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REDUCTIONS OF PARTIALLY INVARIANT SOLUTIONS OF RANK 1 DEFECT 2 FIVE-DIMENSIONAL OVERALGEBRA OF CONICAL SUBALGEBRA

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Abstract. Conic flows are the invariant rank 1 solutions of the gas dynamics equations on the three-dimensional subalgebra defined by the rotation operators, translation by time, and uniform dilatation. The generalization of the conic flows are partially invariant solutions of rank 1 defect 2 on the five-dimensional overalgebra of conic subalgebra extended by the operators of space translations noncommuting with rotation. We prove that the extensions of conic flows are reduced either to function-invariant plane stationary solutions or to a double wave of isobaric motions or to the simple wave.

Keywords: gas dynamics, conic flows, partially invariant solutions.

Introduction

The equations of gas dynamics possesses 11-dimensional Lie algebra of operator. The optimal system of subalgebras was constructed in [1]. A three-dimensional subalgebra in the optimal system with the basis operators $X_7 = \partial_\theta$, $X_{10} = \partial_t$, $X_{11} = t\partial_t + x\partial_x + r\partial_r$ in the cylindrical coordinate system (x, r, θ) generates an invariant rank 1 submodel of conic flows [2]. A fifth-dimensional subalgebra has additional operators of space translations along Cartesian coordinates y, z,

$$X_2 = \partial_y = \cos\theta\partial_r - r^{-1}\sin\theta(\partial_\theta + W\partial_V - V\partial_W),$$

$$X_3 = \partial_z = \sin\theta\partial_r - r^{-1}\cos\theta(\partial_\theta + W\partial_V - V\partial_W).$$

Generalizations of conic flow w.r.t. fifth-dimensional overalgebra are partially partially invariant solutions of rank 1 defect 2. It is convenient to represent the cylindrical coordinates of the velocity $\vec{u}$ as U, $V = Q\cos\vartheta$, $W = Q\sin\vartheta$ ($Q \neq 0$, otherwise we deal with the one-dimensional motion). The invariants of the subalgebra are as follows, U, Q, the density ρ, the entropy S, the pressure is determined by the equation of state $p = f(\rho, S)$.

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The equations of gas dynamics in the mentioned variables cast into the form

$$U_t + UU_x + Q(U_r \cos\vartheta + r^{-1}U_\theta \sin\vartheta) + \rho^{-1}p_x = 0,$$

$$Q_t + UQ_x + Q(Q_r \cos\vartheta + r^{-1}Q_\theta \sin\vartheta) + \rho^{-1}(p_r \cos\vartheta + r^{-1}p_\theta \sin\vartheta) = 0,$$

$$\vartheta_t + U\vartheta_x + Q(\vartheta_r \cos\vartheta + r^{-1}(\vartheta_\theta + 1)\sin\vartheta) +$$

$$+\rho^{-1}Q^{-1}(-p_r \sin\vartheta + r^{-1}p_\theta \cos\vartheta) = 0,$$

$$\rho_t + U\rho_x + Q(\rho_r \cos\vartheta + r^{-1}\rho_\theta \sin\vartheta) +$$

$$+\rho\left[U_x + Q_r \cos\vartheta + r^{-1}Q_\theta \sin\vartheta + Q\big(-\vartheta_r \sin\vartheta + r^{-1}(\vartheta_\theta + 1)\cos\vartheta\big)\right] = 0,$$

$$S_t + US_x + Q(S_r \cos\vartheta + r^{-1}S_\theta \sin\vartheta) = 0.$$

The representation of partially invariant solution of rank 1 defect 2 is as follows; the functions U, Q, ρ, S, p depend on one non-constant parameter α, the functions α, ϑ are generic, i.e., depend on t, x, r, θ.

The substitution of the representation for the solution into the equations of gas dynamics yields an overdetermined system of equations (the main equations of submodel),

$$S_\alpha Y\alpha = 0, \quad U_\alpha Y\alpha + \rho^{-1}p_\alpha \alpha_x = 0,$$

$$Q_\alpha Y\alpha + \rho^{-1}p_\alpha(\alpha_r \cos\vartheta + r^{-1}\alpha_\theta \sin\vartheta) = 0,$$

$$\rho^{-1}\rho_\alpha Y\alpha + U_\alpha \alpha_x + Q_\alpha\left(\alpha_r \cos\vartheta + r^{-1}\alpha_\theta \sin\vartheta\right) +$$

$$+Q\left(-\vartheta_r \sin\vartheta + r^{-1}(\vartheta_\theta + 1)\cos\vartheta\right) = 0,$$

$$\vartheta_t + U\vartheta_x + Q\left(\vartheta_r \cos\vartheta + r^{-1}(\vartheta_\theta + 1)\sin\vartheta\right) +$$

$$+\rho^{-1}Q^{-1}p_\alpha\left(-\alpha_r \sin\vartheta + r^{-1}\alpha_\theta \cos\vartheta\right) = 0,$$

where $Y\alpha = \alpha_t + U\alpha_x + Q\left(\alpha_r \cos\vartheta + r^{-1}\alpha_\theta \sin\vartheta\right)$.

