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INTEGRATION OF HIGHER KORTEWEG-DE VRIES EQUATION WITH SELF-CONSISTENT SOURCE IN CLASS OF PERIODIC FUNCTIONS

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Abstract. In the present work the inverse spectral problem of Sturm-Liouville operator is applied for integrating higher Korteweg-de Vries equation with a self-consistent source in class of periodic functions

Keywords: Sturm-Liouville operator, inverse spectral problem, eigenvalue, eigenfunction, Korteweg-de Vries equation.

1. Introduction

In 1967 in work [1] American scientists C.S. Gardner, J.M. Greene, M.D. Kruskal, and R.M. Miura established the integrability of the Korteweg-de Vries equation (KdV) in the class of "fast decaying" w.r.t. x functions by the method of inverse scattering problem for the Sturm-Liouville equation. In work [2] P. Lax showed an universality of the inverse scattering problem method and generalized the KdV equation by introducing a higher (general) KdV equation.

In works [3-10] KdV equation and higher KdV equation were studied in the class of finite-band and periodic functions.

In the present work we study the higher KdV equation with a self-consistent source in the class of periodic functions.

We note that in [11-15] and other papers the KdV equation with a self-consistent source was considered in the class of fast decaying functions, and nonlinear equations with a source in the class of periodic functions in various formulations were studied in works [16-19].

Let

$$H = -\frac{1}{2}\frac{d^3}{dx^3} + 2q\frac{d}{dx} + q',$$

where $q = q(x,t)$, and the prime denotes the derivative w.r.t. x. According to [20], there exists polynomials P_k (of q and the derivatives of q w.r.t. x) such that

$$HP_k = P'_{k+1}.$$

For example,

$$P_0 = 1, \quad P_1 = q, \quad P_2 = -\frac{1}{2}q_{xx} + \frac{3}{2}q^2, \quad P_3 = \frac{1}{4}q_{xxxx} - \frac{5}{2}qq_{xx} - \frac{5}{4}q_x^2 + \frac{5}{2}q^3.$$

It is easy to prove the following properties of the operator H (see [20]).

Lemma 1. *If $y(x,t)$ is a solution to the following Sturm-Liouville equation*

$$L(t)y \equiv -y'' + q(x,t)y = \lambda y, \quad x \in R^1,$$

the identity

$$H(y^2) = 2\lambda(y^2)'$$

M.M. Matyoqubov, A.B. Yakhshimuratov, Integration of higher Korteweg-de Vries equation with self-consistent source in class of periodic functions.

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holds true.

Lemma 2. *For any $y(x),\ z(x)\in C^3[0,\ \pi]$ the identity*

$$\int_0^\pi Hz\cdot y dx=\left(-\frac{1}{2}z''y+2qzy+\frac{1}{2}z'y'-\frac{1}{2}zy''\right)\Bigg|_0^\pi-\int_0^\pi z\cdot Hy dx$$

holds true.

The following equation

$$q_t=HP_N[q],\quad x\in R^1,\quad t>0,$$

is called a higher KdV equation. Employing the properties of the operator H, we can rewrite this equation as

$$q_t=P'_{N+1}[q],\quad x\in R^1,\quad t>0.$$

For instance, as $N=0,\ 1,\ 2$ we respectively have

$$q_t=q_x,\quad q_t=-\frac{1}{2}q_{xxx}+3qq_x,\quad q_t=\frac{1}{4}q_{xxxxx}-5q_xq_{xx}-\frac{5}{2}qq_{xxx}+\frac{15}{2}q^2q_x.$$

2. Formulation of problem

In this work we consider the following higher KdV equation with a self-consistent source

$$q_t=P'_{N+1}[q]+2\int_0^\infty\beta(\lambda,t)s(\pi,\lambda,t)\left(\psi_+(x,\lambda,t)\psi_-(x,\lambda,t)\right)_x d\lambda,\ t>0,\ x\in R^1, \tag{1}$$

subject to initial condition

$$q(x,t)|_{t=0}=q_0(x), \tag{2}$$

where $q_0(x)\in C^{2N+1}\left(R^1\right)$ is a given real function. It is required to find a real function $q(x,t)$ being π-periodic w.r.t. x,

