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ON SOME NONLINEAR INTEGRAL AND INTEGRO-DIFFERENTIAL EQUATIONS WITH NONCOMPACT OPERATORS ON POSITIVE HALF-LINE

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Abstract. The paper is devoted to the studying certain classes of nonlinear integral and integro-differential with non-compact Hammerstein type operators. These equations have important applications in the kinetic theory of gases and in the wealth distribution theory of a one product economics.

Keywords:integral equation, Hammerstein operator, Sobolev space, convergence, monotonicity.

1. Introduction

The work is devoted to the solvability in certain functional spaces of the following classes of nonlinear integral and integro-differential equations with a non-compact Hammerstein-Wiener-Hopf type operator,

$$f(x) = \int_0^\infty K_0(x-t)N_0(t, f(t))dt + \int_0^\infty K_1(x+t)N_1(t, f(t))dt, \quad x > 0, \tag{1}$$

$$\begin{cases} \dfrac{d\varphi}{dx} + \lambda\varphi(x) = \displaystyle\int_0^\infty T(x-t)H(t,\varphi(t))dt + \int_0^\infty T_1(x+t)H_1(t,\varphi(t))dt, \quad x > 0, & (2) \\ \varphi(0) = 0 & (3), \end{cases}$$

w.r.t. the functions $f(x)$ and $\varphi(x)$, respectively.

Apart from a mathematical interest, these classes of equations have direct applications in the kinetic theory of gases (equation (1)) and in the econometric theory (problem (2)-(3)) (see [1]-[4]).

For equation (1) we suppose

$$K_0(x) \geq 0, \quad x \in \mathbb{R}, \quad K_0 \in L_1(\mathbb{R}) \cap L_\infty(\mathbb{R}), \quad \int_{-\infty}^{+\infty} K_0(x)dx = 1, \tag{4}$$

$$K_1(x) \geq 0, \quad K_1 \not\equiv 0, \quad \int_x^\infty K_1(\tau)d\tau \leqslant \int_x^\infty K_0(\tau)d\tau, \quad x \in \mathbb{R}^+ \equiv (0, +\infty). \tag{5}$$

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In problem (2)-(3), λ is a positive scalar parameter of equation (2), and kernels T and T_1 satisfy the following conditions,

$$T_1(x) \geq 0, \quad T_1 \not\equiv 0, \quad x \in \mathbb{R}^+, \quad T_1 \in L_1(\mathbb{R}^+), \tag{6}$$

$$T(x) \geq 0, \quad x \in \mathbb{R}, \quad T \in L_1(\mathbb{R}), \quad \int\limits_{-\infty}^{+\infty} T(x)dx = \lambda, \tag{7}$$

$$\int\limits_{x}^{\infty} T_1(z)dz \leqslant \int\limits_{x}^{\infty} T(z)dz, \quad x \in \mathbb{R}^+, \tag{8}$$

$$\nu(T) \equiv \int\limits_{-\infty}^{+\infty} \tau T(\tau)d\tau < -1, \quad \int\limits_{-\infty}^{+\infty} |\tau|^j T(\tau)d\tau < +\infty, \quad j = 1, 2. \tag{9}$$

N_0, N_1, H, and H_1 are real functions defined on the set $\mathbb{R}^+ \times \mathbb{R}$ and satisfying certain conditions (see Theorems 1-3).

In the linear case, as $N_0(t,z) \equiv N_1(t,z) \equiv z$, numerous papers were devoted to studying equation (1) (see [5]–[8] and the references therein).

In the case $K_0(x) = K_1(x) = \dfrac{1}{\sqrt{\pi}} e^{-x^2}$ and $N_0(t,z) = N_1(t,z) = z^p$, $p \in (0,1)$, due to an important application in the p-adic string theory, equation (1) was studied in works [9]-[12].

In the case $N_0(t,z) \equiv G(z)$, $N_1(t,z) \equiv G_1(z)$, $\forall t \in \mathbb{R}^+$, where $G, G_1 \in C[0,\eta]$, $G(z) \geq z$, $G_1(z) \geq 0$, $z \in [0,\eta]$, $G, G_1 \uparrow$ on $[0,\eta]$ and $G(\eta) = G_1(\eta) = \eta$ for some $\eta > 0$, equation (1) was studied in work [13] and the existence of a positive and bounded solution tending to η at infinity was proven.