1. Non-isentropic motion

If $S_\alpha \neq 0$, the main equations of the submodel become

$$\begin{gathered} Y\alpha = 0, \quad p_\alpha \alpha_x = 0, \quad p_\alpha\left(\alpha_r \cos\vartheta + r^{-1}\alpha_\theta \sin\vartheta\right) = 0, \\ U_\alpha \alpha_x + Q_\alpha\left(\alpha_r \cos\vartheta + r^{-1}\alpha_\theta \sin\vartheta\right) + \\ +Q\left(-\vartheta_r \sin\vartheta + r^{-1}(\vartheta_\theta + 1)\cos\vartheta\right) = 0, \\ \vartheta_t + U\vartheta_x + Q\left(\vartheta_r \cos\vartheta + r^{-1}(\vartheta_\theta + 1)\sin\vartheta\right) + \\ +\rho^{-1}Q^{-1}p_\alpha\left(-\alpha_r \sin\vartheta + r^{-1}\alpha_\theta \cos\vartheta\right) = 0. \end{gathered} \tag{1.1}$$

1.1. Non-isobaric motion. If $p_\alpha \neq 0$, it follows from (1.1) that

$$\alpha_t = \alpha_x = 0, \quad \alpha_r \cos\vartheta + r^{-1}\alpha_\theta \sin\vartheta = 0 \quad \Rightarrow$$

$$\left(-\alpha_r \sin\vartheta + r^{-1}\alpha_\theta \cos\vartheta\right)\vartheta_\lambda = 0, \quad \lambda = t, x.$$

It implies $\vartheta_t = \vartheta_x = 0$, since the function α is non-constant. There happens a reduction to a plane stationary solution which an invariant solution on a subalgebra $\{\partial_t, \partial_x\}$. The system (1.1) reduces to three equations

$$\vartheta_\theta + 1 = \operatorname{tg}\vartheta\vartheta_\tau, \quad n_\tau = \operatorname{tg}\vartheta\vartheta_\tau, \quad n_\tau + \operatorname{tg}\vartheta n_\theta = 0,$$

where $n(\alpha) = \int p_\alpha \rho^{-1} Q^{-2}\, d\alpha$, $\tau = \ln r$. One of the equation is integrable

$$n = -\ln|\cos\vartheta| + k(\theta),$$

other two cast into the form

$$\cos^{-2}\vartheta\vartheta_\tau = -k' + \operatorname{tg}\vartheta, \quad \cos^{-2}\vartheta\vartheta_\theta = -1 - \operatorname{tg}\vartheta k'. \tag{1.2}$$

The compatibility conditions give the equation for the function $k(\theta)$, $k'' + k'^2 + 1 = 0$, whose solution $k = \ln|\cos\theta| + k_0$ is defined up to the translation w.r.t. θ admitted by system (1.1), k_0 is a constant.

Integration of system (1.2) gives a family of functionally-invariant solutions

$$\operatorname{tg}\vartheta + \operatorname{tg}\theta = \mu_0 r \cos^{-1}\theta, \quad n(\alpha) = k_0 + \ln\left|\frac{\cos\theta}{\cos\vartheta}\right|$$

depending on two constants μ_0, k_0 and three arbitrary functions $S(\alpha)$, $\rho(\alpha)$, $Q(\alpha)$.

