end{bmatrix},$$

then

$$\begin{bmatrix} \alpha_1 \\ \beta_1 \end{bmatrix} = -\frac{1}{2\pi q} \begin{bmatrix} 2\pi (b_2, e) \\ (b_2, g) \end{bmatrix}.$$

Here we have taken into account that in our case the identities $e = e^*$ and $g = g^*$ hold true. Thus, the numbers α_1 and β_1 defined by identities (65) and calculated for equation (8) lead us to the numbers (16).

Consider now the vector e_1 defined by the first identity in (65). By Lemma 7 we have

$$e_1 = \Gamma_0 b_2 = J_\alpha(b_2) e + J_\beta(b_2) g = \alpha_1 e + \beta_1 g\,,$$

i.e., we obtain formula (17).

To complete the proof of Theorem 3, it remains to argue in the same way and show that the numbers α_2 and β_2 and the vector e_2 defined by identities (66) and calculated for equation (8) lead us to corresponding numbers (19) and vector (20). These arguments follow the same lines as the proof of formulae (16) and (17).

At that, in addition it should be shown that vector (67) calculated for equation (8) leads us to vector (18). In other words, it should be also shown that the identity $\varphi = \chi$ holds true for equation (8), where φ and χ are respectively vectors (67) and (18). To prove this fact, together with formulas (74) we employ the identities

$$B''_{\alpha\alpha}(0,0) = q(q-1)I\,, \quad B''_{\beta\beta}(0,0) = -(2\pi q)^2 I\,, \quad B''_{\alpha\beta}(0,0) = 2\pi q^2 \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$

$$\Gamma_0 b_2 = \alpha_1 e + \beta_1 g\,.$$

Substituting these formulae into (67) leads us to the desired identity $\varphi = \chi$.

Remark 2. *We have not employed the parity of q in the proof of Theorem 3. In other words, this theorem is valid for each q. However, for even q, formulae (21) in Theorem 3 possess specific properties that leads us to a qualitative difference between the bifurcation properties of q-cycles of system (5) for even and odd q.*

7.6. Proof of Lemma 3. By (3) we have

$$Q = Q(0) = \begin{bmatrix} \cos 2\pi\theta_0 & -\sin 2\pi\theta_0 \\ \sin 2\pi\theta_0 & \cos 2\pi\theta_0 \end{bmatrix}, \tag{76}$$

where $\theta_0 = p/q$ is an irreducible fraction. Suppose q is even. Then p is odd and hence $Q^{q/2} = -I$; this is why, the identities

$$Q^j = -Q^{j+q/2}\,, \quad j = 0, 1, 2, \ldots \tag{77}$$

hold true.

In the case of even q, the vector b_2 defined by identity (15) involves an even number of terms satisfying by (77) the identities

$$Q^j a_2(Q^{q-1-j}e, \mu_0) = -Q^{j+q/2} a_2(Q^{q/2-1-j}e, \mu_0)\,.$$

Here we have taken into consideration that the nonlinearity $a_2(x,\mu)$ contains only quadratic w.r.t. x terms. This is why $b_2 = 0$. The proof is complete.

7.7. Proof of Theorem 4. The proof of Theorem 4 is reduced to the substitution of identities (24) into formulas (21) and (22).

7.8. Proof of Lemma 4. We note that for each even q, this lemma can be proven in the same way as Lemma 3. For arbitrary $q \geqslant 5$, in the proof of the lemma we shall need auxiliary constructions.

7.8.1. Auxiliary constructions. Let $f(t)$ be a continuous 2π-periodic function. For a natural number n we let $h = \dfrac{2\pi}{n}$ and define a function,

$$G_f^{(n)}(t) = [f(t) + f(t+h) + f(t+2h) + \cdots + f(t+(n-1)h)]h\,. \tag{78}$$

This function is 2π-periodic and for a fixed t, it is a rectangular approximation for calculating an integral of the function $f(s)$, namely, for each fixed t the approximate identity

$$\int\limits_0^{2\pi} f(s)\,ds \approx G_f^{(n)}(t) \tag{79}$$

holds true. In this approximate formula the value t determines the choice of n points $t\,,\ t+h\,,\ t+2h\,,\ \ldots\,,\ t+(n-1)h$ in the segment of length 2π at which the values of the function $f(s)$ are calculated.

