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ON ANALYTIC PROPERTIES OF WEYL FUNCTION OF STURM–LIOUVILLE OPERATOR WITH A DECAYING COMPLEX POTENTIAL

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Abstract. We study the spectral properties of the operator L_β associated with the quadratic form $\mathcal{L}_\beta = \int\limits_0^\infty (|y'|^2 - \beta x^{-\gamma}|y|^2)dx$ with the domain $Q_0 = \{y \in W_2^1(0,+\infty) : y(0) = 0\}$, $0 < \gamma < 2$, $\beta \in \mathbf{C}$, as well as of the perturbed operator $M_\beta = L_\beta + W$. Under the assumption $(1 + x^{\gamma/2})W \in L^1(0,+\infty)$ we prove the existence of the finite quantum defect of the discrete spectrum that was established earlier by L.A. Sakhnovich for $\beta > 0$, $\gamma = 1$ and for real W satisfying a stricter decay condition at infinity. The main result of the paper is the proof of necessity (with some reservations) of the sufficient conditions for $W(x)$ obtained earlier by Kh.Kh. Murtazin under which the Weyl function of the operator M_β possesses an analytic continuation on some angle from non-physical sheet.

Keywords: spectral instability, localization of spectrum, quantum defect, Weyl function, Darboux transformation.

1. Introduction

We shall call an operator L acting in some Hilbert space as *close to a self-adjoint one* if $L = L_0 + V$, where L_0 is self-adjoint, V is relatively compact w.r.t. L_0, i.e., $D(V) \supset D(L_0)$ and the operator $V(L_0 + i)^{-1}$ is compact. If the operator L_0 is lower-semibounded and for some $r > 0$ the operator $(L_0 + r)^{-1/2}V(L_0 + r)^{-1/2}$ is compact, then the operator $L = L_0 + V$, where the sum is understood in the quadratic forms sense, will be called *close to a self-adjoint one in the quadratic forms sense.* The operators close to self-adjoint ones form a natural class of non-self-adjoint operators to that the methods of abstract perturbation theory are applicable and it allows one to obtain rather general results on the asymptotic behavior of spectrum and on the properties of root vectors systems. For example, according to M.V. Keldysh theorem [1], if L_0 is a self-adjoint operator with discrete spectrum whose spectral counting function $N(r, L_0)$ (the number of eigenvalues counting multiplicities in the interval $(-r, r)$) satisfies certain condition (K)[1], then each operator L close to L_0 possesses the properties

a) the root vectors system of L is complete in H;

b) the spectrum of the operator L has the same asymptotics as that of the operator L_0, i.e., for each $\varepsilon > 0$ the spectrum of the operator L outside the angles $\{|\arg \lambda| < \varepsilon\}$ and $\{|\arg \lambda - \pi| < \varepsilon\}$ is finite and for the function $\widetilde{N}(r, L)$, which is the number of the eigenvalues of the operator L

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[1]This condition is the existence of a function $\varphi(r)$ such that $N(r, L_0) \sim \varphi(r)$ as $r \to +\infty$ and $\varphi(r)$ satisfies Keldysh's tauberian condition [1, 2] which were generalized by B.I. Korenblum [3].

counting multiplicities in the disk $|\lambda| < r$, the relation

$$\widetilde{N}(r, L) \sim N(r, L_0), \quad r \to +\infty. \tag{1}$$

holds true.

Under stricter conditions for the function $N(r, L_0)$ and the order of smallness of V one can obtain the statements on the basis property (in some sense) of the root vectors systems, see [4] — [6], and specify the asymptotics for the eigenvalues up to terms allowing calculate the regularized traces (see [7] and the references therein).

Thus, each operator L close to a self-adjoint one L_0 with the spectral counting function $N(r, L_0)$ satisfying condition (K) possesses the property of *spectral stability* in the following sense: each perturbation of L like $M = L + W$, where W is L-compact, possesses properties a) and b), where instead of (1) we have

$$\widetilde{N}(r, M) \sim \widetilde{N}(r, L), \quad r \to +\infty. \tag{2}$$

It is known (see, for instance, [8] and the bibliography therein) that the operators close to self-adjoint ones do not possess such stability. Suppose now that L is close to a self-adjoint operator L_0 with a non-discrete spectrum, i.e., $\sigma(L_0) = \sigma_{\rm disc}(L_0) \cup \sigma_{\rm ess}(L_0)$, where $\sigma_{\rm disc}(L_0)$ and $\sigma_{\rm ess}(L_0)(\neq \varnothing)$ are the discrete and essential parts of the spectrum of L_0, respectively. Since a relatively compact perturbation does not change the spectrum (see [9, c. 306]), then $\sigma_{\rm ess}(L) = \sigma_{\rm ess}(L_0) \neq \varnothing$. Let $\sigma_{\rm disc}(L) = \{\lambda_k\}_{k=1}^{\infty}$, where λ_k are taken counting algebraic multiplicities and let there exists a finite or infinite limit $l = \lim\limits_{k\to\infty} \lambda_k$.