$$q(x+\pi,t)\equiv q(x,t),\quad t\ge 0,\quad x\in R^1, \tag{3}$$

and satisfying the smoothness conditions

$$q(x,t)\in C_x^{2N+1}(t>0)\cap C_t^1(t>0)\cap C(t\ge 0).. \tag{4}$$

Here $\beta(\lambda,t)\in C\left([0,\ \infty)\times[0,\ \infty)\right)$ is a given real function having the uniform asymptotics $\beta(\lambda,t)=O\left(\frac{1}{\lambda}\right),\ \lambda\to\infty,\ \psi_\pm(x,\lambda,t)$ are the Floquet solutions (normalized by the condition $\psi_\pm(0,\lambda,t)=1$) to the Sturm-Liouville equation

$$L(t)y\equiv -y''+q(x,t)y=\lambda\, y,\quad x\in R^1, \tag{5}$$

$s(x,\lambda,t)$ is the solution to equation (5) satisfying the initial conditions $s(0,\lambda,t)=0$, $s'(0,\lambda,t)=1$.

Remark 1. Let us show the uniform convergence of the integral involved in (1). In order to do it, we employ the identity

$$s(\pi,\lambda,t)\psi_+(\tau,\lambda,t)\psi_-(\tau,\lambda,t)=s(\pi,\lambda,t,\tau), \tag{6}$$

where $s(x,\lambda,t,\tau)$ solves the equation

$$-y''+q(x+\tau,t)y=\lambda y\,,\quad x\in R^1,$$

and obeys the initial conditions $s(0,\lambda,t,\tau)=0,\ s'(0,\lambda,t,\tau)=1$.

The asymptotic formulae

$$c(x,\lambda,t)=\cos\sqrt{\lambda}x+O\left(\frac{1}{\sqrt{\lambda}}\right),\quad s(x,\lambda,t)=\frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}}+O\left(\frac{1}{\lambda}\right),$$

$$c'(x,\lambda,t) = -\sqrt{\lambda}\sin\sqrt{\lambda}x + O(1), \quad s'(x,\lambda,t) = \cos\sqrt{\lambda}x + O\left(\frac{1}{\sqrt{\lambda}}\right), \quad (\lambda \to \infty)$$

and identities

$$s(\pi,\lambda,t,\tau) = c(\tau,\lambda,t)s(\pi+\tau,\lambda,t) - s(\tau,\lambda,t)c(\pi+\tau,\lambda,t)$$

imply the estimates

$$s(\pi,\lambda,t,\tau) = O\left(\frac{1}{\sqrt{\lambda}}\right), \quad \frac{\partial s(\pi,\lambda,t,\tau)}{\partial\tau} = O\left(\frac{1}{\sqrt{\lambda}}\right), \quad (\lambda \to \infty).$$

These estimates and identity (6) ensures the uniform convergence of the integral involved in equation (1).

The aim of the present work is to provide the procedure of constructing a solution $q(x,t)$, $\psi_\pm(x,\lambda,t)$ to problem (1)-(5) in the framework of the inverse spectral problem for the Sturm-Liouville operator with a periodic coefficient.

3. Preliminaries

In this section, for the completeness of the content, we present some basic information concerning the inverse spectral problem for the Sturm-Liouville operator with a periodic coefficient (see [21–26]).

Consider the following Sturm-Liouville operator on the axis

$$Ly \equiv -y'' + q(x)y = \lambda\, y, \quad x \in R^1, \tag{7}$$

where $q(x)$ is a real continuous π-periodic function.

By $c(x,\lambda)$ and $s(x,\lambda)$ we denote the solutions to equation (7) satisfying the initial conditions $c(0,\lambda)=1$, $c'(0,\lambda)=0$ and $s(0,\lambda)=0$, $s'(0,\lambda)=1$. The function $\Delta(\lambda) = c(\pi,\,\lambda) + s'(\pi,\,\lambda)$ is called Lyapunov function or Hill discriminant.