1.2. Isobaric motion. Let $p_\alpha = 0$, i.e., $f(\rho, S) = p_0$ is constant. Then system (1.1) becomes

$$\begin{aligned} &\alpha_t + U\alpha_x + Q\left(\alpha_r\cos\vartheta + r^{-1}\alpha_\theta\sin\vartheta\right) = 0,\\ &\vartheta_t + U\vartheta_x + Q\left(\vartheta_r\cos\vartheta + r^{-1}(\vartheta_\theta + 1)\sin\vartheta\right) = 0,\\ &U\alpha_x + Q_\alpha\left(\alpha_r\cos\vartheta + r^{-1}\alpha_\theta\sin\vartheta\right) +\\ &\qquad + Q\left(-\vartheta_r\sin\vartheta + r^{-1}(\vartheta_\theta + 1)\cos\vartheta\right) = 0. \end{aligned} \tag{1.3}$$

The latter equations is equivalent to

$$\operatorname{div}\vec{u} = 0. \tag{1.4}$$

It is convenient to deal with Lagrange variables

$$\begin{gathered} \frac{dx}{dt} = U(\alpha), \quad \frac{dr}{dt} = Q(\alpha)\cos\vartheta, \quad r\frac{d\theta}{dt} = Q(\alpha)\sin\vartheta,\\ x\big|_{t=0} = x_0, \quad r\big|_{t=0} = r_0, \quad \theta\big|_{t=0} = \theta_0. \end{gathered} \tag{1.5}$$

Each solution to system (1.3) can be written by Cauchy problem (1.5) as $\alpha(t,x,r,\theta) = \beta(t,x_0,r_0,\theta_0)$, $\vartheta(t,x,r,\theta) = \sigma(t,x_0,r_0,\theta_0)$.

By (1.3) we have the identities

$$\beta_t = 0, \quad \sigma_t + \theta_t = 0 \quad \Rightarrow \quad \beta = \beta(x_0,r_0,\theta_0), \quad \sigma + \theta = \gamma(x_0,r_0,\theta_0).$$

Due to these identities the solution to problem (1.5) reads as

$$x = U(\beta)t + x_0, \quad r\cos(\gamma - \theta) = Q(\beta)t\cos(\gamma - \theta_0), \quad r\sin(\gamma - \theta) = r_0\sin(\gamma - \theta_0).$$

In Cartesian coordinates two latter identities are written as

$$y = r\cos\theta = Q(\beta)t\cos\gamma + y_0, \quad z = r\sin\theta = Q(\beta)t\sin\gamma + z_0, \tag{1.6}$$

where $y_0 = r_0\cos\theta_0$, $z_0 = r_0\sin\theta_0$.

Thus, the world line are straight ones. The velocities in Cartesian coordinates are represented by the formulae

$$u = U(\beta) = u_0,$$

$$v = V\cos\theta - W\sin\theta = Q(\beta)\cos\gamma = v_0, \tag{1.7}$$

$$w = V\sin\theta + W\cos\theta = Q(\beta)\sin\gamma = w_0.$$

By Euler formula $J_t = J\mathrm{div}\,\vec{u}$ and identity (1.4), the Jacobian of the passage from Lagrange coordinates to Euler ones equals one $J = 1$ or by (1.6), (1.7)

$$1 = \left| I + t\frac{\partial \vec{u}_0}{\partial \vec{x}_0} \right|,$$

where I is the unit matrix, $\partial\vec{u}_0/\partial\vec{x}_0$ is the matrix of partial derivatives, the variable t is free.

It implies that all the invariants of the matrix $\partial\vec{u}_0/\partial\vec{x}_0$ vanish,

$$u_{0x_0} + v_{0y_0} + w_{0z_0} = 0,$$

$$\begin{vmatrix} u_{0x_0} & u_{0y_0} \\ v_{0x_0} & v_{0y_0} \end{vmatrix} + \begin{vmatrix} u_{0x_0} & u_{0z_0} \\ w_{0x_0} & w_{0z_0} \end{vmatrix} + \begin{vmatrix} v_{0y_0} & v_{0z_0} \\ w_{0y_0} & w_{0z_0} \end{vmatrix} = 0, \tag{1.8}$$

$$\det \tfrac{\partial\vec{u}_0}{\partial\vec{x}_0} = 0.$$

The general solution to this system was obtained in [3]. In our case we obtain particular solutions, namely, the solutions of a double wave type,

$$\big(\vec{a}\cdot\nabla\beta\big)\big(\vec{b}\cdot\nabla\gamma\big) = \big(\vec{b}\cdot\nabla\beta\big)\big(\vec{a}\cdot\nabla\gamma\big), \quad \vec{a}\cdot\nabla\beta = \vec{b}\cdot\nabla\gamma,$$

where $\vec{a} = (U', Q'\cos\gamma, Q'\sin\gamma)$, $\vec{b} = (0, -Q\sin\gamma, Q\cos\gamma)$, $\vec{a}\cdot\vec{b} = 0$. As it follows from [3], the level lines of the double wave are second order planar curves.