We pose a question: what is the class of perturbations W preserving the asymptotics of the discrete spectrum of L in the following sense: the eigenvalues μ_k of the operator $M = L + W$ can be ordered so that

$$\mu_k \sim \lambda_k, \quad k \to \infty? \tag{3}$$

According to Weyl-von Neumann theorem [9, Ch. X, Sec. 2.1], each self-adjoint operator L_0 in a separable Hilbert space H can be converted to a self-adjoint operator $L_0 + V$ with pure point spectrum by adding a Hilbert-Schmidt operator V of arbitrarily small norm. This is why it is natural to expect that the classes of perturbations preserving the asymptotics of the discrete spectrum of the operators L_1 and L_2 can differ substantially even in the case when L_1 and L_2 are close to the same self-adjoint operator. Moreover, the perturbations can be uniquely determined by the spectrum (or its part) only in exceptional cases (see [10] and Theorem 5 below). Hence, a more correct problem seems to be the following

Problem 1. *There given an operator L whose spectrum possesses the properties $P = P_{\rm disc} \wedge P_{\rm ess}$, where $P_{\rm disc}$ and $P_{\rm ess}$ are certain properties of discrete and essential parts of the spectrum of L, respectively. It is required to find the conditions (necessary and sufficient, if possible) for perturbations W, under which the spectrum of the operator $M = L + W$ possesses the same properties.*

Of course, in such abstract form the problem is unlikely to be solvable; it is unclear how to choose the properties $P_{\rm ess}$ and it is even more unclear how to extract the conditions for W from the properties P. Nevertheless, for some operator classes (for instance, for differential operators) one succeeds to formulate the conditions (quite natural) for the spectrum and to give the exact description of the class of the perturbations preserving these properties [11, 12].

Let

$$q_\beta(x) = \frac{\beta}{x^\gamma}.$$

Consider a family of quadratic forms $\mathcal{L}_\beta[y] = \int\limits_0^\infty (|y'|^2 - q_\beta(x)|y|^2)dx$ with the domain $Q_0 = \{y \in W_2^1(0, \infty) : y(0) = 0\}$, where $0 < \gamma < 2$ is assumed to be fixed; we shall study just the dependence on the parameter $\beta \in \mathbf{C}$ (see Sec. 2).

Lemma 1. *$\mathcal{L}_\beta$ is a holomorphic family of type (A) on $\mathbb{C}$, i.e.,* [9, Ch. VII, Sec. 4.2],
1) for each $\beta \in \mathbb{C}$ the form $\mathcal{L}_\beta$ is sectorial and closed;
2) for each $y \in Q_0$ the function $f(\beta) = \mathcal{L}_\beta[y]$ is entire.

By representation theory [9, Ch. VI, Sec. 2.1] Item 1) of Lemma 1 implies that for each $\beta \in \mathbb{C}$ there exists an m-sectorial operator L_β associated with the form $\mathcal{L}_\beta$. The family L_β is called analytic family of type (B) (see [9, Ch. VII, Sec. 4.2]).

Lemma 2. *The operator L_β is determined as follows,*

$$\begin{aligned} L_\beta y &= -y'' - q_\beta y, \\ D(L_\beta) &= \{y \in L^2(0,+\infty) : y' \in AC[0,b] \ \forall\ b > 0, -y'' - q_\beta y \in L^2(0,+\infty),\ y(0) = 0\}. \end{aligned} \tag{4}$$

For each $\beta \in \mathbb{C}$ the operator L_β is close (in the quadratic form sense) to the self-adjoint operator $L_0 := L_\beta|_{\beta=0}$ (see Sec. 2, Lemma 3), and thus [13, c. 133] $\sigma_{\text{ess}}(L_\beta) = \sigma_{\text{ess}}(L_0) = [0,+\infty)\ \forall\ \beta \in \mathbb{C}$. However, for the discrete spectrum the picture is completely different (Theorem 1);

as $0 \le |\arg\beta| < \frac{2-\gamma}{2}\pi$, $\sigma_{\text{disc}}(L_\beta)$ consists of infinitely many simple (of algebraic multiplicity 1) eigenvalues being [1] $\lambda_k(\beta) = -\beta^{2/(2-\gamma)} r_k,\ k = 1,2,\dots$, where $r_k \searrow 0,\ k \to +\infty$,

as $\frac{2-\gamma}{2}\pi \le |\arg\beta| \le \pi$, the discrete spectrum of the operator $L(\beta)$ is empty.
This is why the property P in Problem 1 is likely to depend somehow on β.