In the case $N_0(t,z) \equiv z - \omega(z)$, $N_1(t,z) \equiv 0$, and $K_0(-x) = K_0(x)$, $x > 0$, $\int\limits_{-\infty}^{+\infty} |x|^j K_0(x)dx < +\infty$, $j = 1, 2$, where $0 \leqslant \omega \downarrow$ w.r.t. z on $[A, +\infty)$, $A > 0$, $\omega \in C[A, +\infty) \cap L_1(0, +\infty)$, in work [14], the existence of a one-parametric family of positive solutions with the asymptotic behavior $O(x)$ as $x \to +\infty$ was proven. Later, in works [15, 16], this result was generalized first for the case $\nu(K_0) \leqslant 0$, $N_0(t,z) \equiv \mu(t)(z - \overset{\circ}{\omega}(t,z))$, $N_1(t,z) \equiv z$, where $0 < \mu(t) \leqslant 1$, $t \in \mathbb{R}^+$, $1 - \mu \in L_1(\mathbb{R}^+)$, $\overset{\circ}{\omega}(t,z) \geq 0$, $\overset{\circ}{\omega}(t,z) \leqslant \omega(z)$, $(t,z) \in \mathbb{R}^+ \times [A, +\infty)$, $\overset{\circ}{\omega} \downarrow$ w.r.t. z on $[A, +\infty)$, and after that, in [17, 18], for the cases $N_0(t,z) \equiv \mu(t)(G(z) - \overset{\circ}{\omega}(t,z))$, $N_1(t,z) \equiv G_1(z)$.

Recently, in [19], problem (2)-(3) was studied in the case $H(t,z) = G(z)$, $H_1 \equiv 0$. In [19], a nonnegative and monotonically growing nonzero solution in the Sobolev space $W^1_\infty(\mathbb{R}^+)$ was constructed.

In the present work we construct nonzero and nonnegative solutions to equations (1) and (2) for completely different conditions for N_0, N_1, H, and H_1. We note also that for various values of $\nu(K_0)$, a solution to equation (1) is constructed in the spaces $L_1(\mathbb{R}^+) \cap L^0_\infty(\mathbb{R}^+)$ and $L^0_\infty(\mathbb{R}^+) \equiv \{\varphi(x) : \varphi \in L_\infty(\mathbb{R}^+), \lim\limits_{x\to\infty} \varphi(x) = 0\}$, and under conditions (6)-(9), a solution to problem (2)-(3) is constructed in the Sobolev space $W^1_1(\mathbb{R}^+)$.

2. Solvability of equation (1) in case of negativity of first moment for kernel K_0

Suppose for the functions $N_0(t,z)$ and $N_1(t,z)$ there exist numbers $\eta>0$ and $\eta_0\in(0,\eta)$ such that

1) $N_0(t,z), N_1(t,z)\uparrow$ w.r.t. z on $[\Phi_{\eta_0}(t),\eta]$, for each fixed $t\in\mathbb{R}^+$, where

$$\Phi_{\eta_0}(t)\equiv\eta_0\int\limits_t^\infty K_1(\tau)d\tau,\quad t\in\mathbb{R}^+. \tag{10}$$

2) N_0 and N_1 satisfy Caratheodory condition on the set $\mathbb{R}^+\times[0,\eta]$ w.r.t. z. In what follows, we write briefly this condition as

$$N_0,N_1\in Carat_z(\mathbb{R}^+\times[0,\eta]), \tag{11}$$

$$3)\quad N_0(t,0)\equiv 0,\quad N_1(t,0)\equiv 0,\quad t\in\mathbb{R}^+ \tag{12}$$

$$4)\quad 0\leqslant N_0(t,z)\leqslant z,\quad (t,z)\in\mathbb{R}^+\times[\Phi_{\eta_0}(t),\eta] \tag{13}$$

$$5)\quad N_1(t,\Phi_{\eta_0}(t))\geq\eta_0,\quad N_1(t,\eta)\leqslant\eta. \tag{14}$$

The following theorem holds true.

Theorem 1. *Suppose kernels K_0 and K_1 satisfy conditions (4)-(5) and $\nu(K_0)\equiv\int\limits_{-\infty}^{+\infty}\tau K_0(\tau)d\tau<0$, $\int\limits_{-\infty}^{+\infty}|\tau|^jK_0(\tau)d\tau<+\infty$, $j=1,2$. Then equation (1) has a positive solution in the space $L_1(\mathbb{R}^+)\cap L_\infty^0(\mathbb{R}^+)$.*

Proof. We first consider the Wiener-Hopf integral equation,

$$S(x)=\int\limits_0^\infty K_0(x-t)S(t)dt,\quad x>0, \tag{15}$$

for a real measurable function $S(x)$, with a kernel K_0 obeying the assumptions of the theorem.