The spectrum of operator (7) is pure continuous and coincides with the set

$$E = \{\lambda \in R^1 : \ -2 \le \Delta(\lambda) \le 2\ \} = [\lambda_0, \lambda_1] \cup [\lambda_2,\ \lambda_3] \cup \ldots \cup [\lambda_{2n}, \lambda_{2n+1}] \cup \ldots .$$

The intervals $(-\infty,\ \lambda_0)$, $(\lambda_{2n-1},\ \lambda_{2n})$, $n = 1,\ 2,\ \ldots$ are called gaps. Here λ_0, λ_{4k-1}, λ_{4k} are the eigenvalues of the periodic problem ($y(0) = y(\pi)$, $y'(0) = y'(\pi)$), and λ_{4k+1}, λ_{4k+2} are that of the antiperiodic problem ($y(0) = -y(\pi)$, $y'(0) = -y'(\pi)$) for equation (7).

Let ξ_n, $n = 1, 2, \ldots$, be the roots to the equation $s(\pi,\lambda) = 0$. We observe that ξ_n, $n = 1, 2, \ldots$, coincide with the eigenvalues of the Dirichlet problem ($y(0) = y(\pi) = 0$) for equation (7), and moreover, the belongings $\xi_n \in [\lambda_{2n-1}, \lambda_{2n}]$, $n = 1,\ 2,\ \ldots$ are fulfilled.

The numbers ξ_n, $n = 1,\ 2,\ \ldots$ with the signs $\sigma_n = sign\{s'(\pi,\xi_n) - c(\pi,\xi_n)\}$, $n = 1,\ 2,\ \ldots$ are called spectral parameters of problem (7). The spectral parameters ξ_n, σ_n, $n = 1,\ 2,\ \ldots$ and the edges λ_n, $n = 0,\ 1,\ 2,\ \ldots$ of the spectrum are called spectral data of operator (7). Recovering of the coefficient $q(x)$ by the spectral data is called the inverse spectral problem for operator (7).

The spectrum of the Sturm-Liouville operator with the coefficient $q(x+\tau)$ is independent of the real parameter τ, and the spectral parameters depend on τ; $\xi_n(\tau)$, $\sigma_n(\tau)$, $n = 1,\ 2,\ \ldots$. The spectral parameters satisfy the following Dubrovin system of equations

$$\frac{d\xi_n}{d\tau} = 2(-1)^{n-1}\sigma_n(\tau)\sqrt{(\xi_n - \lambda_{2n-1})(\lambda_{2n} - \xi_n)}\times$$
$$\times\sqrt{(\xi_n - \lambda_0)\prod_{\substack{k=1\\k\neq n}}^{\infty}\frac{(\lambda_{2k-1} - \xi_n)(\lambda_{2k} - \xi_n)}{(\xi_k - \xi_n)^2}}, \quad n \ge 1. \tag{8}$$

The Dubrovin system of equations and the following trace formula

$$q(\tau,t) = \lambda_0 + \sum_{k=1}^{\infty}(\lambda_{2k-1} + \lambda_{2k} - 2\xi_k(\tau,t))$$

give the method for solving the inverse spectral problem.

There are also other trace formulae, for instance, the second and third trace formulae read as

$$q^2(\tau) - \frac{1}{2}q_{\tau\tau}(\tau) = \lambda_0^2 + \sum_{k=1}^{\infty}(\lambda_{2k-1}^2 + \lambda_{2k}^2 - 2\xi_k^2(\tau)),$$

$$\frac{3}{16}q_{\tau\tau\tau\tau}(\tau) - \frac{3}{2}q(\tau)q_{\tau\tau}(\tau) - \frac{15}{16}q_\tau^2(\tau) + q^3(\tau) =$$

$$= \lambda_0^3 + \sum_{k=1}^{\infty}(\lambda_{2k-1}^3 + \lambda_{2k}^3 - 2\xi_k^3(\tau)).$$

Employing Dubrovin system of equations and the first trace formula, E. Trubowitz [25] succeeded to prove theorems relating the analyticity of the potential and the decay of the gaps lengths for the periodic potential of the Sturm-Liouville operator (7); if $q(x)$ is a real analytic π-periodic function, the lengths $\lambda_{2n} - \lambda_{2n-1}$ of the gaps exponentially tend to zero, i.e., there exist the constants $a > 0$, $b > 0$ such that $\lambda_{2n} - \lambda_{2n-1} < ae^{-bn}$, $n \geq 1$; and vice-versa, if $q(x) \in C^2(R^1)$ is a real π-periodic function and the lengths $\lambda_{2n} - \lambda_{2n-1}$ of the gaps exponentially tend to zero, then $q(x)$ is an analytic function.

