**ON GROWTH CHARACTERISTICS OF OPERATOR-VALUED FUNCTIONS**

S.N. MISHIN

**Abstract.** In the work we generalize Liouville theorem and the concept of order and type of entire function to the case of an operator-valued function with values in the space \( \text{Lec}(H_1, H) \) of all linear continuous operators acting from a locally convex space \( H_1 \) to a locally convex space \( H \) with an equicontinuous bornology. We find the formulae expressing the order and type of an operator-valued function in terms of the characteristics for the sequence of the coefficients. Some properties of the order and type of an operator-valued function are established.

**Keywords:** locally convex space, order and type of sequence of operators, order and type of entire function, equicontinuous bornology, convergence by bornology, operator-valued function.

**INTRODUCTION**

It is known [3, 4] that if an entire scalar function \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) is not a polynomial, the maximum of its modulus \( M_f(r) = \max_{|z| \leq r} |f(z)| \) grows faster than any positive power of \( r \) as \( r \to \infty \) (Liouville theorem). To estimate the growth of such functions, one usually uses the characteristics (order and type),

\[
\rho = \lim_{r \to \infty} \frac{\ln \ln M_f(r)}{\ln r}, \quad \sigma = \lim_{r \to \infty} \frac{\ln M_f(r)}{r^\rho}.
\]

At that, the formulae expressing these characteristics in terms of the coefficients

\[
\rho = \lim_{n \to \infty} \frac{n \ln n}{-\ln |a_n|}, \quad (\rho e\sigma)^{\frac{1}{\rho}} = \lim_{n \to \infty} \frac{1}{n^{\frac{1}{\rho}}} \sqrt[n]{|a_n|}
\]

are known. This work is devoted to the generalization of these formulae and the Liouville theorem for the case of an entire operator-valued function \( F(t) = \sum_{n=0}^{\infty} A_n t^n \) with the values in the space \( \text{Lec}(H_1, H) \) of all linear continuous operators acting from a locally convex space \( H_1 \) into a locally convex space \( H \). The spaces \( H_1 \) and \( H \) are in general not normable.

1. **ENTIRE OPERATOR-VALUED FUNCTIONS AND ANALOGUE TO LIOUVILLE THEOREM**

\( H_1 \) and \( H \) are separable locally convex spaces over the field of complex numbers with the topologies defined respectively by the multinorms \( \{\|\cdot\|_q\} \), \( q \in Q \) and \( \{\|\cdot\|_p\} \), \( p \in P \). Without loss of generality one can regard the multinorms in \( H_1 \) and \( H \) as majorant [2]. By \( A = \{A_n\}_{n=0}^{\infty} \) we denote a sequence of linear continuous operators acting from the locally convex space \( H_1 \).
into the locally convex space $\mathbf{H}$. The sequence $\mathcal{A}$ is called as having an order $[1, 5]$, if there exists a sequence of positive numbers $\{c_n\}_{n=0}^{\infty}$ such that
\[ \forall p \in \mathcal{P} \exists C_p > 0 \exists q(p) \in \mathbb{Q} \forall x \in \mathbf{H}_1 \forall n \in \mathbb{N} : \|c_n A_n(x)\|_p \leq C_p \|x\|_q, \quad (3) \]
eq \infty is not excluded). We denote
\[
\beta_{p,q}(\mathcal{A}) = \lim_{n \to \infty} \frac{\ln \theta_{\mathcal{A}}(p, q, n)}{n \ln n}.
\]

**Definition 1.** The number $\beta_p(\mathcal{A}) = \inf_{q \in \mathbb{Q}} \beta_{p,q}(\mathcal{A})$, $(p \in \mathcal{P})$ is called a $p$-order of the sequence of the operators $\mathcal{A}$, and the number $\beta(\mathcal{A}) = \sup_{p \in \mathcal{P}} \{\beta_p(\mathcal{A})\}$ is called its order.

If $\beta(\mathcal{A}) = \pm \infty$ and at that the sequence $\mathcal{A}$ has an order, then it is called a sequence of an infinite order.

**Remark.** Let us note that there is an essential difference between the sequences having an order $\beta(\mathcal{A}) = +\infty$, and that having no order (despite formally $\beta(\mathcal{A}) = +\infty$). If $\beta(\mathcal{A}) = +\infty$, but the sequence $\mathcal{A} = \{A_n\}$ has an order, it is possible to select a sequence of positive numbers $\{c_n\}$ such that condition $[3]$ holds. And one can not select such a sequence for the sequences having no order.

If a sequence of operators $\mathcal{A}$ has a $p$-order $\beta_p(\mathcal{A}) \neq \pm \infty$, one introduces for it a finer characteristics. Denote
\[
\alpha_{p,q}(\mathcal{A}) = \lim_{n \to \infty} n^{-\beta_p(\mathcal{A})} \sqrt[n]{\theta_{\mathcal{A}}(p, q, n)}. \]

**Definition 2.** The number $\alpha_p(\mathcal{A}) = \inf_{q \in \mathbb{Q}} \alpha_{p,q}(\mathcal{A})$, $(p \in \mathcal{P})$ is called a $p$-type of a sequence of operators $\mathcal{A}$ at the $p$-order $\beta_p(\mathcal{A})$.

It is obvious that $\beta_p(\mathcal{A}) \leq \beta(\mathcal{A})$, $\forall p$. It is possible to show $[7]$ that the case when the identity $\beta_p(\mathcal{A}) = \beta(\mathcal{A})$ is valid not for all $p$, but just for some $p$, is reduced to the case $\beta_p(\mathcal{A}) = \beta(\mathcal{A})$, $\forall p$ by replacing the multinorm to an equivalent one. This replacement changes neither the order nor the type of a sequence of operators. This is why (without loss of generality) we shall consider two cases, either $\beta_p(\mathcal{A}) = \beta(\mathcal{A})$, $\forall p$, or $\beta_p(\mathcal{A}) < \beta(\mathcal{A})$, $\forall p$.

**Definition 3.** Let a sequence of operators $\mathcal{A}$ has the $p$-orders $\beta_p(\mathcal{A})$ and the order $\beta(\mathcal{A}) \neq \pm \infty$. The number
\[
\alpha(\mathcal{A}) = \left\{ \begin{array}{ll}
\sup_{p \in \mathcal{P}} \{\alpha_p(\mathcal{A})\}, & \beta_p(\mathcal{A}) = \beta(\mathcal{A}), \forall p \\
0, & \beta_p(\mathcal{A}) < \beta(\mathcal{A}), \forall p
\end{array} \right.
\]
is called a type of the sequence of operators $\mathcal{A}$ at the order $\beta(\mathcal{A})$.