2. Isentropic motion

Let $S = S_0$ be constant. Then the main equations can be written as

$$\begin{gathered} \vartheta_t + U\vartheta_x + Q\left(\vartheta_r\cos\vartheta + r^{-1}(\vartheta_\theta + 1)\sin\vartheta\right) + \\ +\rho^{-1}Q^{-1}p'\left(-\alpha_r\sin\vartheta + r^{-1}\alpha_\theta\cos\vartheta\right) = 0, \end{gathered} \tag{2.1}$$

$$-\vartheta_r\sin\vartheta + r^{-1}(\vartheta_\theta + 1)\cos\vartheta = c(\alpha)\alpha_x, \tag{2.2}$$

$$\alpha_r\cos\vartheta + r^{-1}\alpha_\theta\sin\vartheta = Q'U'^{-1}\alpha_x, \tag{2.3}$$

$$\alpha_t + b(\alpha)\alpha_x = 0, \tag{2.4}$$

where $b(\alpha) = U + \rho^{-1}U'^{-1}p' + QQ'U'^{-1}$, $c(\alpha) = \rho^{-2}Q^{-1}U'^{-1}\left(p'\rho' - \rho^2\big(U'^2 + Q'^2\big)\right)$.

The general solution to equation (2.4) can be written implicitly as

$$x - b(\alpha)t = g(\alpha, r, \theta), \tag{2.5}$$

where g is an arbitrary function. We introduce new independent variables α, r, θ, x. The derivatives w.r.t. old variables are expressed in terms of the derivatives of the functions g and $\bar{\vartheta}(\alpha, t, r, \theta) = \vartheta(t, x, r, \theta)$ by the formulae

$$\alpha_t = -\frac{b}{g_\alpha + tb'}, \quad \alpha_x = \frac{1}{g_\alpha + tb'}, \quad \alpha_r = -\frac{g_r}{g_\alpha + tb'}, \quad \alpha_\theta = -\frac{g_\theta}{g_\alpha + tb'};$$

$$\vartheta_x = \bar{\vartheta}_\alpha\alpha_x, \quad \vartheta_\lambda = \bar{\vartheta}_\lambda + \bar{\vartheta}_\alpha\alpha_\lambda, \quad \lambda = t, r, \theta,$$

where $t = (x - g)b^{-1}$.

Equation (2.3) becomes

$$g_r \cos\bar\vartheta + r^{-1} g_\theta \sin\bar\vartheta = -Q'U'^{-1}. \tag{2.6}$$

It yields $\bar\vartheta_t = 0$.

Equation (2.2) in new variables

$$\left[-\bar\vartheta_r(g_\alpha + tb' + \bar v_\alpha g_r)\right]\sin\bar\vartheta + r^{-1}\cos\bar\vartheta\left[(\bar\vartheta_\theta + 1)(g_\alpha + tb') - g_\theta\bar\vartheta_\alpha\right] = c(\alpha)$$

involves a free variable t. Equating the coefficient at the free variable to zero implies the identities

$$-\bar\vartheta_r \sin\bar\vartheta + r^{-1}(\bar\vartheta_\theta + 1)\cos\bar\vartheta = 0, \tag{2.7}$$

$$g_r \sin\bar\vartheta - r^{-1} g_\theta \cos\bar\vartheta = c(\alpha)\bar\vartheta_\alpha^{-1}. \tag{2.8}$$

Similar procedure for equation (2.1) yield the identities

$$\bar\vartheta_r \cos\bar\vartheta + r^{-1}(\bar\vartheta_\theta + 1)\sin\bar\vartheta = 0, \tag{2.9}$$

$$\bar\vartheta_\alpha = k'(\alpha) = \left(\frac{cp'U'}{\rho Q(bU' - UU' - QQ')}\right)^{1/2}. \tag{2.10}$$

It follows from identities (2.7), (2.9), (2.10) that $\bar\vartheta_r = 0$, $\bar\vartheta_\theta = -1 \Rightarrow \bar\vartheta = k(\alpha) - \theta$. In view of the obtained identity, equations (2.8), (2.6) can be integrated

$$g = h(\alpha) + r\left(ck'^{-1}\sin(k-\theta) - Q'U'^{-1}\cos(k-\theta)\right),$$

and general solution (2.6) casts into the form

$$x - b(\alpha)t = h(\alpha) + y\left(ck'^{-1}\sin k - Q'U'^{-1}\cos k\right) + z\left(-ck'^{-1}\cos k - Q'U'^{-1}\sin k\right).$$

It follows that level surface ($\alpha = \text{const}$) is a plane as for the simple wave [2].

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