As it is known (see [20]), equation (15) has a positive bounded solution with the following properties,

$$S(x)\geq\eta(1-\gamma_+),\quad S(x)\uparrow\text{ w.r.t. }\ x\ \text{ on }\ \mathbb{R}^+ \tag{16}$$

$$\lim_{x\to\infty}S(x)=\eta, \tag{17}$$

$$\gamma_+\equiv\int\limits_0^\infty v_+(x)dx\in(0,1). \tag{18}$$

Here the functions $v_\pm(x)\geq 0$, $v_\pm(x)\in L_1(\mathbb{R}^+)$ are determined by Engibaryan's nonlinear factorization equations,

$$v_\pm(x)=K_0(\pm x)+\int\limits_0^\infty v_\mp(t)v_\pm(x+t)dt,\quad x>0, \tag{19}$$

and

$$\gamma_-\equiv\int\limits_0^\infty v_-(x)dx=1,\quad\gamma_+\in(0,1). \tag{20}$$

In the recent work of one of the authors [21], as an auxiliary statement, the following additional properties of the function $S(x)$

$$\eta-S(x)\in L_1(\mathbb{R}^+)\cap L_\infty^0(\mathbb{R}^+), \tag{21}$$

$$\eta - S(x) \geq \eta \int\limits_x^\infty K_0(\tau)d\tau, \quad x \in \mathbb{R}^+ \tag{22}$$

were proven. In what follows, we shall make use of inclusion (21) and inequality (22). We introduce the following successive approximations,

$$f_0(x) = \eta - S(x), \tag{23}$$

$$\begin{gathered} f_{n+1} = \int\limits_0^\infty K_0(x-t)N_0(t, f_n(t))dt + \int\limits_0^\infty K_1(x+t)N_1(t, f_n(t))dt, \\ n = 0, 1, 2, \ldots, \quad x \in \mathbb{R}^+. \end{gathered} \tag{24}$$

By induction on n, let us prove the following properties of the sequence $\{f_n(x)\}_{n=0}^\infty$,

$$a) \quad f_n(x) \downarrow \text{w.r.t.} \quad n, \qquad b) \quad f_n(x) \geq \Phi_{\eta_0}(x), \quad n = 0, 1, 2, \ldots. \tag{25}$$

We note that it follows immediately from (22) and $\eta_0 \in (0, \eta)$ that

$$\eta \geq f_0(x) \geq \eta \int\limits_x^\infty K_0(\tau)d\tau \geq \eta_0 \int\limits_x^\infty K_1(\tau)d\tau = \Phi_{\eta_0}(x). \tag{26}$$

By (26) and the properties of the functions N_0 and N_1, in (24) we get

$$f_1(x) = \int\limits_0^\infty K_0(x-t)N_0(t, \eta - S(t))dt + \int\limits_0^\infty K_1(x+t)N_1(t, \eta - S(t))dt \leqslant$$

$$\leqslant \int\limits_0^\infty K_0(x-t)(\eta - S(t))dt + \int\limits_0^\infty K_1(x+t)N_1(t, \eta)dt \leqslant$$

$$\leqslant \eta \int\limits_{-\infty}^x K_0(\tau)d\tau - \int\limits_0^\infty K_0(x-t)S(t)dt + \eta \int\limits_x^\infty K_1(\tau)d\tau \leqslant \eta - S(x) = f_0(x),$$

$$f_1(x) \geq \int\limits_0^\infty K_1(x+t)N_1(t, f_0(t))dt \geq \int\limits_0^\infty K_1(x+t)N_1(t, \Phi_{\eta_0}(t))dt \geq$$

$$\geq \eta_0 \int\limits_x^\infty K_1(\tau)d\tau = \Phi_{\eta_0}(x).$$

Suppose now that $\Phi_{\eta_0}(x) \leqslant f_n(x) \leqslant f_{n-1}(x)$ for some $n \in \mathbb{N}$, $x \in \mathbb{R}^+$. Then by (24), (14), and the monotonicity of N_0 and N_1 we have

$$f_{n+1}(x) \leqslant \int\limits_0^\infty K_0(x-t)N_0(t, f_{n-1}(t))dt + \int\limits_0^\infty K_1(x+t)N_1(t, f_{n-1}(t))dt = f_n(x),$$

$$f_{n+1}(x) \geq \int\limits_0^\infty K_1(x+t)N_1(t, \Phi_{\eta_0}(t))dt \geq \Phi_{\eta_0}(x).$$

Therefore, the sequence of the functions $\{f_n(x)\}_{n=0}^\infty$ has a pointwise limit as $n \to \infty$, $\lim\limits_{n\to\infty} f_n(x) = f(x)$.