In the present paper we formulate a certain property P_β (in terms of Weyl function for the operator L_β) and obtain necessary and sufficient condition for the function $W(x)$ under which the operator M_β obtained from L_β by the replacement of the potential $q_\beta(x)$ by $q_\beta(x) + W(x)$ also possesses the property P_β. For a complex parameter β this condition happens to differs substantially from the corresponding condition in the case of real β.

The main result of the paper is Theorem 6 (Sec. 5). Before we formulate it, in Secs. 3 and 4 we establish some properties of analytic (Theorems 2, 3) and compactly supported (Theorems 4, 5) perturbations of the operator L_β hinting in some sense the main result.

Our choice of L_β as the unperturbed operator is due to the following reason: the Sturm-Liouville operators with a complex decaying potential are quite well studied (see [14] —[19] and the references therein), at the same time, the question on necessity of known sufficient condition for the potentials under which one can obtain the asymptotics of the discrete spectrum (see Remark 2) is still open. In what follows we shall show that this question is a part of Problem 1.

2. Properties of operators L_β

Proof of Lemma 1. Let $\varepsilon > 0$. We have $\dfrac{1}{x^\gamma} = q_1(x) + q_2(x)$, where

$$q_1(x) = \begin{cases} \frac{1}{x^\gamma}, & 0 < x \le \delta_\varepsilon, \\ 0, & x > \delta_\varepsilon, \end{cases} \qquad q_2(x) = \frac{1}{x^\gamma} - q_1,$$

and the number δ_ε is chosen so that $\dfrac{1}{x^\gamma} < \dfrac{\varepsilon}{4x^2}$ for $x \in (0, \delta_\varepsilon]$. Then by a known inequality (see, for instance, [20, Ch. II, Sec. 26]) for all $y \in Q_0$

$$\left(\frac{1}{x^\gamma} y, y\right) < \varepsilon \|y'\|^2 + C_\varepsilon \|y\|^2, \tag{5}$$

with some constant $C_\varepsilon > 0$.

[1]Hereinafter, if else is not said, the branch of the function z^α ($\alpha \in \mathbb{R}$) is fixed so that $z^\alpha > 0$ as $z > 0$.

Hence, the quadratic form $\left(\dfrac{1}{x^\gamma}y, y\right)$ is relatively bounded w.r.t. a closed positive form $\mathcal{L}_0[y] = \|y'\|^2$, $D(\mathcal{L}_0) = Q_0$ and its relative bound is zero. It yields (see [9, Ch. VI, Sec. 3.2] Item 1).

Item 2) is obvious. The proof is complete. □

Proof of Lemma 2. We denote by $\mathcal{L}_\beta[y, v]$ the sesquilinear form determined by the quadratic form $\mathcal{L}_\beta[y]$ by polarization identity (see [9, p. 387]):

$$\mathcal{L}_\beta[y, v] = \frac{1}{4}\sum_{k=0}^{3} i^{-k}\mathcal{L}_\beta[y + i^k v, y + i^k v].$$

It is clear that

$$\mathcal{L}_\beta[y, v] = \int_0^\infty (y'\overline{v}' - q_\beta(x)y\overline{v})\,dx, \quad y, v \in Q_0.$$

Further, we indicate by D_β the right hand side of (4) and let us prove that $D(L_\beta) \subset D_\beta$.

Let $y \in D(L_\beta)$ and $L_\beta y = f$. Then by the representation theorem

$$(f, v) = \mathcal{L}_\beta[y, v] := \int_0^\infty \left(y'\overline{v'} - q_\beta(x)y\overline{v}\right)dx, v \in Q_0. \tag{6}$$

Let $(a, b) \subset (0, +\infty)$. Then identity (6) is valid for all v belonging to the set $Q'_{ab} = \{y \in Q_0 : y(x) \equiv 0 \text{ as } x \notin (a, b)\}$.

Let h be a primitive for the function $-f - q_\beta(x)y$ on the interval (a, b),

$$h' = -f - q_\beta(x)y \quad \text{a.e. on } (a, b).$$

Then for all $v \in Q'_{ab}$

$$\int_0^\infty (f + q_\beta(x)y)\,\overline{v}dx = -\int_a^b h'\overline{v}dx = \int_a^b h\overline{v}'dx.$$

On the other hand, it follows from (6) that

$$\int_a^b (f + q_\beta(x)y)\,\overline{v}dx = \int_a^b y'\overline{v}'dx.$$

Therefore,

$$\int_a^b (h - y')\overline{v}'dx = 0 \quad \text{for all} \quad v \in Q'_{ab}. \tag{7}$$

Denote by φ_{ab} the restriction of $h - y'$ on (a, b). Then (7) means

$$\varphi_{ab} \perp \mathrm{Ran}T_{ab}, \tag{8}$$

where T_{ab} is the operator $\frac{d}{dx}$ with the domain $D(T_{ab}) = \{v \in W_2^1(a, b) : v(a) = v(b) = 0\}$.