A sequence of operators $\mathcal{A}$ is called belonging to the class $\mathfrak{L}_{H_1, H}[b,a]$, (cf. $[1, 5]$) if its order is less than $b$ or equal to $b$, but then the type does not exceed $a$.

Let $\mathbf{H}$ be a complete space. It is known $[8]$ that in this case the space Lec($\mathbf{H}_1, \mathbf{H}$) of linear continuous operators acting from $\mathbf{H}_1$ into $\mathbf{H}$ equipped with an equicontinuous bornology is a complete bornological vector convex space.

**Definition 4.** An operator-valued function $F : \mathbb{C} \to \text{Lec}(\mathbf{H}_1, \mathbf{H})$ is called differentiable at a point $t_0 \in \mathbb{C}$ if there exists a limit (w.r.t. the bornology of the space Lec($\mathbf{H}_1, \mathbf{H}$))
\[
\lim_{t \to t_0} \frac{F(t) - F(t_0)}{t - t_0}. \quad (4)
\]
This limit is called a derivative of the operator-valued function \( F \) at the point \( t_0 \) and is indicated by \( F'(t_0) \).

**Definition 5.** An operator-valued function \( F : \mathbb{C} \to \text{Lec}(\mathcal{H}_1, \mathcal{H}) \) is called entire if its defined and differentiable at each point \( t \in \mathbb{C} \).

An entire operator-valued function is obviously continuous everywhere (w.r.t. the bornology of the space \( \text{Lec}(\mathcal{H}_1, \mathcal{H}) \)).

Let

\[
\theta_F(p, q, t) = \sup_{\|x\|_q \neq 0} \left\{ \frac{\|F(t)(x)\|_p}{\|x\|_q^r} \right\}, \quad t \in \mathbb{C}
\]

(the case \( \theta_F(p, q, t) = +\infty \) is not excluded).

**Theorem 1.** An entire operator-valued function \( F(t) \) is bounded on each closed disk, i.e., the family of the operators \( \{F(t)\}_{|t| \leq r} \) is equicontinuous for each \( r > 0 \).

**Proof.** We fix an arbitrary \( r > 0 \). Suppose the function \( F(t) \) is entire, and the family \( \{F(t)\}_{|t| \leq r} \) is not equicontinuous, i.e., there exists \( p_0 \in \mathcal{P} \) such that for each \( C > 0 \) and for each \( q \in \mathcal{Q} \) there exists \( t_C = t_C(q) \) such that \( |t_C| \leq r \) and \( \theta_F(p_0, q, t_C) > C \). We fix an arbitrary \( q \in \mathcal{Q} \) and take \( C = n, n \in \mathbb{N} \). We obtain then a sequence of complex numbers \( t_n = t_n(q) \) lying within the disk \( |t| \leq r \). At that,

\[
\theta_F(p_0, q, t_n) > n, \quad \forall n.
\]

By the boundedness of the sequence \( \{t_n\} \) there exists a converging subsequence \( \{t_{n_k}\} \). It follows from \( (5) \) that \( \theta_F(p_0, q, t_{n_k}) > n_k, \quad \forall k \), i.e., the sequence \( \{F(t_{n_k})\} \) is not equicontinuous and thus diverges. But by the continuity of the function \( F \) it must converges. We obtain the contradiction. 

If the function \( F(t) \) is entire, then for each fixed \( x \in \mathcal{H}_1 \), \( F(t)(x) \) is an entire function with values in \( \mathcal{H} \). Such function is represented as a power series

\[
F(t)(x) = \sum_{n=0}^{\infty} x_n t^n, \quad x \in \mathcal{H}_1, \quad \{x_n\} \subset \mathcal{H}
\]

(the sequence \( \{x_n\} \) depends on \( x \)). We let

\[
M_F(p, q, r) = \sup_{|t| \leq r} \theta_F(p, q, t).
\]

We define a sequence of operators \( A_n : \mathcal{H}_1 \to \mathcal{H} \) as follows, \( A_n(x) = x_n, \quad \forall x \in \mathcal{H}_1 \). We obtain the expansion of the function \( F(t) \) as a power series

\[
F(t) = \sum_{n=0}^{\infty} A_n t^n.
\]

At that, series \( (6) \) everywhere pointwise converges to the function \( F(t) \) (for each fixed \( x \in \mathcal{H}_1 \) the series \( \sum_{n=0}^{\infty} A_n(x) t^n \) converges to the function \( F(t)(x) \) everywhere). Let us show that \( \{A_n\} \subset \text{Lec}(\mathcal{H}_1, \mathcal{H}) \) and series \( (6) \) converges everywhere to the function \( F(t) \) w.r.t. the bornology. First we prove the following theorem.

**Theorem 2 (Analogue of Cauchy inequality).** The inequality

\[
\theta_A(p, q, n) \leq \frac{M_F(p, q, r)}{r^n}, \quad \forall p \quad \forall q \quad \forall n \quad \forall r > 0
\]

holds true.
Proof. Let $p \in \mathcal{P}$, $q \in \mathcal{Q}$, $r > 0$. If $M_F(p, q, r) = \infty$, then inequality (7) holds true. Let $M_F(p, q, r) < \infty$. Since for each fixed $x$ the vector-function $F(t)(x) = \sum_{n=0}^{\infty} A_n(x)t^n$ is entire, then (see, for instance, [9])

$$A_n(x) = \frac{1}{2\pi i} \int_{|\xi| = r} \frac{F(\xi)(x)d\xi}{\xi^{n+1}}, \quad n \in \mathbb{N}.$$ 

Hence, $\forall p \in \mathcal{P}$ $\forall x \in H_1 \forall r > 0 \forall n \in \mathbb{N}$ we have

$$\|A_n(x)\|_p \leq \sup_{|\xi| \leq r} \frac{\|F(\xi)(x)\|_p}{r^n} \leq \frac{\sup_{|\xi| \leq r} \theta_F(p, q, \xi)}{r^n} \|x\|_q = \frac{M_F(p, q, r)}{r^n} \|x\|_q'$$ 

that yields inequality (7).