We denote by P_n the set of continuous 2π-periodic functions $f(s)$ such that the identity

$$\int_0^{2\pi} f(s)\,ds \equiv G_f^{(n)}(t) \tag{80}$$

holds true. In other words, the set P_n consists of the functions for which approximate formula (79) is exact for each t.

Lemma 10. *Let natural numbers n and m be such that*

$$\frac{2m}{n} \neq k\,, \quad k = 1, 2, 3, \dots\,. \tag{81}$$

Then $\sin mt \in P_n$ *and* $\cos mt \in P_n$.

In other words, under condition (81), the functions $f(t) = \sin mt$ and $f(t) = \cos mt$ satisfy the identity $G_f^{(n)}(t) \equiv 0$.

Proof. We restrict ourselves by considering the function $f(t) = \cos mt$. By (78) we define the auxiliary function

$$F(t) = f(t) + f(t+h) + f(t+2h) + \dots + f(t+(n-1)h) = \tag{82}$$
$$= \cos mt + \cos m(t+h) + \cos m(t+2h) + \dots + \cos m(t+(n-1)h) =$$
$$= \cos\tau + \cos(\tau+\nu) + \cos(\tau+2\nu) + \dots + \cos(\tau+(n-1)\nu)\,,$$

where it is denoted $\tau = mt$ and $\nu = mh$. The lemma will be proven if we prove the identity $F(t) \equiv 0$.

By (81) we have

$$\nu = mh = m\frac{2\pi}{n} \neq \pi k\,, \quad k = 0, 1, 2, \dots\,.$$

Therefore, $\sin\nu \neq 0$. This is why function (82) can be represented as

$$F(t) = \cos\tau + \frac{\cos(\tau+\nu) + \cos(\tau+2\nu) + \dots + \cos(\tau+(n-1)\nu)}{\sin\nu} \cdot \sin\nu\,.$$

Employing now the formula $\cos\alpha\sin\beta = \frac{1}{2}[\sin(\alpha+\beta) - \sin(\alpha-\beta)]$, by simple transformations we arrive to the identity $F(t) \equiv 0$. The proof is complete. □

In what follows, we shall be interested in Lemma 10 only for $m = 1, 2, 3, 4$. For these numbers, Lemma 10 can be deepened.

Let, for instance, $m = 1$; it follows from Lemma 10 that if $n \neq 1$ and $n \neq 2$, the $\sin mt \in P_n$ and $\cos mt \in P_n$. A straightforward checking shows that these belongings hold true as $n = 2$, while for $n = 1$ they are not true. Similar arguments for the numbers $m = 2$, $m = 3$, and $m = 4$ lead us to the following auxiliary statement.

Lemma 11. *Let $f(t) = \sin mt$ or $f(t) = \cos mt$. Then*

- *if $m = 1$, then $f(t) \in P_n \iff n \neq 1$;*
- *if $m = 2$, then $f(t) \in P_n \iff n \neq 1$ and $n \neq 2$;*
- *if $m = 3$, then $f(t) \in P_n \iff n \neq 1$ and $n \neq 3$;*
- *if $m = 4$, then $f(t) \in P_n \iff n \neq 1$, $n \neq 2$, and $n \neq 4$.*

We define the functions

$$f_2(t) = (a_2(e(t), \mu_0), Qe(t))\,, \quad g_2(t) = (a_2(e(t), \mu_0), Qg(t))\,, \tag{83}$$

and let us show that the following lemma holds true.

Lemma 12. *Functions (34) and (83) are related by the identities*

$$G_{f_2}^{(q)}(t) = -2\pi\alpha_1(t)\,, \quad G_{g_2}^{(q)}(t) = -\beta_1(t)\,. \tag{84}$$