In its turn, (8) is equivalent to $\varphi_{ab} \in \mathrm{Ker}(T^*_{ab})$. We have $T^*_{ab} = -\frac{d}{dx}$, $D(T^*_{ab}) = W_2^1(a, b)$, so, $\varphi_{ab} = c = \text{const}$ a.e. on (a, b) that by the arbitrariness of a, b yields $y' = h - c$ a.e. on $(0, +\infty)$. Hence, $y' \in AC[0, b]\ \forall\ b > 0$ and $-y'' = f + q_\beta(x)y$, i.e., $L_\beta y = -y'' + q_\beta(x)y$.

Let us prove now that $D_\beta \subset D(L_\beta)$. By the definition of the operator associated with the quadratic form (see [9, Ch. VI, Sec. 2.1]), if $y \in Q_0$, $w \in L^2(0, +\infty)$, and the identity

$$\mathcal{L}_\beta[y, v] = (w, v) \tag{9}$$

holds true for all v in the core[1] of the form $\mathcal{L}_\beta$, then $y \in D(L_\beta)$ and $L_\beta y = w$.

Let us show that $C_0^\infty(0,+\infty)$ is the core for $\mathcal{L}_\beta$. The closure of $C_0^\infty(0,+\infty)$ w.r.t. the norm of $W_2^1(0,+\infty)$ is Q_0, and this is why $C_0^\infty(0,+\infty)$ is the core for the form $\mathcal{L}_0[y,v] = \int\limits_0^\infty y'\overline{v}'dx$ with $D(\mathcal{L}_0) = Q_0$. By inequality (5) it implies that $C_0^\infty(0,+\infty)$ is the core for $\mathcal{L}_\beta[y,v]$ for all $\beta \in \mathbb{C}$.

Let $y \in D_\beta$ and $f = -y'' - q_\beta(x)y$. According to (9),

$$\mathcal{L}_\beta[y,v] = \int\limits_0^\infty \left(y'\overline{v}' - q_\beta(x)y\right)\overline{v}dx = (f,v), \quad \text{for any } v \in C_0^\infty(0,+\infty).$$

On the other hand, integrating by parts, we get

$$\mathcal{L}_\beta[y,v] = \int\limits_0^\infty \left(-y'' - q_\beta(x)y\right)\overline{v}dx = (f,v),$$

hence, $(f - w, v) = 0$ for all $v \in C_0^\infty(0,+\infty)$. But $C_0^\infty(0,+\infty)$ is dense in $L^2(0,+\infty)$ and thus $w = f$ a.e. on (a,b). It yields that $y \in D(L_\beta)$ and $L_\beta y = -y'' - q_\beta(x)y$. The proof is complete. □

Let $L_0 = L_\beta|_{\beta=0}$, i.e., $L_0 y = -y''$, $y \in D(L_0) = \{y \in W_2^2(0,+\infty) : y(0) = 0\}$. The following lemma holds true.

Lemma 3. *Let q be the operator of multiplication by the function $x^{-\gamma}$. Then for each $r > 0$ the operator $K = (L_0 + r)^{-\frac{1}{2}} q (L_0 + r)^{-\frac{1}{2}}$ is compact.*

Proof. Let $\delta > 0$, χ_1, χ_2, χ_3 be characteristic functions of the segments $(0,\delta)$, $(\delta, \frac{1}{\delta})$, and $(\frac{1}{\delta}, +\infty)$, respectively. Then $K = K_1 + K_2 + K_3$, where $K_i = (L_0 + r)^{-\frac{1}{2}} q\chi_i (L_0 + r)^{-\frac{1}{2}}$, $i = \overline{1,3}$. Since the kernel of the resolvent $(L_0 + 1)^{-1}$ reads as

$$G(x,t) = \begin{cases} \mathrm{sh}x e^{-t}, & 0 \le x < t, \\ e^{-x}\mathrm{sh}t, & 0 \le t \le x, \end{cases}$$

$q\chi_2(L_0 + 1)^{-1}$ is a Hilbert-Schmidt operator. It is known [13, Sec. Problems] that if H_0 is a positive self-adjoint operator, V is symmetric operator with $D(V) \supset D(H_0)$, then the compactness of the operator $V(H_0 + 1)^{-1}$ implies that of $(H_0 + 1)^{-\frac{1}{2}} V (H_0 + 1)^{-\frac{1}{2}}$. This is why the operator K_2 is compact.

Further, since $\|K_3\| < \sup \mid q\chi_3 \mid = \delta^\gamma \to 0, \delta \to 0$, to prove the lemma, it is sufficient to make sure that $\|K_1\| \to 0,\ \delta \to 0$.