Since the function $F(t)$ is entire, by Theorem 1 for each $r > 0$ the family $\{F(t)\}_{|t| \leq r}$ is equicontinuous, i.e.,

$$\forall p \in \mathcal{P} \exists C_p > 0 \exists q_p \in \mathcal{Q} \forall x \in H_1 \forall t |t| \leq r \Rightarrow \|F(t)(x)\|_p \leq C_p \|x\|_q'.$$ 

For each $p$ we choose $q_0 = q_0(p)$ such that $\|x\|_{q_0} \geq \|x\|_{q_0}'$ $\forall x \in H_1$ (it is always possible since the multinorm is majorant). Then

$$\theta_F(p, q_0, t) = \sup_{\|x\|_{q_0}' \neq 0} \left\{ \frac{\|F(t)(x)\|_p}{\|x\|_{q_0}'} \right\} \leq \sup_{\|x\|_{q_0}' \neq 0} \left\{ \frac{C_p \|x\|_{q_0}'}{\|x\|_{q_0}'} \right\} = \tilde{C}_p(q_0), \ |t| \leq r.$$ 

Thus, for each $r > 0$ and each $p \in \mathcal{P}$ there exists $q_0 \in \mathcal{Q}$ such that $\theta_F(p, q_0, t)$ (as functions of $t$) are bounded in the disk $|t| \leq r$ And it means that

$$\forall r \forall p \exists q_0(p, r): \ M_F(p, q_0, r) < \infty.$$ 

Hence, by Theorem 2

$$\lim_{n \to \infty} \sqrt[n]{\theta_A(p, q_0, n)} \leq \frac{1}{r}, \ r > 0. \ (8)$$ 

It follows from (8) that either $\beta_p(A) < 0$ or $\beta_p(A) = 0$, but then by the arbitrariness of $r$

$$\alpha_p(A) = \inf \lim_{q \in \mathcal{Q} \ n \to \infty} \sqrt[n]{\theta_A(p, q, n)} = 0.$$ 

Thus, the sequence $\{A_n\}$ belongs to the class $\mathcal{L}_{H_1}$ and therefore series (6) everywhere converges to the function $F(t)$ w.r.t. bornology (see [1, 5]).

**Theorem 3** (Analogue of Liouville theorem). Suppose function (6) is entire and satisfies the condition

$$\exists k \forall p \exists K_p > 0 \exists q(p) \forall r > 0: \ M_F(p, q, r) \leq K_p r^k. \ (9)$$ 

Then $F$ is an operator-valued polynomial of degree at most $k$, i.e.,

$$F(t) = \sum_{n=0}^{[k]} A_n t^n.$$ 

Proof. By inequalities (7), (9) and the definition of the numbers $\theta_A(p, q, n)$ we have

$$\|A_n(x)\|_p \leq \theta_A(p, q, n) \|x\|_q' \leq K_p r^{k-n} \|x\|_q', \forall p \forall x \in H_1 \forall r > 0 \forall n, \ q = q(p).$$ 

By the arbitrariness of $r,$

$$\|A_n(x)\|_p = 0, \forall n > k \forall p \forall x \in H_1,$$ 

thus, $A_n = 0, \forall n > k.$

Theorem 3 shows that if $F$ is an entire transcendental function, then the quantities $M_F(p, q, r)$ grows faster than any positive power as $r \to \infty.$
2. Growth characteristics for entire function and formulae for their calculation

**Definition 6.** Let $F : \mathbb{C} \rightarrow \text{Lec}(H_1, H)$ be an entire transcendental function. The number

$$\rho_p(F) = \inf_{q \in \mathbb{Q}} \rho_{p,q}(F),$$

where

$$\rho_{p,q}(F) = \lim_{r \to \infty} \frac{\ln \ln M_F(p, q, r)}{\ln r},$$

will be called a $p$-order of the function $F$, and the number $\rho(F) = \sup_{p \in \mathbb{P}} \{\rho_p(F)\}$ will be called its order.

If $0 < \rho_p(F) < \infty$, the number $\sigma_p(F) = \inf_{q \in \mathbb{Q}} \sigma_{p,q}(F)$, where

$$\sigma_{p,q}(F) = \lim_{r \to \infty} \frac{\ln M_F(p, q, r)}{r_{\rho_p(F)}},$$

will be called a $p$-type of the function $f$ at $p$-order $\rho(F)$.

It can be shown that the case when for some $p$, $\rho_p(F) < \rho(F)$, while for other $\rho_p(F) = \rho(F)$, is reduced to the case $\rho_p(F) = \rho(F)$, $\forall p$ by the replacement of the multinorm to an equivalent one. This is why (without loss of generality) we shall consider two cases, either $\rho_p(F) < \rho(F)$, $\forall p$, or $\rho_p(F) = \rho(F)$, $\forall p$.

**Definition 7.** Suppose a function $F(t)$ has $p$-orders $\rho_p(F)$ and order $0 < \rho(F) < \infty$. The number

$$\sigma(F) = \begin{cases} 
0 & \rho_p(F) < \rho(F), \forall p \\
\sup \{\sigma_p(F)\} & \rho_p(F) = \rho(F), \forall p
\end{cases}$$

will be called a type of the function $f$ at the order $\rho(F)$.

**Lemma 1.** Suppose

$$\forall p \exists q_p \exists a_p, b_p > 0 \exists r_0(p) \forall r > r_0 : M_F(p, q_p, r) < e^{a_p r^{b_p}}. \tag{10}$$

Then

$$\forall p \exists n_0(p) \forall n > n_0 : \sqrt[n]{\theta_A(p, q_p, n)} < \left(\frac{a_p b_p e}{n}\right)^{\frac{1}{b_p}}. \tag{11}$$

**Proof.** Suppose inequality (10) holds true, then by (7) we have

$$\theta_A(p, q_p, n) < \frac{e^{a_p r^{b_p}}}{r^n} ; \forall p \forall r > r_0(p) \forall n. \tag{12}$$

We denote $\mu_p(r) = e^{a_p r^{b_p}} r^{-n}$. It is obvious that

$$\forall p : \mu_p(0) = \mu_p(+\infty) = +\infty.$$

Let us find $\min_{r > 0}\{\mu_p(r)\}$,

$$\mu_p'(r) = \mu_p(r) \ln' \mu_p(r),$$

$$\mu_p'(r) = \mu_p(r) \left(a_p r^{b_p} - n \ln r\right)^',$$

$$\mu_p'(r) = \mu_p(r) \left(a_p b_p r^{b_p-1} - \frac{n}{r}\right)$$

$\mu_p'(r) = 0$ as $r = r_1 = \left(\frac{n}{a_p b_p}\right)^{\frac{1}{b_p}}$. Substituting $r_1$ in inequality (12), we obtain (11). \qed
Lemma 2. Suppose
\[ \forall p \exists q_p \exists a_p, b_p > 0 \exists n_0(p) \forall n > n_0 : \sqrt[n]{\theta_A(p, q_p, n)} < \left(\frac{a_p b_p e}{n}\right)^{\frac{1}{p}}. \] (13)