In 1946 G. Borg proved a unique theorem (Borg's inverse theorem) on the period of the potential of the Hill equation (see [27]): the number $\pi/2$ is a period of the potential $q(x)$ of the Sturm-Liouville equation (7), if and only if all the roots to the equation $\Delta(\lambda) + 2 = 0$ are double, i.e., if and only if all the gaps with odd indices disappear.

In 1977 (see [28]) H. Hochstadt gave a short proof, and in 1984 a generalization of the Borg's theorem (see [29]). Let $q(x) \in C^1(R^1)$ be a real π-periodic function. The number π/n is the period of the potential $q(x)$ of Sturm-Liouville equation (7), if and only if all the gaps whose indices are not divisible by n disappear. Here $n \geq 2$ is a natural number.

4. Main theorem

The main result of the present work is the following theorem.

Theorem 1. *Let $q(x,t)$, $\psi_\pm(x,\lambda,t)$ be a solution to problem (1)-(5). Then the spectrum of operator (5) is independent of the parameter t, and the spectral parameters $\xi_n(t)$, $n = 1, 2, \ldots$ satisfies an analogue of Dubrovin system of equations,*

$$\dot{\xi}_n = 2(-1)^{n-1}\sigma_n(t)\left\{\sum_{k=0}^{N}(2\xi_n)^{N-k}\cdot P_k[q(0,t)] + \int_0^\infty \frac{s(\pi,\lambda,t)\beta(\lambda,t)}{\lambda-\xi_n}d\lambda\right\}\times$$

$$\times\sqrt{(\xi_n-\lambda_{2n-1})(\lambda_{2n}-\xi_n)}\times\sqrt{(\xi_n-\lambda_0)\prod_{\substack{k=1\\k\neq n}}^{\infty}\frac{(\lambda_{2k-1}-\xi_n)(\lambda_{2k}-\xi_n)}{(\xi_k-\xi_n)^2}},\quad n\geq 1, \tag{9}$$

where the sign of $\sigma_n(t)$ changes to the opposite under each collision of the point $\xi_n(t)$ and the edges of the gap $[\lambda_{2n-1}, \lambda_{2n}]$. Moreover, the initial conditions

$$\xi_n(t)|_{t=0} = \xi_n^0,\quad \sigma_n(t)|_{t=0} = \sigma_n^0,\quad n\geq 1,$$

hold true, where ξ_n^0, σ_n^0, $n \geq 1$ are spectral parameters to the Sturm-Liouville operator with the coefficient $q_0(x)$.

Proof. We introduce the notation

$$G(x,t) = 2\int_0^\infty \beta(\lambda,t)s(\pi,\lambda,t)\left(\psi_+(x,\lambda,t)\cdot\psi_-(x,\lambda,t)\right)_x d\lambda,$$

and rewrite equation (1) as

$$q_t = P'_{N+1}[q] + G(x,t). \tag{10}$$

By $y_n(x,t)$, $n = 1,\ 2,\ \dots$ we denote orthonormalized eigenfunctions to the Dirichlet problem $(y(0) = 0,\ \ y(\pi) = 0)$ for equation (5) with the π-periodic potential $q(x,t)$ being a solution to equation (10); these eigenfunctions are associated with the eigenvalues $\xi_n(t)$, $n = 1,\ 2,\ \dots$.

Differentiating the identity $(L(t)y_n, y_n) = \xi_n$ w.r.t. t and employing the symmetricity of the operator $L(t)$, we have

$$\dot{\xi}_n = (L\dot{y}_n + q_t y_n, y_n) + (Ly_n, \dot{y}_n) = (\dot{y}_n, Ly_n) + (Ly_n, \dot{y}_n) + (q_t y_n, y_n) =$$

$$= \xi_n((y_n, y_n))\dot{} + (q_t y_n, y_n) = \int\limits_0^\pi q_t(x,t) y_n^2(x,t)dx. \tag{11}$$

Here $(\cdot\ ,\ \cdot)$ is a scalar product in the space $L_2(0,\ \pi)$.