If $\delta < 1$, for each $u \in L^2(0,+\infty)$ we have

$$(K_1 u, u) < 4\delta^{2-\gamma} \left(\frac{1}{4} x^{-2} (L_0 + 1)^{-\frac{1}{2}} u, (L_0 + 1)^{-\frac{1}{2}} u\right).$$

By the uncertainty principle [21, Ch. X, Sec. 2]

$$\frac{1}{4}\left(x^{-2}y, y\right) < \int_0^\infty |y'|^2 dx = \|L_0^{\frac{1}{2}} y\|^2, \quad \forall\ y \in Q_0,$$

and thus $(K_1 u, u) < 4\delta^{2-\gamma} \|L_0^{\frac{1}{2}} (L_0 + 1)^{-\frac{1}{2}} u\|^2 < 4\delta^{2-\gamma}\|u\|^2$. Since for each bounded self-adjoint operator A on the whole Hilbert space H $\|A\| = \sup|(Au,u)|$ [22, Ch. VI, Sec. Problems], it yields $\|K_1\| < 4\delta^{2-\gamma} \to 0$, $\delta \to 0$. The proof is complete. □

[1]By the definition (see [9, Ch. VI, Sec. 1.4]), a linear subspace Q' of the set Q_0 is called a core of the form $\mathcal{L}_\beta$ if the closure of the restriction of $\mathcal{L}_\beta$ on Q' coincides with $\mathcal{L}_\beta$.

Theorem 1. *The following statements hold true,*
1) as $0 \le |\arg\beta| < \frac{2-\gamma}{2}\pi$, $\sigma_{\rm disc}(L_\beta)$ *consists of infinitely many simple (of geometric multiplicity* 1*) eigenvalues lying on the ray* $\arg(-\lambda) = \frac{2\arg\beta}{2-\gamma}$, *namely,*

$$\sigma_{\rm disc}(L_\beta) = \bigcup_{k=1}^{\infty} \lambda_k(\beta)$$

and

$$\lambda_k(\beta) = -\beta^{2/(2-\gamma)} r_k, \tag{10}$$

where $-r_k$ *are taken in the ascending order eigenvalues of the self-adjoint operator* L_1 *(i.e.,* $L_\beta|_{\beta=1}$*) having the asymptotics*

$$r_k \sim C\cdot(k-1/4)^{-2\gamma/(2-\gamma)}, \quad k\to+\infty, \quad C = \left[\frac{\Gamma(-\frac{1}{2}+\frac{1}{\gamma})}{2\sqrt{\pi}\Gamma(\frac{1}{\gamma})}\right]^{\frac{2\gamma}{2-\gamma}}; \tag{11}$$

2) as $\frac{2-\gamma}{2}\pi \le |\arg\beta| \le \pi$, *the discrete spectrum of the operator* L_β *is empty;*
3) for all $\beta\in\mathbb{C}$ *the operator* L_β *on the semi-axis* $[0,+\infty)$ *has neither eigenvalues no spectral singularities* [23, c. 456]*:* $v_\beta(0,\lambda)\ne 0\ \forall\ \lambda\ge 0$, *where* $v_\beta(x,\lambda)$ *is the solution to equation* (19) *satisfying estimate* (20) .

Proof. The identity $\sigma_{\rm disc}(L_{\overline{\beta}}) = \overline{\sigma_{\rm disc}(L_\beta)}$ yields that it is sufficient to prove statements 1)–3) for $0\le\arg\beta\le\pi$.

Let us prove 1). Consider a one-parametric family of unitary dilations in $L^2(0,+\infty)$, $[U_\omega\varphi](x) = e^{\frac{\omega}{2}}\varphi(e^\omega x)$, where $\omega\in\mathbb{R}$. We have

$$U_\omega L_1 U_\omega^{-1} = e^{-2\omega} L_{e^{(2-\gamma)\omega}}, \omega\in\mathbb{C}. \tag{12}$$

By Lemma 1 it follows that the family of the operators $T(\omega) = U_\omega L_1 U_\omega^{-1}$ is an analytic one of type (B) on the whole complex plane $\mathbb{C}$. Since $-r_k$ is a simple eigenvalue of the operator $T(0)$, by Theorem XII.13 in [13], for small $\omega\in\mathbb{C}$ in a vicinity of $-r_k$ there exists the unique eigenvalue $\lambda_k(\omega)$ of the operator $T(\omega)$ being analytic in a vicinity of $\omega=0$. On the other hand, for real ω the operator $T(\omega)$ is unitarily equivalent to the operator L_1, so, $\lambda_k(\omega)\equiv -r_k$ for all small real ω. The analyticity of $\lambda_k(\omega)$ yields that $\lambda_k(\omega)\equiv -r_k$ for all sufficiently small $\omega\in\mathbb{C}$. This statement obviously remains true, if one replaces 0 by any $\omega_0\in\mathbb{C}$ such that $-r_k\in\sigma_{\rm disc}(T(\omega_0))$.