Then
\[ \forall p \forall \epsilon > 0 \exists r_0(p, \epsilon) \forall r > r_0 : M_F(p, q_p, r) < e^{(a_p e)^{\frac{1}{p}}}. \] (14)

Proof. By condition (13) \( A \in \mathfrak{L}_{\mathbb{H}_1, \mathbb{H}}[0, 0], \) thus, \( F \) is an entire operator-valued function. Let us fix an arbitrary \( p \) (and fix by this depending on it \( q_p, a_p, b_p \)) and consider the inequality
\[ \theta_A(p, q_p, n)r^n < \left(\frac{a_p b_p e}{n}\right)^{\frac{1}{p}} r^n. \]

For sufficiently large \( n \)
\[ \left(\frac{a_p b_p e}{n}\right)^{\frac{1}{p}} r^n < \frac{1}{2}. \] (15)

By \( N_p(r) \) we denote the lowest of natural numbers \( n \) for which inequality (15) holds true. Let us find the dependence of \( N_p(r) \) on \( r \). We have
\[ 2r \left(\frac{a_p b_p e}{n}\right)^{\frac{1}{p}} < 1, \text{ as } n > (2r)^{bp}(a_p b_p e). \]

Therefore, we can let \( N_p(r) = \lfloor (2r)^{bp}(a_p b_p e) \rfloor + 1. \)

Further, for each fixed \( p \in \mathcal{P}, t \in \mathbb{C} \) and \( x \in \mathbb{H}_1 \) we have
\[ \|F(t)(x)\|_p \leq \sum_{n=0}^{\infty} \|A_n(x)\|_p |t|^n \leq \sum_{n=0}^{\infty} \theta_A(p, q_p, n)|t|^n \|x\|_{q_p}, \]
hence,
\[ \theta_F(p, q_p, t) \leq \sum_{n=0}^{\infty} \theta_A(p, q_p, n)|t|^n, \]
i.e.,
\[ \forall p \forall r > 0 : M_F(p, q_p, r) \leq \sum_{n=0}^{\infty} \theta_A(p, q_p, n)r^n = \sum_{n=0}^{N_p(r)-1} \theta_A(p, q_p, n)r^n + \sum_{n=N_p(r)}^{\infty} \theta_A(p, q_p, n)r^n. \]

For \( n \geq N_p(r) \) the inequality \( \theta_A(p, q_p, n)r^n < \left(\frac{1}{2}\right)^n \) holds true and hence
\[ \sum_{n=N_p(r)}^{\infty} \theta_A(p, q_p, n)r^n < \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n < \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = 2. \]

Since for each fixed \( p \) and \( r \)
\[ \lim_{n \to \infty} \theta_A(p, q_p, n)r^n = 0, \]
the sequence \( \{\theta_A(p, q_p, n)r^n\} \) has a maximal term. Let
\[ m_p(r) = \max_{n \geq 0} \{\theta_A(p, q_p, n)r^n\}, \]
then
\[ \sum_{n=0}^{N_p(r)-1} \theta_A(p, q_p, n)r^n \leq m_p(r)N_p(r). \]

Let us estimate \( m_p(r) \). Let \( \theta_A(p, q_p, s)r^s \) be a maximal term. Under an unbound increasing of \( r \) the index \( s \) of maximal term increases unboundedly as well, i.e., \( s \to \infty \) as \( r \to \infty \). If \( r \) is sufficiently large, then \( s > n_0 \), where \( n_0 \) is a number in (13).
This is why
\[ m_p(r) = \theta_A(p, q_p, s) r^s < \left( \frac{a_p b_p e}{s} \right) \frac{\xi_p}{s} r^s \leq \max_{\xi \geq 0} \left\{ \left( \frac{a_p b_p e}{\xi} \right) \frac{\xi}{s} r^\xi \right\}. \]

We denote
\[ \nu_p(\xi) = \left( \frac{a_p b_p e}{\xi} \right) \frac{\xi}{s} r^\xi. \]

Clearly,
\[ \forall p : \nu_p(0) = 1, \nu_p(+\infty) = 0. \]

Let us find \( \max_{\xi \geq 0} \{\nu_p(\xi)\} \). We have
\[ \nu'_p(\xi) = \nu_p(\xi) \left( \frac{\ln(a_p b_p e)}{b_p} - \ln \xi \right) \frac{1}{b_p} + \ln r. \]

\[ \nu'_p(\xi) = 0 \text{ as } \xi = \xi_1 = (a_p b_p) r^{b_p}. \]

\[ \nu_p(\xi_1) = e^{a_p r^{b_p}}. \]

Therefore (for sufficiently large \( r \)), \( m_p(r) < e^{a_p r^{b_p}} \).

Thus,
\[ M_F(p, q_p, r) \leq N_p(r) m_p(r) + 2 \leq ((2r)^{b_p}(a_p b_p e) + 1)e^{a_p r^{b_p}} + 2 < e^{(a_p + \epsilon)r^{b_p}}. \]

\[ \square \]

**Theorem 4.** The growth characteristics of function \[ (6) \] are calculated by the formulae
\[ \rho_p(F) = -\frac{1}{\beta_p(A)}, \forall p, \] (16)
\[ \sigma_p(F) = -\frac{\beta_p(A)}{e^{(\alpha(A))^{-\frac{1}{p^p(A)}}}}, \forall p, \] (17)
\[ \rho(F) = \frac{1}{\beta(A)}, \] (18)
\[ \sigma(F) = \left\{ \begin{array}{ll}
0 & , \beta_p(A) < \beta(A), \forall p \\
-\frac{\alpha(A)}{e^{(\alpha(A))^{-\frac{1}{p^p(A)}}}} & , \beta_p(A) = \beta(A), \forall p
\end{array} \right. \] (19)

**Proof.** We fix an arbitrary \( p \). Suppose the \( p \)-order of the function \( F \) equals \( \rho_p(F) \). Then
\[ \forall p, \forall \epsilon > 0 \exists q_p(\epsilon) \exists \rho_0(p, \epsilon) \forall r > \rho_0 : M_F(p, q_p, r) \leq \exp \left\{ p^{\rho_p(F) + \epsilon} \right\}. \]

By Lemma [1] \( (b_p = \rho_p(F) + \epsilon, \ a_p = 1) \)
\[ \sqrt[\rho_p(F) + \epsilon]{\theta_A(p, q_p, n) < \left( \frac{\rho_p(F) + \epsilon}{n} \right)^{\frac{1}{\rho_p(F) + \epsilon}}, \forall n > n_0. \]