Employing (10) and identity $HP_k = P'_{k+1}$, we rewrite identity (11) as

$$\dot{\xi}_n = \int\limits_0^\pi y_n^2(x,t)HP_N dx + \int\limits_0^\pi y_n^2(x,t)G(x,t)dx. \tag{12}$$

Employing Lemmata 1 and 2, we convert the following integral

$$J_k = \int\limits_0^\pi y_n^2(x,t)HP_k dx = \left(-\frac{1}{2}P''_k \cdot y_n^2 + 2qP_k \cdot y_n^2 + \frac{1}{2}P'_k \cdot (y_n^2)' - \frac{1}{2}P_k \cdot (y_n^2)''\right)\Bigg|_0^\pi -$$

$$-\int\limits_0^\pi P_k \cdot H(y_n^2)dx = -P_k[q(0,t)] \cdot [{y'_n}^2(\pi,t) - {y'_n}^2(0,t)] - \int\limits_0^\pi P_k \cdot 2\xi_n (y_n^2)' dx =$$

$$= -P_k[q(0,t)] \cdot [{y'_n}^2(\pi,t) - {y'_n}^2(0,t)] + 2\xi_n \int\limits_0^\pi P'_k \cdot y_n^2 dx,$$

i.e.,

$$J_k - 2\xi_n \cdot J_{k-1} = -P_k[q(0,t)] \cdot [{y'_n}^2(\pi,t) - {y'_n}^2(0,t)].$$

Calculating the following sum

$$J_N - (2\xi_n)^N \cdot J_0 = \sum_{k=1}^N (2\xi_n)^{N-k} \cdot (J_k - 2\xi_n \cdot J_{k-1}) =$$

$$= -[{y'_n}^2(\pi,t) - {y'_n}^2(0,t)] \cdot \sum_{k=1}^N (2\xi_n)^{N-k} \cdot P_k[q(0,t)]$$

and the integral

$$J_0 = \int\limits_0^\pi y_n^2(x,t)HP_0 dx = \int\limits_0^\pi y_n^2(x,t) q_x dx = -[{y'_n}^2(\pi,t) - {y'_n}^2(0,t)],$$

we deduce the identity

$$J_N = -[{y'_n}^2(\pi,t) - {y'_n}^2(0,t)] \cdot \sum_{k=0}^N (2\xi_n)^{N-k} \cdot P_k[q(0,t)]. \tag{13}$$

Now we proceed to calculating the second integral in identity (12),

$$\int_0^\pi G\cdot y_n^2 dx = \int_0^\infty s(\pi,\lambda,t)\beta(\lambda,t)\left\{2\int_0^\pi y_n^2\cdot(\psi_+\psi_-)' dx\right\}d\lambda.$$

Integrating by parts, it is easy to see that

$$I = 2\int_0^\pi y_n^2\cdot(\psi_+\psi_-)' dx = \int_0^\pi \{y_n^2\cdot(\psi_+\psi_-)' - (y_n^2)'\cdot(\psi_+\psi_-)\}dx =$$

$$= \int_0^\pi \{y_n\psi_-(y_n\psi_+' - y_n'\psi_+) + y_n\psi_+(y_n\psi_-' - y_n'\psi_-)\}dx.$$

It yields

$$I = \frac{1}{\xi_n - \lambda}\cdot[{y_n'}^2(\pi,t) - {y_n'}^2(0,t)].$$

Hence,

$$\int_0^\pi G\cdot y_n^2 dx = [{y_n'}^2(\pi,t) - {y_n'}^2(0,t)]\cdot\int_0^\infty \frac{s(\pi,\lambda,t)\beta(\lambda,t)}{\xi_n-\lambda}d\lambda. \tag{14}$$

Substituting expressions (13) and (14) into (12), we obtain the identity

$$\dot{\xi}_n = [{y_n'}^2(\pi,t) - {y_n'}^2(0,t)]\times$$

$$\times\left\{-\sum_{k=0}^N (2\xi_n)^{N-k}\cdot P_k[q(0,t)] + \int_0^\infty \frac{s(\pi,\lambda,t)\beta(\lambda,t)}{\xi_n-\lambda}d\lambda\right\}.. \tag{15}$$