By condition (11) and the Lebesgue's dominated convergence theorem (see [22]) it follows that $f(x)$ satisfies equation (1). Moreover, properties (25) imply the following inequalities for the limiting function $f(x)$,

$$\Phi_{\eta_0}(x) \leqslant f(x) \leqslant \eta - S(x). \tag{27}$$

Since $\eta - S(x) \in L_1(\mathbb{R}^+) \cap L^0_\infty(\mathbb{R}^+)$, by (27) we obtain that $f(x) > 0$, $f \in L_1(\mathbb{R}^+) \cap L^0_\infty(\mathbb{R}^+)$. The proof is complete. □

3. Solvability of equation (1) for even kernel K_0

We proceed to solving equation (1) under other assumptions for the functions N_0 and N_1 in the case

$$K_0(-x) = K_0(x), \quad x \in \mathbb{R}^+. \tag{28}$$

The following theorem holds true.

Theorem 2. *Given a measurable function $Q : \mathbb{R} \to \mathbb{R}$, let ζ and η be the lowest positive roots of the equations $Q(x) = 2x$ and $Q(x) = x$, respectively, and $2\zeta < \eta$, $Q \in C[0, \eta]$, $Q(x) \uparrow$ w.r.t. x on $[0, \eta]$. Suppose that*
a) $0 \leqslant N_0(t, z) \leqslant \eta - Q(\eta - z)$ as $(t, z) \in \mathbb{R}^+ \times [0, \eta]$,
b) $N_0, N_1 \in Carat_z(\mathbb{R}^+ \times [0, \eta])$,
c) $N_0, N_1 \uparrow$ w.r.t. z on the segment $[0, \eta]$ for each fixed $t \in \mathbb{R}^+$,
d) there exists $\eta_0 \in (0, \eta)$ such that

$$N_1(t, \Phi_{\eta_0}(t)) \geq \eta_0, \quad N_1(t, \eta) \geq \eta.$$

Then under conditions (4), (5), (28), equation (1) has a positive solution in the space $L^0_\infty(\mathbb{R}^+)$.

Proof. We consider first the following auxiliary nonlinear Hammerstein type integral equation

$$\psi(x) = \int_0^\infty K_0(x - t) Q(\psi(t)) dt, \quad x \in \mathbb{R}^+ \tag{29}$$

w.r.t. the function $\psi(x)$. We define the iterations,

$$\psi_{n+1}(x) = \int_0^\infty K_0(x - t) Q(\psi_n(t)) dt, \quad \psi_0(x) \equiv \eta, \quad n = 0, 1, 2, \ldots \tag{30}$$

Due to the properties of the functions Q and K_0, by the induction on n, one can easily make sure that

$$\psi_n(x) \quad \downarrow \text{w.r.t.} \quad n, \quad \psi_n(x) \geq \zeta, \quad n = 0, 1, 2, \ldots, \quad x \in \mathbb{R}^+.$$

Therefore, the sequence of the functions $\{\psi_n(x)\}_{n=0}^\infty$ has a pointwise limit $\lim\limits_{n\to\infty} \psi_n(x) = \psi(x)$ and by the Levi's theorem the limiting function satisfies equation (29) and the relation

$$\zeta \leqslant \psi(x) \leqslant \eta, \quad x \in \mathbb{R}^+. \tag{31}$$

By the induction it is also possible to prove that

$$\psi_n(x) \quad \uparrow \text{w.r.t.} \quad x \quad \text{on} \quad \mathbb{R}^+, \quad n = 0, 1, 2, \ldots \tag{32}$$

if one write iterations (30) as follows,

$$\psi_{n+1}(x) = \int_{-\infty}^x K_0(\tau) Q(\psi_n(x - \tau)) d\tau, \quad \psi_0(x) \equiv \eta, \quad n = 0, 1, 2, \ldots. \tag{33}$$

Hence, in view of (32), we obtain that

$$\psi(x) \quad \uparrow \text{w.r.t.} \quad x \quad \text{on} \quad \mathbb{R}^+. \tag{34}$$

Thus, due to (31) and (34) we can say that there exists

$$\lim_{x\to\infty} \psi(x) \equiv \eta^* \leqslant \eta, \quad \eta^* > 0. \tag{35}$$

Passing in (29) to the limit as $x \to \infty$, by employing the known property of the convolutions and formula (4), we get $\eta^* = Q(\eta^*)$. Since η is a first positive root of the equation $Q(x) = x$ and $0 < \eta^* \leqslant \eta$, we have $\eta^* = \eta$.