Let $0<|\arg\beta|<\frac{2-\gamma}{2}\pi$ (as $\arg\beta=0$, identity (10) becomes an identical equation). We let $\omega_\beta = \frac{1}{2-\gamma}(\ln|\beta| + i(\arg\beta))$. Let us show that $e^{2\omega_\beta}(-r_k)$ is an eigenvalue of the operator L_β. It will then imply (10).

Since $e^{(2-\gamma)\omega_\beta}=\beta$, by (12) it is sufficient to show that $-r_k\in\sigma_{\rm disc}(T(\omega_\beta))$. Let $I_\beta=[0,\omega_\beta]$. Denote by J_β the set of all $\omega\in I_\beta$, for which $-r_k\in\sigma_{\rm disc}(T(\omega))$. Since $0\in J_\beta$, then $J_\beta\ne\varnothing$. It follows from the above that J_β is open in I_β. On the other hand, since L_β is an analytic family of type (B), due to (12) the same is true for the family $T(\omega)$, $\omega\in\mathbb{C}$, and this is why if $\omega_n\to\omega$ and $-r_k\in\sigma_{\rm disc}(T(\omega_n))$ for all n, then $-r_k\in\sigma(T(\omega))$. But by (12) and Lemma 3

$$\sigma_{\rm ess}(T(\omega)) = e^{-2(\operatorname{Im}\omega)i}[0,+\infty) \tag{13}$$

and $\operatorname{Im}\omega_\beta = -\frac{\pi-arg\beta}{2-\gamma} > -\frac{\pi}{2}$, thus, for all $\omega\in I_\beta$ the point $-r_k$ lies outside $\sigma_{\rm ess}(T(\omega_n))$. Hence, if $\omega\in I_\beta$ and $\omega_n\to\omega, -r_k\in\sigma_{\rm disc}(T(\omega_n))$, then $-r_k\in\sigma_{\rm disc}(T(\omega))$. It means that the set J_β is closed in I_β. Thus, J_β is a closed and open non-empty subset of I_β. Therefore, $J_\beta=I_\beta$. Identity (10) is proven.

Let us prove (11). Let $-r$, $r>0$, be an eigenvalue of the operator L_1. Then for the associated eigenfunction f we have

$$-f''(x)-\frac{1}{x^\gamma}f(x)=-r\cdot f(x),\ x>0,$$
$$f(0)=0.$$

After the change

$$\xi=\left(\sqrt{r}x\right)^{(2-\gamma)/2},\ f=x^{\gamma/4}g(\xi,\mu),\ \mu=\left(\frac{2}{2-\gamma}\right)^2 r^{-(2-\gamma)/2}$$

we arrive at the problem

$$-\frac{d^2g}{d\xi^2}+p(\xi)g \;=\; \mu\cdot g, \tag{14}$$
$$g(0) \;=\; 0, \tag{15}$$

where $p(\xi)=4\nu^2\xi^\alpha-\left(\frac14-\nu^2\right)\xi^{-2},\ \nu=1/(2-\gamma),\ \alpha=2\gamma/(2-\gamma)$.

Hence, $r\in\sigma_{\rm disc}(L_1)$ if and only if

$$\mu=\left(\frac{2}{2-\gamma}\right)^2 r^{-(2-\gamma)/2} \tag{16}$$

is the eigenvalue of problem (14)–(15). The asymptotics for the spectrum of problem (14)–(15) is well-known [24],

$$\mu_k\sim(C_0\pi k)^{2\alpha/(2+\alpha)},\quad k\to+\infty,\quad C_0=\frac{\alpha\Gamma\left(\frac32+\frac1\alpha\right)}{\Gamma\left(\frac32\right)\Gamma\left(\frac1\alpha\right)}.$$

Together with (16) it implies (11).

2) Let us prove first that $\sigma_{\rm disc}(L_\beta)=\varnothing$ for $\arg\beta=\frac{2-\gamma}{2}\pi$. Suppose the opposite; there exists $\lambda_0\in\sigma_{\rm disc}(L_\beta)$ for some $\beta=b\cdot e^{\frac{2-\gamma}{2}\pi i}, b>0$. Let f be a normalized eigenfunction associated with λ_0, in view of the relation $\lambda_0=\|f'\|^2-\beta(x^{-\gamma}f,f)$ we obtain that

$$-\frac{\gamma\pi}{2}<\arg\lambda_0<0. \tag{17}$$

We let $T(\omega)=U(\omega)L_\beta U^{-1}(\omega)$. We have $T(\omega)=e^{-2\omega}L_{\beta e^{(2-\gamma)\omega}}$, so that

$$T\left(-\frac{i\pi}{2}\right)=-L_\beta. \tag{18}$$

Let $I=[0,-\frac{i\pi}{2}]$. It follows from relation (13) that for all $\omega\in I\quad \lambda_0\notin\sigma_{\rm ess}(T(\omega))$, and arguing as in the proof of Item 1), we obtain that $\lambda_0\in\sigma_{\rm disc}(T(-\frac{i\pi}{2}))$. By (18) it yields $\lambda_0>0$ that contradicts (17).