Proof. For the sake of simplicity, in the proof of this lemma we shall denote the nonlinearity $a_2(x,\mu_0)$ by $a_2(x)$, i.e., we shall omit the notation μ_0. By (78) we have

$$G_{f_2}^{(q)}(t) = [f_2(t) + f_2(t+h) + \cdots + f_2(t+(q-1)h)]h = \tag{85}$$

$$= [(a_2(e(t)), Qe(t)) + (a_2(e(t+h)), Qe(t+h)) +$$

$$+ \cdots + (a_2(e(t+(q-1)h)), Qe(t+(q-1)h))]h\,;$$

here $h = 2\pi/q$. On the other hand, by (33) and (34) we get

$$\alpha_1(t) = -\frac{1}{q}(b_2(t), e(t)) = -\frac{1}{q}\left(Q^{q-1}a_2(e(t)) + Q^{q-2}a_2(Qe(t)) + \cdots +\right.$$

$$\left. + a_2(Q^{q-1}e(t)), e(t)\right) = -\frac{1}{q}\left[(a_2(e(t)), (Q^*)^{q-1}e(t)) +\right.$$

$$\left. + (a_2(Qe(t)), (Q^*)^{q-2}e(t)) + \cdots + (a_2(Q^{q-1}e(t)), e(t))\right]\,,$$

where Q^* is the transposed matrix. Matrix (76) satisfies the identities

$$(Q^*)^k = Q^{q-k}\,, \quad k = 0, 1, 2, \ldots$$

Hence,

$$\alpha_1(t) = -\frac{1}{q}\left[(a_2(e(t)), Qe(t)) + (a_2(Qe(t)), Q^2e(t)) +\right. \tag{86}$$

$$\left. + \cdots + (a_2(Q^{q-1}e(t)), e(t))\right]\,,$$

Let us compare identities (85) and (86). Suppose first $p = 1$, i.e., $\theta_0 = 1/q$. In this case the identities

$$e(t+h) = Qe(t)\,, \; e(t+2h) = Q^2e(t)\,, \; \ldots$$

hold true and it implies that the corresponding terms in the brackets in the right hand sides of formulae (85) and (86) coincide. In the case $p > 1$, the terms in the right hand side of formulae (85) and (86) coincide as well but after appropriate permutations. It means the validity of the first identity (84). The second identity can be proven in the same way. The proof is complete. □

7.8.2. End of proof for Lemma 4. We observe first that since the nonlinearity $a_2(x,\mu)$ is quadratic, the Fourier series of 2π-periodic functions $f_2(t)$ and $g_2(t)$ defined by identities (83) contain only the functions $\sin mt$ and $\cos mt$ as $m = 1$ and $m = 3$. Since $q \geqslant 5$, by Lemma 11 we get that $f_2(t) \in P_q$ and $g_2(t) \in P_q$. In other words, the functions $f_2(t)$ and $g_2(t)$ satisfy the identities

$$\int\limits_0^{2\pi} f_2(s)\,ds \equiv G_{f_2}^{(q)}(t)\,, \quad \int\limits_0^{2\pi} g_2(s)\,ds \equiv G_{g_2}^{(q)}(t)\,.$$

By the aforementioned property of the Fourier series for the functions $f_2(t)$ and $g_2(t)$, the integrals in the obtained identities vanish. Together with Lemma 12 it imply the identities $\alpha_1(t) \equiv 0$ and $\beta_1(t) \equiv 0$. The proof is complete.

7.9. Proof of Lemma 5. By Corollary 4 given in Subsection 5.3, functions (36) and (42) coincide, namely, they read as follows,

$$\alpha_2(t) = -\frac{1}{q}(b_3(t), e(t)) , \quad \beta_2(t) = -\frac{1}{2\pi q}(b_3(t), g(t)) .$$

The rest of the proof of Lemma 5 follows the same lines as that of Lemma 4. At the first step we define the analogues of functions (83),

$$f_3(t) = (a_3(e(t), \mu_0), Qe(t)) , \quad g_3(t) = (a_3(e(t), \mu_0), Qg(t)) . \tag{87}$$

and we show that the analogues of identities (84)

$$G^{(q)}_{f_3}(t) = -2\pi\alpha_2(t) , \quad G^{(q)}_{g_3}(t) = -\beta_2(t)$$

hold true. At the second step we note that since the nonlinearity $a_3(x, \mu)$ is cubic, the Fourier series of 2π-periodic functions $f_3(t)$ and $g_3(t)$ defined by identities (87) involve only the functions $\sin mt$ and $\cos mt$ as $m = 0$, $m = 2$, and $m = 4$. Together with Lemma 12 it implies Lemma 5.

7.10. Proof of Theorems 7-9. The validity of Theorem 7 follows from Theorems 5 and 6 and Lemma 5. Theorem 8 follows from Theorem 6, and Theorem 9 is implied by Theorem 5.

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