Therefore,

$$0 \leqslant \eta - \psi \in L_\infty^0(\mathbb{R}^+). \tag{36}$$

Let us prove the following auxiliary inequality,

$$\eta - \psi(x) \geq \eta \int_x^\infty K_0(\tau)d\tau, \quad x \in \mathbb{R}^+. \tag{37}$$

By (29), (4), and the properties of the function Q we have

$$\eta - \psi(x) = \eta - \int_0^\infty K_0(x-t)Q(\psi(t))dt = \eta \int_x^\infty K_0(\tau)d\tau + \eta \int_{-\infty}^x K_0(\tau)d\tau -$$

$$-\int_0^\infty K_0(x-t)Q(\psi(t))dt = \eta \int_x^\infty K_0(\tau)d\tau + \int_0^\infty K_0(x-t)(Q(\eta) - Q(\psi(t)))dt \geq \eta \int_x^\infty K_0(t)dt.$$

Consider the following iterations for equation (1),

$$\begin{cases} f_{n+1}(x) = \int_0^\infty K_0(x-t)N_0(t, f_n(t))dt + \int_0^\infty K_1(x+t)N_1(t, f_n(t))dt, & (38) \\ f_0(x) = \Phi_{\eta_0}(x), \quad n = 0, 1, 2, \dots \quad x \in \mathbb{R}^+. & (39) \end{cases}$$

By induction, we first prove that

$$f_n(x) \quad \uparrow \text{w.r.t.} \quad n. \tag{40}$$

Since

$$0 \leqslant f_0(x) \leqslant \eta \int_x^\infty K_1(z)dz \leqslant \eta \int_x^\infty K_0(z)dz,$$

then

$$f_1(x) \geq \int_0^\infty K_1(x+t)N_1(t, f_0(t))dt \geq \Phi_{\eta_0}(x) \equiv f_0(x),$$

$$f_1(x) \leqslant \int_0^\infty K_0(x-t)N_0(t,\eta)dt + \int_0^\infty K_1(x+t)N_1(t,\eta)dt \leqslant \eta \int_{-\infty}^x K_0(\tau)d\tau +$$

$$+\eta \int_x^\infty K_1(\tau)d\tau \leqslant \eta.$$

Assuming $\eta \geq f_n(x) \geq f_{n-1}(x)$ for some $n \in \mathbb{N}$, by (38), conditions c) and d) we get

$$f_{n+1}(x) \geq \int_0^\infty K_0(x-t)N_0(t, f_{n-1}(t))dt + \int_0^\infty K_1(x+t)N_1(t, f_{n-1}(t))dt = f_n(x)$$

and

$$f_{n+1}(x) \leqslant \int_0^\infty K_0(x-t)N_0(t,\eta)dt + \int_0^\infty K_1(x+t)N_1(t,\eta)dt \leqslant \eta.$$

Let us make sure that the inequality

$$f_n(x) \leqslant \eta - \psi(x), \quad n = 0, 1, 2, \dots, \quad x \in \mathbb{R}^+ \tag{41}$$

holds true. Indeed, as $n = 0$, (37) implies immediately (41). Let $f_n(x) \leqslant \eta - \psi(x)$ for some $n \in \mathbb{N}$. Then by (38) and conditions a) and d) we get

$$f_{n+1}(x) \leqslant \int_0^\infty K_0(x-t)N_0(t, \eta - \psi(t))dt + \int_0^\infty K_1(x+t)N_1(t, \eta - \psi(t))dt \leqslant$$

$$\leqslant \int_0^\infty K_0(x-t)(\eta - Q(\psi(t)))dt + \int_0^\infty K_1(x+t)N_1(t, \eta)dt \leqslant$$

$$\leqslant \eta \int_{-\infty}^x K_0(\tau)d\tau - \psi(x) + \eta \int_x^\infty K_1(\tau)d\tau \leqslant \eta - \psi(x).$$

Therefore, (40) and (41) yield the pointwise convergence of the sequence $\{f_n(x)\}_{n=0}^\infty$: $\lim\limits_{n\to\infty} f_n(x) = f(x)$ and

$$0 \leqslant \Phi_{\eta_0}(x) \leqslant f(x) \leqslant \eta - \psi(x) \in L_\infty^0(\mathbb{R}^+), \quad x > 0. \tag{42}$$

By Levi's theorem, $f(x)$ solves equation (1). It follows from (42) that $f \in L_\infty^0(\mathbb{R}^+)$. The proof is complete. □

Remark 1. *The results of Theorem 2 remain true if we replace condition* (28) *by a weaker one,* $\int\limits_{-\infty}^0 K_0(\tau)d\tau \geq \frac{1}{2}$.