Let us prove now that $\sigma_{\rm disc}(L_\beta)=\varnothing$ as $\frac{2-\gamma}{2}\pi<\arg\beta\le\pi$. We again arguing by contradiction; let for some $\beta_0=b_0\cdot e^{i\theta_0}$, $b_0>0$, $\frac{2-\gamma}{2}\pi<\theta_0<\pi$, the operator L_{β_0} has an eigenvalue λ_0. Then $-\pi+\theta_0<\arg\lambda_0<0$ and arguing as in the case $\arg\beta=\frac{2-\gamma}{2}\pi$, we obtain $\lambda_0\in\sigma_{\rm disc}(T(\omega))$, $\forall\omega\in[0,-\frac{i\pi}{2}]$, i.e.,$\sigma_{\rm disc}(L_\beta)\ne\varnothing$ as $\theta_0-\frac{2-\gamma}{2}\pi\le\arg\beta\le\theta_0$. Since $\sigma_{\rm disc}(L_\beta)=\varnothing$ as $\arg\beta=\frac{2-\gamma}{2}\pi$, it implies $\sigma_{\rm disc}(L_\beta)=\varnothing$ as $\frac{2-\gamma}{2}\pi\le\arg\beta\le\min\{\pi,(2-\gamma)\pi\}$. If $\gamma\le1$, then the proof of 2) is completed. If $\gamma\le2(1-1/(k+2)),\ k\in\mathbb{N}$, repeating the previous procedure k more times, we show that $\sigma_{\rm disc}(L_\beta)=\varnothing$ as $\frac{2-\gamma}{2}\pi\le\arg\beta\le\pi$. Statement 2) is proven.

3) Let $\lambda\in\mathbb{C}\backslash[0,+\infty)$ and $\beta\in\mathbb{C}$ and let $a=a(\lambda,\beta)>0$ be so that $\lambda+\beta x^{-\gamma}\ne0$ as $x\ge a$. Then (see, for instance, [25, c. 34]) the equation

$$-y''(x)-q_\beta(x)y(x)=\lambda\cdot y(x), \tag{19}$$

has a solution $v_\beta(x,\lambda)$ obeying WKB-estimate

$$v_\beta(x,\lambda) \sim (\lambda + q_\beta(x))^{-1/4} \exp\left(i\int_0^x (\lambda + q_\beta(t))^{1/2}\, dt\right)\left[1 + O\left(x^{\gamma/2-1}\right)\right], \; x \to +\infty. \tag{20}$$

Then for Weyl function $m_\beta(\lambda)$ of the operator L_β the formula

$$m_\beta(\lambda) = \frac{v'_\beta(0,\lambda)}{v_\beta(0,\lambda)}$$

holds true. In work [18] it was shown that the function $m_\beta(\lambda)$ possesses an analytic continuation via the cut on the semi-axis $[0,+\infty)$ on infinitely-sheeted Riemann manifold by the formula

$$m_\beta(\lambda) = e^{-i\varphi} m_{\beta e^{i(2-\gamma)}}(e^{2i\varphi}\lambda), \; v_b(0,\lambda) \neq 0. \tag{21}$$

Suppose that $0 < \arg\beta < \pi$ and $\lambda_0 > 0$ is pole of $m_\beta(\lambda)$. Then it follows from (21) with $\varphi = -\arg\beta/(2-\gamma)$ that the point $e^{-2i\arg\beta/(2-\gamma)}\lambda_0$ is a pole of the function $m_{|\beta|}(\lambda)$, i.e., is an eigenvalue of the self-adjoint operator $L_{|\beta|}$ that is impossible.

If $\beta \in \mathbb{R}$ and $v_\beta(0,\lambda_0) = 0$ for some $\lambda_0 > 0$, then $\overline{v_\beta}(0,\lambda_0) = 0$ and therefore the Wronskian of the functions v_β and $\overline{v_\beta}$ vanishes at the origin. But the function $\overline{v_\beta}(x,\lambda_0)$ is also a solution to equation (19), and by (20) the Wronskian of v_β and $\overline{v_\beta}$ equals $2i$.