By this we successively find
\[ \frac{1}{n} \ln \theta_A(p, q_p, n) < \left( \frac{1}{\rho_p(F) + \epsilon} \right) \ln \left( (\rho_p(F) + \epsilon) e \right) \left( \frac{\rho_p(F) + \epsilon}{n} \right) - \frac{\ln n}{\rho_p(F) + \epsilon} = C_p(\epsilon) - \frac{\ln n}{\rho_p(F) + \epsilon}, \]
\[ \ln \theta_A(p, q_p, n) < C_p(\epsilon)n - \frac{n \ln n}{\rho_p(F) + \epsilon}, \]
\[ \ln \frac{1}{\theta_A(p, q_p, n)} > \frac{n \ln n}{\rho_p(F) + \epsilon} - C_p(\epsilon)n = n \ln \left( \frac{1}{\rho_p(F) + \epsilon} - \frac{C_p(\epsilon)}{\ln n} \right), \forall n > n_0. \] (20)
As \( n \to \infty \), the expression in parentheses in (20) tends to \( \frac{1}{\rho_p(F) + \varepsilon} \), and for large \( n \)
\[
\ln \frac{1}{\theta_A(p, q_p, n)} > \frac{n \ln n}{\rho_p(F) + 2\varepsilon},
\]
i.e.,
\[
\rho_p(F) + 2\varepsilon > \frac{n \ln n}{-\ln \theta_A(p, q_p, n)}.
\]

By the arbitrariness of \( \varepsilon \),
\[
-\frac{1}{\beta_{p,q}(A)} = \lim_{n \to \infty} \frac{n \ln n}{-\ln \theta_A(p, q_p, n)} \leq \rho_p(F).
\]

Since \( \beta_p(A) = \inf_q \{ \beta_{p,q}(A) \} \), then
\[
-\frac{1}{\beta_p(A)} \leq -\frac{1}{\beta_{p,q}(A)} \leq \rho_p(F).
\]

Hence, \( \rho_p(F) \geq -\frac{1}{\beta_p(A)}, \forall p \).

Vice-versa, since
\[
-\frac{1}{\beta_{p,q}(A)} = \lim_{n \to \infty} \frac{n \ln n}{-\ln \theta_A(p, q, n)},
\]
then
\[
\frac{n \ln n}{-\ln \theta_A(p, q, n)} < -\frac{1}{\beta_{p,q}(A)} + \frac{\varepsilon}{2}, \forall p \ \forall \varepsilon > 0 \ \forall q \ \forall n > n_0(p, q, \varepsilon).
\]

And since \( \beta_p(A) = \inf_q \{ \beta_{p,q}(A) \} \), then
\[
\forall p \ \forall \varepsilon > 0 \ \exists q_p(\varepsilon) : -\frac{1}{\beta_{p,q}(A)} \leq -\frac{1}{\beta_p(A)} + \frac{\varepsilon}{2}.
\]

Thus,
\[
\forall p \ \forall \varepsilon > 0 \ \exists q_p(\varepsilon) \ \exists n_0(p, \varepsilon) \ \forall n > n_0 : \frac{n \ln n}{-\ln \theta_A(p, q_p, n)} < -\frac{1}{\beta_p(A)} + \varepsilon,
\]
therefore,
\[
\forall p \ \forall \varepsilon > 0 \ \exists q_p(\varepsilon) \ \exists n_0(p, \varepsilon) \ \forall n > n_0 : \sqrt{n} \theta_A(p, q_p, n) < n - \rho_p(F)^{1+\varepsilon}.
\]

By Lemma 2 \( \left( b_p = -\frac{1}{\beta_p(A)} + \varepsilon, \ a_p = \frac{1}{e^{\frac{1}{\beta_p(A)} + \varepsilon}} \right) \)
\[
\forall p \ \forall \varepsilon > 0 \ \exists q_p(\varepsilon) \ \exists n_0(p, \varepsilon) \ \forall r > r_0 : M_F(p, q_p, r) \leq \exp \left\{ (a_p + \varepsilon)r^{\rho_p(F)^{1+\varepsilon}} \right\}.
\]

It means that \( \rho_p(F) \leq -\frac{1}{\beta_p(A)}, \forall p \).

Thus, identity (16) is proven. Identity (18) follows immediately from (16).

Let us prove identity (17).

Suppose the function \( F \) has the \( p \)-order \( 0 < \rho_p(F) < \infty \) and the \( p \)-type \( \sigma_p(F) \). Then
\[
\forall p \ \forall \varepsilon > 0 \ \exists q_p(\varepsilon) \ \forall r > r_0 : M_F(p, q_p, r) < \exp \left\{ (\sigma_p(F) + \varepsilon)r^{\rho_p(F)} \right\}.
\]

By Lemma 1 \( \left( a_p = \sigma_p(F) + \varepsilon, \ b_p = \rho_p(F) \right) \) we have
\[
\sqrt{n} \theta_A(p, q_p, n) < \left( \frac{(\sigma_p(F) + \varepsilon)\rho_p(F)e}{n} \right)^{\frac{1}{\rho_p(F)}}, \forall n > n_0,
\]
\[
n^{\frac{1}{\rho_p(F)}} \sqrt{n} \theta_A(p, q_p, n) < \left( (\sigma_p(F) + \varepsilon)\rho_p(F)e \right)^{\frac{1}{\rho_p(F)}}, \forall n > n_0.
\]
By the arbitrariness of \( \varepsilon \)

\[
\alpha_{p,q}(A) = \lim_{n \to \infty} n^{-\beta_p(A)} \sqrt[n]{\theta_A(p, q, n)} = \lim_{n \to \infty} n^{\frac{1}{n \rho_p(F)}} \sqrt[n]{\theta_A(p, q, n)} \leq (\sigma_p(F)\rho_p(F)e)^{\frac{1}{\rho_p(F)}}
\]

Since \( \alpha_p(A) = \inf_q \{\alpha_{p,q}(A)\} \), then

\[
\alpha_p(A) \leq \alpha_{p,q}(A) \leq (\sigma_p(F)\rho_p(F)e)^{\frac{1}{\rho_p(F)}}, \quad \forall p.
\]

Vice-versa, since

\[
\alpha_{p,q}(A) = \lim_{n \to \infty} n^{-\beta_p(A)} \sqrt[n]{\theta_A(p, q, n)} = \lim_{n \to \infty} n^{\frac{1}{n \rho_p(F)}} \sqrt[n]{\theta_A(p, q, n)}, \quad \forall p, \forall q,
\]

then

\[
\forall \varepsilon > 0 \quad \forall p \exists q(p, \varepsilon) \exists n_0(p, \varepsilon) \forall n > n_0,
\]

\[
\sqrt[n]{\theta_A(p, q, n)} < \left( \frac{(\alpha_{p,q}(A) + \varepsilon)\rho_p(F)}{n} \right)^{\frac{1}{\rho_p(F)}} < \left( \frac{(\alpha_p(A) + 2\varepsilon)\rho_p(F)}{n} \right)^{\frac{1}{\rho_p(F)}}.
\]