4. Examples of functions N_0, N_1, and Q

In what follows we give several examples of functions N_0, N_1, and Q subject to the assumptions of the proven theorems.

Examples for Theorem 1.

I) $N_0(t, z) \equiv h(t, z)\widetilde{N}(z)$, where the function h is continuous w.r.t. all its arguments on the set $\mathbb{R}^+ \times [0, \eta]$, $0 \leqslant h(t, z) \leqslant 1$, $(t, z) \in \mathbb{R}^+ \times [0, \eta]$, $h \uparrow$ in z on $[0, \eta]$, $\widetilde{N} \in C[0, \eta]$, $\widetilde{N} \uparrow$ in z on $[0, \eta]$, $0 \leqslant \widetilde{N}(z) \leqslant z$, $z \in [0, \eta]$. As the functions h and $\widetilde{N}$, we can take the following examples,

- $h(t, z) = ze^{-z} \cdot sin^2 t, \quad \widetilde{N}(z) = z^p, \quad p > 1, \quad \eta = 1.$
- $h(t, z) = \eta e^{\frac{z}{\eta} - 1}, \quad \widetilde{N}(z) = sin z.$

II)

$$N_1(t, z) = \frac{\alpha z}{z + (\frac{\alpha}{\eta_0} - 1)\Phi_{\eta_0}(t)}, \quad \eta > \alpha > \eta_0 > 0, \tag{43a}$$

$$N_1(t, z) = \frac{\alpha z}{z + (\frac{\alpha}{\eta_0} - 1)\Phi_{\eta_0}(t)} + \frac{1}{2\eta^{p-1}} z^p, \quad p > 1, \quad \eta \geq 2\alpha, \quad \alpha > \eta_0. \tag{43b}$$

Examples for Theorem 2.

III) $Q(z) = \frac{z^\alpha}{\eta^{\alpha-1}}, \quad \alpha \in (0, 1),$

IV) $Q(z) = \eta e^{\frac{z}{\eta} - 1}$

V) $Q(z) = \sqrt{ze^{z-1}}, \quad \eta = 1$

VI) $N_0(t,z)=\frac{(\eta-Q(\eta-z))^\beta}{\eta^{\beta-1}}, \quad \beta\geq 1$

VII) $N_0(t,z)=sin(\eta-Q(\eta-z))$
As $N_1(t,z)$, in Theorem 2 we can consider examples $(43a)$ and $(43b)$.

5. On solvability of problem (2)-(3) in Sobolev space $W_1^1(\mathbb{R}^+)$

The following theorem holds true.

Theorem 3. *Suppose the function $H(t,z)$ in equation (2) satisfies all the assumptions for the function $N_0(t,z)$ in Theorem 1, and $H_1(t,z)$ is a real function defined on the set $\mathbb{R}^+\times\mathbb{R}$ and there exist positive numbers $\eta>0$, $\eta_0\in(0,\eta)$, $\xi\in(0,\frac{1}{\lambda})$, $\theta\in(0,1)$ such that*

$$i_1)\quad H_1(t,\xi\rho^\sigma_{\eta_0}(t))\geq\eta_0,\quad H_1(t,\eta)\leqslant\eta, \tag{44}$$

where

$$\rho^\sigma_{\eta_0}(t)=\eta_0\int\limits_{t+\sigma}^{\infty}T_1(z)dz,\quad \sigma=\frac{1}{\lambda\theta}ln\frac{1}{1-\lambda\xi} \tag{45}$$

$$i_2)\quad H_1(t,0)\equiv 0,\quad H_1\in Carat_z(\mathbb{R}^+\times[0,\eta]). \tag{46}$$

$i_3)$ $H_1(t,z)\uparrow$ w.r.t. z on $[0,\eta]$ for each fixed $t\in\mathbb{R}^+$.

Then under conditions (6)-(9), problem (2)-(3) has a nonnegative nontrivial solution in the Sobolev space $W_1^1(\mathbb{R}^+)$.