Employing the identities

$$y_n(x,t) = \frac{1}{c_n(t)}s(x,\xi_n(t),t),$$

$$c_n^2(t) \equiv \int_0^\pi s^2(x,\xi_n(t),t)dx = s'(\pi,\xi_n(t),t)\frac{\partial s(\pi,\xi_n(t),t)}{\partial\lambda},$$

we have

$${y_n'}^2(\pi,t) - {y_n'}^2(0,t) = \frac{1}{\frac{\partial s(\pi,\xi_n(t),t)}{\partial\lambda}}\left(s'(\pi,\xi_n(t),t) - \frac{1}{s'(\pi,\xi_n(t),t)}\right).$$

Substituting here the expression

$$s'(\pi,\xi_n,t) - \frac{1}{s'(\pi,\xi_n,t)} = \sigma_n(t)\sqrt{\Delta^2(\xi_n(t)) - 4},$$

we obtain

$${y_n'}^2(\pi,t) - {y_n'}^2(0,t) = \frac{\sigma_n(t)\sqrt{\Delta^2(\xi_n(t)) - 4}}{\frac{\partial s(\pi,\xi_n(t),t)}{\partial\lambda}}.$$

Here $\sigma_n(t) = sign\{s'(\pi,\xi_n(t),t) - c(\pi,\xi_n(t),t)\}$.

The expansions

$$\Delta^2(\lambda) - 4 = 4\pi^2(\lambda_0-\lambda)\prod_{k=1}^\infty \frac{(\lambda_{2k-1}-\lambda)(\lambda_{2k}-\lambda)}{k^4},$$

$$s(\pi,\lambda,t) = \pi\prod_{k=1}^\infty \frac{\xi_k(t)-\lambda}{k^2}$$

imply

$$y_n'^2(\pi,t) - y_n'^2(0,t) = 2(-1)^n\sigma_n(t)\sqrt{(\xi_n-\lambda_{2n-1})(\lambda_{2n}-\xi_n)}\times$$
$$\times\sqrt{(\xi_n-\lambda_0)\prod_{\substack{k=1\\k\neq n}}^{\infty}\frac{(\lambda_{2k-1}-\xi_n)(\lambda_{2k}-\xi_n)}{(\xi_k-\xi_n)^2}}. \tag{16}$$

By (15) and (16) we get (9).

Let us prove the independence of t for the eigenvalues λ_n, $n = 0,\ 1,\ 2,\ \dots$ of the periodic and anti-periodic problems for Sturm-Liouville equation (5). By analogy with formula (15) one can show that

$$\dot{\lambda}_n(t) = \int_0^\pi G(x,t)v_n^2(x,t)dx,$$

where $v_n(x,t)$ is a normalized eigenfunction of either periodic or antiperiodic problem for Sturm-Liouville equation (5). Taking into consideration for structure of the function $G(x,t)$ and proceeding as above, we obtain $\dot{\lambda}_n(t) = 0$. The proof is complete. □

5. Corollaries of main theorem

Corollary 1. *If instead of $q(x,t)$ we consider $q(x+\tau,t)$, the eigenvalues to periodic and antiperiodic problem are independent of the parameters τ and t, and the eigenvalues ξ_n of the Dirichlet problem and the signs σ_n depend on τ and t; $\xi_n = \xi_n(\tau,t)$, $\sigma_n = \sigma_n(\tau,t) = \pm 1$, $n \geq 1$. In this case system (9) casts into the form*

$$\frac{\partial \xi_n}{\partial t} = 2(-1)^{n-1}\sigma_n(\tau,t)\left\{\sum_{k=0}^{N}(2\xi_n)^{N-k}\cdot P_k[q(\tau,t)] + \int_0^\infty \frac{s(\pi,\lambda,t,\tau)\beta(\lambda,t)}{\lambda-\xi_n}d\lambda\right\}\times$$
$$\times\sqrt{(\xi_n-\lambda_{2n-1})(\lambda_{2n}-\xi_n)}\times\sqrt{(\xi_n-\lambda_0)\prod_{\substack{k=1\\k\neq n}}^{\infty}\frac{(\lambda_{2k-1}-\xi_n)(\lambda_{2k}-\xi_n)}{(\xi_k-\xi_n)^2}},\quad n\geq 1. \tag{17}$$