By Lemma 2

\[
\left( b_p = \rho_p(F), \right. \quad \left. a_p = \frac{(\alpha_p(A) + 2\varepsilon)\rho_p(F)}{\rho_p(F)e} \right)
\]

we obtain

\[
\forall \forall \varepsilon > 0 \exists q_p(\varepsilon) \exists n_0(p, \varepsilon) \forall r > r_0 : \quad M_F(p, q_p, r) < \exp\left\{ (a_p + \varepsilon)r^{\rho_p(F)} \right\}.
\]

It implies

\[
\sigma_p(F) \leq a_p = \frac{(\alpha_p(A) + 2\varepsilon)\rho_p(F)}{\rho_p(F)e}.
\]

By the arbitrariness of \( \varepsilon \)

\[
\sigma_p(F)\rho_p(F)e \leq (\alpha_p(A))^{\rho_p(F)},
\]

therefore,

\[
\alpha_p(A) \geq (\sigma_p(F)\rho_p(F)e)^{\frac{1}{\rho_p(F)}},
\]

i.e.,

\[
\sigma_p(F) = -\frac{\beta_p(A)}{e} (\alpha_p(A))^{\frac{1}{\rho_p(F)}}, \quad \forall p.
\]

Hence, identity (17) is proven.

Let us prove identity (19).

If \( \beta_p(A) < \beta(A), \forall p \), from identity (16) it follows \( \rho_p(F) < \rho(F), \forall p \) and by the definition \( \sigma(F) = 0 \).

If \( \beta_p(A) = \beta(A), \forall p \), identity (16) yields \( \rho_p(F) = \rho(F), \forall p \) and by the definition

\[
\sigma(F) = \sup_p \{\sigma_p(F)\} = -\frac{\beta(A)}{e} \sup_p \{ (\alpha_p(A))^{\frac{1}{\rho_p(F)}} \} = -\frac{\beta(A)}{e} (\alpha(A))^{\frac{1}{\rho_p(F)}},
\]

Remark. We observe that relation (16) is true also for \( \rho_p(F) = \infty \). If we suppose \( \rho_p(F) = \infty \) and \( \beta_p(A) < 0 \), by (above proven) \( \rho_p(F) < \infty \) that is false. Similarly, identity (17) holds also for \( \sigma_p(F) = \infty \).

Remark. Formulae (16) and (17) show that \( p \)-orders and \( p \)-type of an entire operator-valued function are completely determined by the characteristics of the sequence of its coefficients.

Examples.
1. Let $H_1 = H = H(C)$ be the space of all entire functions with the topology of uniform convergence on the compacts

$$\|x(z)\|_p = \max_{|z| \leq p} |x(z)|, \ p > 0.$$ 

Let us find the characteristics of the function

$$F(t) = e^{\int \frac{dt}{dz}} = \sum_{n=0}^{\infty} \frac{t^n}{n!} \frac{d^n}{dz^n} : C \to \text{Lec}(H(C)).$$

The sequence $A = \{ \frac{d^n}{dz^n} \}$ has the following characteristics $[\Pi]$,

$$\beta_p(A) = \alpha_p(A) = 0, \ \forall p.$$ 

Therefore, $\rho_p(F) = \infty, \ \forall p.$

2. Let $H_1 = [\rho, \sigma], \ H = [\rho, \theta], \ \theta \geq \sigma$. The topologies on these spaces are determined by the multinorms

$$\|x(z)\|_\varepsilon = \sup_{p > 0} \left\{ \max_{|z| \leq p} |x(z)| e^{-(\sigma + \varepsilon)p^p} \right\}, \ \varepsilon > 0, \ x \in [\rho, \sigma].$$

$$\|y(z)\|_\varepsilon = \sup_{p > 0} \left\{ \max_{|z| \leq p} |y(z)| e^{-(\theta + \varepsilon)p^p} \right\}, \ \varepsilon > 0, \ y \in [\rho, \theta].$$

Let us find the characteristics of the function

$$F(t) = e^{\int \frac{dt}{dz}} = \sum_{n=0}^{\infty} \frac{t^n}{n!} \frac{d^n}{dz^n} : C \to \text{Lec}([\rho, \sigma], [\rho, \theta]).$$

The sequence $A = \{ \frac{d^n}{dz^n} \}$ has the following characteristics $[\Pi]$:

$$\beta_\varepsilon(A) = -\frac{1}{\rho}, \ \alpha_\varepsilon(A) = (\rho e \sigma \Omega_\varepsilon)^{\frac{1}{\varepsilon}}, \ \forall \varepsilon,$$

where

$$\Omega_\varepsilon = \left\{ \begin{array}{ll} 
1 - \left(\frac{\rho}{\theta + \varepsilon}\right)^{\frac{1}{\rho - 1}} & , \ \rho > 1 \\
1 & , \ \rho \leq 1
\end{array} \right.$$ 

Therefore, $\rho_\varepsilon(F) = \rho, \ \sigma_\varepsilon(F) = \sigma \Omega_\varepsilon, \ \forall \varepsilon.$

3. Let $H_1 = H = H(C)$ be the space of all entire functions with the topology of uniform convergence on the compacts

$$\|x(z)\|_p = \max_{|z| \leq p} |x(z)|, \ p > 0.$$ 

Let us find the characteristics of the function

$$F(t)(x) = x(z) + t \int_0^z e^{(z-\xi)t} x(\xi) d\xi =$$

$$= x(z) + t \sum_{n=0}^{\infty} \int_0^z \frac{(z-\xi)^n t^n}{n!} x(\xi) d\xi,$$

$$F(t) : C \to \text{Lec}(H(C)).$$

Here

$$A_n(x) = \int_0^z \frac{(z-\xi)^{n-1}}{(n-1)!} x(\xi) d\xi, \ n = 1, 2, \ldots, \ A_0 = E.$$
The sequence $\mathcal{A} = \{A_n\}$ has the following characteristics $\\beta$:

$$\beta_p(\mathcal{A}) = -1, \quad \alpha_p(\mathcal{A}) = p, \quad \forall p.$$ 

Therefore, $\rho_p(F) = 1, \quad \sigma_p(F) = \frac{\rho_p}{\epsilon}, \quad \forall p.$

3. Properties of Growth Characteristics for Operator-Valued Functions

Let us note certain properties of the growth characteristics for operator-valued functions implied by Theorem $\\beta$.