Proof. We introduce the function

$$K_0(x)=\int\limits_0^\infty e^{-\lambda z}T(x-z)dz,\quad x\in\mathbb{R}. \tag{47}$$

By the Fubini theorem, the function $K_0(x)$ possesses the following "splendid" properties,

$$K_0(x)\geq 0,\quad \int\limits_{-\infty}^{+\infty}K_0(x)dx=1,\quad K_0\in L_1(\mathbb{R})\cap L_\infty(\mathbb{R}), \tag{48}$$

$$\nu(K_0)<0,\quad \int\limits_{-\infty}^{+\infty}\tau^2K_0(\tau)d\tau<+\infty. \tag{49}$$

Let us prove the following inequality

$$\int\limits_x^\infty K_0(t)dt\geq\frac{1}{\lambda}\int\limits_x^\infty T(t)dt,\quad x\in\mathbb{R}^+. \tag{50}$$

We have

$$\int\limits_x^\infty K_0(t)dt=\int\limits_x^\infty\int\limits_0^\infty e^{-\lambda z}T(t-z)dzdt=\int\limits_0^\infty e^{-\lambda z}\int\limits_x^\infty T(t-z)dtdz=$$

$$=\int\limits_0^\infty e^{-\lambda z}\int\limits_{x-z}^\infty T(y)dydz\geq\frac{1}{\lambda}\int\limits_x^\infty T(t)dt.$$

Consider the homogeneous Wiener-Hopf equation

$$S(x)=\int\limits_0^\infty K_0(x-t)S(t)dt,\quad x\in\mathbb{R}^+, \tag{51}$$

with a kernel of the form (47). As it was noted, (48), (49) imply the existence of a positive solution with properties (16), (17), (21), (22).

Denote

$$F(x)=\frac{d\varphi}{dx}+\lambda\varphi(x). \tag{52}$$

Then equation (2) with initial condition (3) casts into the form

$$\begin{aligned}F(x)=&\int\limits_0^\infty T(x-t)H\left(t,\int\limits_0^t e^{-\lambda(t-\tau)}F(\tau)d\tau\right)dt+\\ +&\int\limits_0^\infty T_1(x+t)H_1\left(t,\int\limits_0^t e^{-\lambda(t-\tau)}F(\tau)d\tau\right)dt,\quad x\in\mathbb{R}^+.\end{aligned} \tag{53}$$

Consider the iterations

$$\begin{aligned}F_{n+1}(x)=&\int\limits_0^\infty T(x-t)H\left(t,\int\limits_0^t e^{-\lambda(t-\tau)}F_n(\tau)d\tau\right)dt+\\ +&\int\limits_0^\infty T_1(x+t)H_1\left(t,\int\limits_0^t e^{-\lambda(t-\tau)}F_n(\tau)d\tau\right)dt\\ F_0(x)=&\lambda(\eta-S(x)),\quad n=0,1,2,\dots,\quad x\in\mathbb{R}^+.\end{aligned} \tag{54}$$

In what follows we shall show that

$$j_1)\quad F_n(x)\downarrow \text{w.r.t.}\quad n, \tag{55}$$

$$j_2)\quad F_n(x)\geq\rho^\sigma_{\eta_0}(x),\quad n=0,1,2,\dots,\quad x\in\mathbb{R}^+. \tag{56}$$

By (22) and (50) we have

$$F_0(x)=\lambda(\eta-S(x))\geq\lambda\eta\int\limits_x^\infty K_0(t)dt\geq\eta\int\limits_x^\infty T(t)dt\geq\eta\int\limits_x^\infty T_1(t)dt\geq$$

$$\geq\eta_0\int\limits_{x+\sigma}^\infty T_1(t)dt=\rho^\sigma_{\eta_0}(x).$$

In particular, it implies that

$$\rho^\sigma_{\eta_0}(x)\leqslant\lambda\eta,\quad x\in\mathbb{R}^+. \tag{57}$$

Employing the properties of the functions H, H_1, T, and T_1, we obtain

$$F_1(x)\leqslant\int\limits_0^\infty T(x-t)H\left(t,\eta-\lambda\int\limits_0^t e^{-\lambda(t-\tau)}S(\tau)d\tau\right)dt+\int\limits_0^\infty T_1(x+t)H_1(t,\eta)dt\leqslant$$

$$\leqslant\eta\int\limits_0^\infty T(x-t)dt-\lambda\int\limits_0^\infty T(x-t)\int\limits_0^t e^{-\lambda(t-\tau)}S(\tau)d\tau dt+\eta\int\limits_x^\infty T_1(z)dz\leqslant$$

$$\leqslant\lambda\eta-\lambda\int\limits_0^\infty K_0(x-\tau)S(\tau)d\tau=\lambda(\eta-S(x))=F_0(x).$$