Here

$$s(\pi,\lambda,t,\tau) = \pi\prod_{k=1}^{\infty}\frac{\xi_k(t,\tau)-\lambda}{k^2}. \tag{18}$$

Corollary 2. *Consider the case $N = 2$. In this case differential equation (1) becomes*

$$q_t = \frac{1}{4}q_{xxxxx} - 5q_xq_{xx} - \frac{5}{2}qq_{xxx} + \frac{15}{2}q^2q_x + G(x,t), \tag{19}$$

and Dubrovin system of differential equations (17) is written as

$$\frac{\partial \xi_n}{\partial t} = 2(-1)^{n-1}\sigma_n(\tau,t)\left\{4\xi_n^2 + 2\xi_n q - \frac{1}{2}q_{\tau\tau} + \frac{3}{2}q^2 + \int_0^\infty \frac{s(\pi,\lambda,t,\tau)\beta(\lambda,t)}{\lambda-\xi_n}d\lambda\right\}\times$$
$$\times\sqrt{(\xi_n-\lambda_{2n-1})(\lambda_{2n}-\xi_n)}\times\sqrt{(\xi_n-\lambda_0)\prod_{\substack{k=1\\k\neq n}}^{\infty}\frac{(\lambda_{2k-1}-\xi_n)(\lambda_{2k}-\xi_n)}{(\xi_k-\xi_n)^2}},\quad n\geq 1. \tag{20}$$

Employing the following trace formulae

$$q(\tau,t) = \lambda_0 + \sum_{k=1}^{\infty}(\lambda_{2k-1}+\lambda_{2k}-2\xi_k(\tau,t)), \tag{21}$$

$$q^2(\tau,t)-\frac{1}{2}q_{\tau\tau}(\tau,t)=\lambda_0^2+\sum_{k=1}^{\infty}(\lambda_{2k-1}^2+\lambda_{2k}^2-2\xi_k^2(\tau,t)), \tag{22}$$

system (20) can be rewritten in a closed form.

Corollary 3. *This theorem provides a method for solving problem (19), (2)-(5).*

Indeed, denote by λ_n, $n=0,\ 1,\ 2,\ \dots$, $\xi_n(\tau,t)$, $\sigma_n(\tau,t)$, $n=1,\ 2,\ \dots$, the spectral data of the problem

$$-y''+q(x+\tau,t)y=\lambda\, y,\quad x\in R^1.$$

Let us find spectral data λ_n, $n=0,\ 1,\ 2,\ \dots$, $\xi_n^0(\tau)$, $\sigma_n^0(\tau)$, $n=1,\ 2,\ \dots$ for the equation

$$-y''+q_0(x+\tau)y=\lambda y,\quad x\in R^1.$$

We solve then the Cauchy problem $\xi_n(\tau,t)|_{t=0}=\xi_n^0(\tau)$, $\sigma_n(\tau,t)|_{t=0}=\sigma_n^0(\tau)$, $n=1,\ 2,\ \dots$ for Dubrovin system of equations (20). By trace formula (21) we find the solution to problem (19), (2)-(5). Then it is easy to find the Floquet solutions $\psi_\pm(x,\lambda,t)$.

Remark 2. Let us show that the constructed function $q(\tau,t)$ satisfies equation (19). In order to do it, we employ the following Dubrovin system of equations

$$\frac{\partial\xi_n}{\partial\tau}=2(-1)^{n-1}\sigma_n(\tau,t)\sqrt{(\xi_n-\lambda_{2n-1})(\lambda_{2n}-\xi_n)}\times$$

$$\times\sqrt{(\xi_n-\lambda_0)\prod_{\substack{k=1\\k\neq n}}^{\infty}\frac{(\lambda_{2k-1}-\xi_n)(\lambda_{2k}-\xi_n)}{(\xi_k-\xi_n)^2}},\quad n=1,\ 2,\ \dots, \tag{23}$$

and trace formulae (21), (22), as well as (see [26])

$$\frac{3}{16}q_{\tau\tau\tau\tau}(\tau,t)-\frac{3}{2}q(\tau,t)q_{\tau\tau}(\tau,t)-\frac{15}{16}q_\tau^2(\tau,t)+q^3(\tau,t)=$$

$$=\lambda_0^3+\sum_{k=1}^{\infty}(\lambda_{2k-1}^3+\lambda_{2k}^3-2\xi_k^3(\tau,t)). \tag{24}$$