1. Entire function $F$ and its $k$th derivative $F^{(k)}$ has the same $p$-orders and $p$-types of growth.

The validity follows from the fact that the sequences $\{A_n\}$ and $\{(n+k)! A_{n+k}\}$ has the same characteristics for each fixed $k$.

2. If a function $F_1$ has the $p$-orders $\rho_p(F_1)$ and the $p$-types $\sigma_p(F_1)$, and a function $F_2$ has the $p$-orders $\rho_p(F_2) > \rho_p(F_1), \quad \forall p$ and the $p$-types $\sigma_p(F_2) > \sigma_p(F_1), \quad \forall p$, then the functions $F = F_1 + F_2$ has the $p$-orders $\rho_p(F) = \rho_p(F_2), \quad \forall p$ and the $p$-types $\sigma_p(F) = \sigma_p(F_2), \quad \forall p$.

The validity is implied by the fact that the characteristics of the sum of operators are equal to the characteristics of the term of the greater order.

3. If a function $F_1$ has the $p$-orders $\rho_p(F_1)$ and the $p$-types $\sigma_p(F_1)$, and a function $F_2$ has the $p$-orders $\rho_p(F_2) = \rho_p(F_1), \quad \forall p$ and the $p$-types $\sigma_p(F_2) > \sigma_p(F_1), \quad \forall p$, then the function $F = F_1 + F_2$ has the $p$-orders $\rho_p(F) = \rho_p(F_2), \quad \forall p$ and the $p$-types $\sigma_p(F) = \sigma_p(F_2), \quad \forall p$.

The validity follows from the fact that the characteristics of the sum of operators of same orders are equal to the characteristics of the term of the greater type.

4. (The case $\mathbf{H}_1 = \mathbf{H}$. ) Suppose a function $F_1$ has the order $\rho(F_1)$ and the type $\sigma(F_1)$, and a function $F_2$ has the order $\rho(F_2) > \rho(F_1)$ and the type $\sigma(F_2)$. Then the function $F = F_1 F_2$ has the order $\rho(F) \leq \rho(F_2)$ and the type $\sigma(F)$. If $\rho(F) = \rho(F_2)$, then $\sigma(F) \leq \sigma(F_2)$. A similar statement holds for the function $F = F_2 F_1$.

The proof is based on the following lemma.

Lemma 3. Suppose a sequence of operators $\mathcal{A} = \{A_n\}$ has the order $\beta(\mathcal{A})$ and the type $\alpha(\mathcal{A})$, and a sequence of the operators $\mathcal{B} = \{B_n\}$ has the order $\beta(\mathcal{B}) > \beta(\mathcal{A})$ and the type $\alpha(\mathcal{B})$. Then the sequence of the operators $\mathcal{C} = \{C_n\}$, where $C_n = \sum_{k=0}^{n} A_k B_{n-k}$ has the order $\beta(\mathcal{C}) \leq \beta(\mathcal{B})$ and the type $\alpha(\mathcal{C})$. If $\beta(\mathcal{C}) = \beta(\mathcal{B})$, then $\alpha(\mathcal{C}) \leq \alpha(\mathcal{B})$.

Proof. Denote $a = \alpha(\mathcal{A}) e^{\beta(\mathcal{A})}, \quad b = \alpha(\mathcal{B}) e^{\beta(\mathcal{B})}$.

The definition of the order and type of a sequence of operators implies $\\beta$

$$\forall \epsilon, \epsilon_1 > 0, \quad \forall p, \quad \exists M_p, \quad \exists q, \quad \forall n, \quad \forall x \in \mathbf{H} :$$

$$\|C_n(x)\|_p \leq M_p \left( (b + \epsilon)^n n! \beta(\mathcal{B}) + (a + \epsilon_1)(b + \epsilon)^{n-1} 1! \beta(\mathcal{A})(n - 1)! \beta(\mathcal{B}) + \cdots + \right.$$

$$\left. + (a + \epsilon_1)^n (b + \epsilon)(n - 1)! \beta(\mathcal{A}) 1! \beta(\mathcal{B}) + (a + \epsilon_1)^n n! \beta(\mathcal{A}) \right) \|x\|_q \leq$$

$$\leq M_p (b + \epsilon)^n n! \beta(\mathcal{B}) \left[ 1 + \left( \frac{n}{1} \right)^{-\beta(\mathcal{B})} \left( \frac{a + \epsilon_1}{b + \epsilon} \right) 1! \nu \beta(\mathcal{B}) + \left( \frac{n}{2} \right)^{-\beta(\mathcal{B})} \left( \frac{a + \epsilon_1}{b + \epsilon} \right)^2 2! \nu \beta(\mathcal{B}) + \cdots + \right. \right.$$

$$\left. + \left( \frac{n}{n} \right)^{-\beta(\mathcal{B})} \left( \frac{a + \epsilon_1}{b + \epsilon} \right)^{n} n! \nu \beta(\mathcal{B}) \right] \|x\|_q, \quad (21)$$

where $\nu = \beta(\mathcal{A}) - \beta(\mathcal{B})$. 

If $\beta(\mathcal{B}) > \beta(\mathcal{A})$ ($\nu < 0$), for large $n$ the expression in brackets in (21) does not exceed $(1 + \varepsilon_2)^n$, $\forall \varepsilon_2 > 0$ and thus $\beta(\mathcal{C}) \leq \beta(\mathcal{B})$, and if $\beta(\mathcal{C}) = \beta(\mathcal{B})$, then $\alpha(\mathcal{C}) \leq \alpha(\mathcal{B})$. \hfill \Box

5°. (The case $\mathbf{H}_1 = \mathbf{H}$.) Suppose a function $F_1$ has the order $\rho(F_1)$ and the type $\sigma(F_1)$, and a function $F_2$ has the order $\rho(F_2) = \rho(F_1)$ and the type $\sigma(F_2) \geq \sigma(F_1)$. Then the function $F = F_1F_2$ has the order $\rho(F) \leq \rho(F_1)$ and the type $\sigma(F)$. If $\rho(F) = \rho(F_1)$, then $\sigma(F) \leq 2\sigma(F_2)$.

A similar statement holds true for the function $\tilde{F} = F_2F_1$.

The proof is based on the following lemma.