Let $F_n(x) \geq \rho^\sigma_{\eta_0}(x)$ for some $n \in \mathbb{N}$. Then by (44), (45), (54), i_3), monotonicity of $H(t,z)$ we obtain

$$F_{n+1}(x) \geq \int\limits_0^\infty T(x-t)H\left(t, \int\limits_0^t e^{-\lambda(t-\tau)}\rho^\sigma_{\eta_0}(\tau)d\tau\right)dt+$$

$$+\int\limits_0^\infty T_1(x+t)H_1\left(t, \int\limits_0^t e^{-\lambda(t-\tau)}\rho^\sigma_{\eta_0}(\tau)d\tau\right)dt \geq$$

$$\geq \int\limits_0^\infty T_1(x+t)H_1\left(t, \int\limits_0^t e^{-\lambda(t-\tau)}\rho^\sigma_{\eta_0}(\tau)d\tau\right)dt \geq$$

$$\geq \int\limits_\sigma^\infty T_1(x+t)H_1\left(t, \int\limits_{(1-\theta)\sigma}^t e^{-\lambda(t-\tau)}\rho^\sigma_{\eta_0}(\tau)d\tau\right)dt \geq$$

$$\geq \int\limits_\sigma^\infty T_1(x+t)H_1\left(t, \rho^\sigma_{\eta_0}(t)\int\limits_{(1-\theta)\sigma}^\sigma e^{-\lambda(\sigma-\tau)}d\tau\right)dt \geq$$

$$\geq \int\limits_\sigma^\infty T_1(x+t)H_1\left(t, \rho^\sigma_{\eta_0}(t)\frac{(1-e^{-\lambda\theta\sigma})}{\lambda}\right)dt =$$

$$= \int\limits_\sigma^\infty T_1(x+t)H_1\left(t, \xi\rho^\sigma_{\eta_0}(t)\right)dt \geq \eta_0 \int\limits_{x+\sigma}^\infty T_1(y)dy = \rho^\sigma_{\eta_0}(x).$$

Suppose $F_n(x) \leqslant F_{n-1}(x)$ for some $n \in \mathbb{N}$. Then the monotonicity of H and H_1 immediately yields that $F_{n+1} \leqslant F_n$. Therefore, there exists

$$\lim_{n\to\infty} F_n(x) = F(x) \tag{58}$$

and $F(x)$ satisfies equation (53) and the estimates

$$\rho^\sigma_{\eta_0}(x) \leqslant F(x) \leqslant \lambda(\eta - S(x)) \in L_1(\mathbb{R}^+) \cap L^0_\infty(\mathbb{R}^+). \tag{59}$$

It follows from (59) that $F \in L_1(\mathbb{R}^+) \cap L^0_\infty(\mathbb{R}^+)$.

Solving the simplest Cauchy problem

$$\begin{cases} \dfrac{d\varphi}{dx} + \lambda\varphi(x) = F(x), \quad x \in \mathbb{R}^+, \\ \varphi(0) = 0, \end{cases} \tag{60}$$

we complete the proof. □

Remark 2. *Since a solution to problem* (60) *reads as*

$$\varphi(x) = \int\limits_0^x e^{-\lambda(x-t)}F(t)dt,$$

by (59) *we get the following two-sided estimate*

$$\int\limits_0^x e^{-\lambda(x-t)}\rho^\sigma_{\eta_0}(t)dt \leqslant \varphi(x) \leqslant \lambda\int\limits_0^x e^{-\lambda(x-t)}(\eta - S(t))dt$$

for $\varphi(x)$.

In the end of the work, we give two examples of $H_1(t,z)$,

$$1)\quad H_1(t,z)=\frac{\alpha z}{z+(\frac{\alpha}{\eta_0}-1)\rho^{\sigma}_{\eta_0}(t)},\quad \eta>\alpha>\eta_0>0,$$

$$2)\quad H_1(t,z)=\frac{\alpha z}{z+(\frac{\alpha}{\eta_0}-1)\rho^{\sigma}_{\eta_0}(t)}+\frac{1}{2\eta^{p-1}}z^p,\quad p>1,\quad \eta\geq 2\alpha,\quad \alpha>\eta_0.$$

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