Dubrovin system (20) and (23) imply

$$\frac{\partial\xi_k}{\partial t}=\left\{4\xi_k^2+2\xi_k q-\frac{1}{2}q_{\tau\tau}+\frac{3}{2}q^2+\int_0^{\infty}\frac{s(\pi,\lambda,t,\tau)\beta(\lambda,t)}{\lambda-\xi_k}d\lambda\right\}\frac{\partial\xi_k}{\partial\tau},\quad k\geq 1. \tag{25}$$

First trace formula (21) and (25) yield

$$q_t=-2\sum_{k=1}^{\infty}\frac{\partial\xi_k}{\partial t}=(q_{\tau\tau}-3q^2)\cdot\sum_{k=1}^{\infty}\frac{\partial\xi_k}{\partial\tau}-4q\sum_{k=1}^{\infty}\xi_k\frac{\partial\xi_k}{\partial\tau}-8\sum_{k=1}^{\infty}\xi_k^2\frac{\partial\xi_k}{\partial\tau}+$$

$$+2\int_0^{\infty}\beta(\lambda,t)\left\{\sum_{k=1}^{\infty}\frac{s(\pi,\lambda,t,\tau)}{\xi_k-\lambda}\frac{\partial\xi_k}{\partial\tau}\right\}d\lambda. \tag{26}$$

Differentiating trace formulae (21), (22), and (24) w.r.t. τ, we obtain

$$2\sum_{k=1}^{\infty}\frac{\partial\xi_k}{\partial\tau}=-q_\tau,\quad 4\sum_{k=1}^{\infty}\xi_k\frac{\partial\xi_k}{\partial\tau}=\frac{1}{2}q_{\tau\tau\tau}-2qq_\tau,$$

$$-2\sum_{k=1}^{\infty}\xi_k^2\frac{\partial\xi_k}{\partial\tau}=\frac{1}{16}q_{\tau\tau\tau\tau\tau}-\frac{1}{2}qq_{\tau\tau\tau}-\frac{9}{8}q_\tau q_{\tau\tau}+q^2q_\tau.$$

Employing these identities and expansion (18), by (26) we deduce

$$q_t = \frac{1}{4}q_{\tau\tau\tau\tau\tau} - 5q_\tau q_{\tau\tau} - \frac{5}{2}qq_{\tau\tau\tau} + \frac{15}{2}q^2 q_\tau + 2\int\limits_0^\infty \beta(\lambda, t)\frac{\partial s(\pi, \lambda, t, \tau)}{\partial \tau}d\lambda.$$

It follows from identity (6) that

$$q_t = \frac{1}{4}q_{\tau\tau\tau\tau\tau} - 5q_\tau q_{\tau\tau} - \frac{5}{2}qq_{\tau\tau\tau} + \frac{15}{2}q^2 q_\tau +$$

$$+2\int\limits_0^\infty \beta(\lambda, t)s(\pi, \lambda, t)\frac{\partial}{\partial \tau}\left(\psi_+(\tau, \lambda, t)\cdot\psi_-(\tau, \lambda, t)\right)d\lambda.$$

Corollary 4. *From the results of work [25] we deduce that if the initial function $q_0(x)$ is real and analytic, then the lengths $\lambda_{2n} - \lambda_{2n-1}$ of the gaps corresponding to this coefficients decay exponentially. Since the lengths of the gaps corresponding to the coefficients $q(x, t)$ are independent of t, the function $q(x, t)$ is analytic w.r.t. x.*

Corollary 5. *The generalized Borg's inverse theorem (see [29]) follows that if $q_0(x)$ has the period $\frac{\pi}{n}$, the solution $q(x, t)$ to problem (19), (2)-(5) is $\frac{\pi}{n}$-periodic w.r.t. x.*

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