**Lemma 4.** Let a sequence of operators $\mathcal{A} = \{A_n\}$ has the order $\beta(\mathcal{A})$ and the type $\alpha(\mathcal{A})$, and a sequence of operators $\mathcal{B} = \{B_n\}$ does the order $\beta(\mathcal{B}) = \beta(\mathcal{A})$ and the type $\alpha(\mathcal{B}) \geq \alpha(\mathcal{A})$. The the sequence of the operators $\mathcal{C} = \{C_n\}$, where $C_n = \sum_{k=0}^{n} A_kB_{n-k}$, has the order $\beta(\mathcal{C}) \leq \beta(\mathcal{B})$ and the type $\alpha(\mathcal{C})$. If $\beta(\mathcal{C}) = \beta(\mathcal{B})$, then $\alpha(\mathcal{C}) \leq 2^{-\beta(\mathcal{B})}\alpha(\mathcal{B})$.

**Proof.** Under the hypothesis of the lemma the expression in the brackets in (21) does not exceed $2^{-\beta(\mathcal{B})n}n$ and thus $\beta(\mathcal{C}) \leq \beta(\mathcal{B})$, and if $\beta(\mathcal{C}) = \beta(\mathcal{B})$, then $\alpha(\mathcal{C}) \leq 2^{-\beta(\mathcal{B})}\alpha(\mathcal{B})$. \hfill \Box

**Remark.** As it is known, in the scalar case the theorem on categories [3, Th. 12] is valid. In the case of operator-valued function this question is still open.

6°. (Invariance.) Suppose $\mathbf{H}_1, \tilde{\mathbf{H}}_1, \mathbf{H}$ and $\tilde{\mathbf{H}}$ are four locally convex spaces with the topologies induced respectively by the multinorms $\|\cdot\|_q$, $q \in \mathcal{Q}$, $\|\cdot\|_{\tilde{q}}$, $\tilde{q} \in \tilde{\mathcal{Q}}$, $\|\cdot\|_p$, $p \in \mathcal{P}$, $\|\cdot\|_{\tilde{p}}$, $\tilde{p} \in \tilde{\mathcal{P}}$ and let $T_1 : \mathbf{H}_1 \to \tilde{\mathbf{H}}$, $T : \mathbf{H} \to \tilde{\mathbf{H}}$ are two topological isomorphisms. Then

1) for each operator-valued function

$$F(t) = \sum_{n=0}^{\infty} A_n t^n : \mathbb{C} \to \text{Lec}(\mathbf{H}_1, \mathbf{H})$$

its order and type coincide with the order and type of the function

$$\tilde{F}(t) = \sum_{n=0}^{\infty} T A_n T_1^{-1} t^n : \mathbb{C} \to \text{Lec}(\tilde{\mathbf{H}}_1, \tilde{\mathbf{H}});$$

2) if all the $p$-orders of the function $F$ are strictly less than its order, then all the $\tilde{p}$-orders of the function $\tilde{F}$ are strictly less than its order;

3) if at least one $p$-order of the function $F$ equals to its order, then at least one $\tilde{p}$-order of the function $\tilde{F}$ equals to its order;

4) if the function $F$ has the $p$-orders $\rho_p(F)$, the order $\rho(F)$, the $p$-types $\sigma_p(F)$ and the type $\sigma(F)$, at the set

$$\mathcal{P}_F = \{p \in \mathcal{P} : \rho_p(F) = \rho(F)\}$$

is non-empty and $\forall p \in \mathcal{P}_F : \sigma_p(F) < \sigma(F)$, then the function $\tilde{F}$ has the $\tilde{p}$-orders $\rho_{\tilde{p}}(\tilde{F})$, the order $\rho(\tilde{F})$, the $\tilde{p}$-types $\sigma_{\tilde{p}}(\tilde{F})$ and the type $\sigma(\tilde{F})$, at the set

$$\tilde{\mathcal{P}}_{\tilde{F}} = \{\tilde{p} \in \tilde{\mathcal{P}} : \rho_{\tilde{p}}(\tilde{F}) = \rho(\tilde{F})\}$$

is non-empty and $\forall \tilde{p} \in \tilde{\mathcal{P}}_{\tilde{F}} : \sigma_{\tilde{p}}(\tilde{F}) < \sigma(\tilde{F})$;

5) if under hypothesis of Item 4) $\exists p \in \mathcal{P}_F : \sigma_p(F) = \sigma(F)$, then $\exists \tilde{p} \in \tilde{\mathcal{P}}_{\tilde{F}} : \sigma_{\tilde{p}}(\tilde{F}) = \sigma(\tilde{F})$.

The validity of property 6° follows from analogous properties for the characteristics of a sequence of operators [1, 6].

The invariance property implies that under any replacements of the multinorms in $\mathbf{H}_1$ and $\mathbf{H}$ to equivalent ones ($T_1$ and $T$ are identity operators)

1) the order and type of the operator-valued function $F$ remain the same;
2) if all the $p$-orders of the function $F$ were strictly less than its order before the replacement of the multinorms, after the replacement of the multinorms all its $\tilde{p}$-orders are also strictly less than the order;

3) if at least one $p$-order of the function $F$ equals its order before the replacement of the multinorms, after the replacement at least one its $\tilde{p}$-order (not necessarily the same) is also equal to its order;

4) if the function $F$ has the $p$-order $\rho_p(F)$ before the replacement of the multinorms, the order $\rho(F)$, the $p$-types $\sigma_p(F)$, and the type $\sigma(F)$, at that the set

$$\mathcal{P}_F = \{ p \in \mathcal{P} : \rho_p(F) = \rho(F) \}$$

is non-empty and $\forall p \in \mathcal{P}_F : \sigma_p(F) < \sigma(F)$, then after the replacement of the multinorms this function has the $\tilde{p}$-orders $\rho_{\tilde{p}}(F)$, the order $\rho(F)$, the $\tilde{p}$-types $\sigma_{\tilde{p}}(F)$, and the type $\sigma(F)$, at that the set

$$\tilde{\mathcal{P}}_F = \{ \tilde{p} \in \tilde{\mathcal{P}} : \rho_{\tilde{p}}(F) = \rho(F) \}$$

is non-empty and $\forall \tilde{p} \in \tilde{\mathcal{P}}_F : \sigma_{\tilde{p}}(F) < \sigma(F)$;

5) if under the hypothesis of Item 4) $\exists p \in \mathcal{P}_F : \sigma_p(F) = \sigma(F)$, then $\exists \tilde{p} \in \tilde{\mathcal{P}}_F : \sigma_{\tilde{p}}(F) = \sigma(F)$.

BIBLIOGRAPHY


Sergei Nikolaevich Mishin,
Orel State University,
Komsomolskaya str., 95,
302026, Orel, Russia
E-mail: sergeymishin@rambler.ru