Thus, for each $\beta \in \mathbb{C}$ the operator L_β has neither positive eigenvalues no spectral singularities. The fact that zero is neither eigenvalue no spectral singularity follows from the fact that $v_\beta(x,0)$ coincides with the function $f(x) = \sqrt{x}H_\nu^{(1)}\left(2\nu\sqrt{\beta}x^{(2-\gamma)/2}\right)$ up to a multiplicative constant, where $\nu = 1/(2-\gamma)$, $H_\nu^{(1)}$ is the Hankel function and $f(0) \neq 0$ [25, Ch. IV, Sec. 4]. The proof is complete. $\square$

Remark 1. *Formula* (21) *specifies Statements 1) and 2); as the argument* β *grows (decays) from* 0 *to* $\frac{2-\gamma}{2}\pi$ *(respectively, to* $-\frac{2-\gamma}{2}\pi$ *), for each fixed* k *the eigenvalue* $\lambda_k(\beta)$ *of the operator* L_β *moves along the circle* $|\lambda| = -\lambda_k(|\beta|)$ *from the point* $\lambda_k(|\beta|) < 0$ *counterclockwise (respectively, clockwise) and get to* $[0,+\infty)$ *that is the essential spectrum of* L_β*. Under further grow of* $|\arg\beta|$*,* $\lambda_k(\beta)$ *continues the same movement along the same circle being a pole of the analytic continuation of the Weyl function on the next sheet.*

3. Calculation of quantum defects

We introduce the family of operators

$$M_\beta = L_\beta + W,$$

where W is the operator of multiplication by a complex-valued measurable function $W(x)$ satisfying the condition

$$\int_0^\infty (1 + x^{\gamma/2})|W(x)|dx < \infty. \tag{22}$$

In work [19] for $\beta > 0$, $\gamma = 1$, and real W satisfying the estimate

$$|W(x)| \le \int_{3/2}^2 x^{-t}|d\sigma(t)|, \quad \text{where} \int_{3/2}^2 \frac{|d\sigma(t)|}{(t-3/2)(2-t)} < \infty \tag{23}$$

it was shown that $\{\mu_k(\beta)\}_1^\infty$, which are the eigenvalues of the operator M_β taken in the ascending order, have the asymptotics (cf. (10) and (11))

$$\mu_k(\beta) \sim -\beta^{-2/(2-\gamma)} C(k - 1/4 + \delta_\beta)^{-2\gamma/(2-\gamma)}, \quad k \to +\infty, \tag{24}$$

where $z^{-2/(2-\gamma)} > 0$ as $z > 0$, the constant C is determined by formula (11), δ_β is a real constant independent of k called **quantum defect** [26].

It is easy to check that (23) implies (22).

We shall show that formula (24) remains true also for complex W satisfying (22) and an additional condition, which holds immediately in the case of real W (see Remark 2). We shall also show that to satisfy (24) instead of (22), in the case $0 < \arg\beta < \frac{2-\gamma}{\gamma}\pi$ it is sufficient to require that W has an analytic continuation $\widetilde{W}$ to an angle $-\arg\beta/(2-\gamma) < \arg z < 0$ and that (22) holds true only on the ray $\arg z = -\arg\beta/(2-\gamma)$.

3.1. Case $\beta > 0$. Throughout this subsection the parameter β is supposed to be fixed and this is why in all the notations, if there is no special need, we shall not indicate the dependence on β.

Lemma 4. *Let $\beta > 0$. Then the equation*

$$-y'' + (-q_\beta + W)y = 0 \tag{25}$$

has two linearly independent solutions $e_\pm(x)$ satisfying asymptotic estimates

$$e_\pm^{(\nu)}(x) \sim (q_\beta(x))^{-1/4+\nu/2}(\pm i)^\nu \exp\left(\pm i\int_0^x \sqrt{q_\beta(x)}dx\right), \quad x \to +\infty,\ \nu = 0, 1. \tag{26}$$

Proof. Consider equations

$$e_\pm(x) = u_\pm(x) - \int_x^{+\infty} \sin\left(\int_t^x \sqrt{q_\beta(\tau)}d\tau\right) x^{\gamma/4}t^{\gamma/4}\left(\frac{\gamma(4-\gamma)}{16}t^{-2} + W(t)\right)e_\pm(t)dt, \tag{27}$$

where $u_\pm$ denotes the right hand side of (26). It is easy to check that each solution to (27) solves also (25). Let us show that for sufficiently large $x > 0$ the equation is uniquely solvable and its solutions satisfies (26).

For $\widetilde{e}_\pm = e_\pm/u_\pm$ we have

$$\widetilde{e}_\pm = 1 + A_\pm\widetilde{e}_\pm, \tag{28}$$

where A is the integral operator with kernel

$$A_\pm(x,t) = \begin{cases} \pm\frac{1}{2i}\left(1 - \exp\left(\pm 2i\int_x^t \sqrt{q_\beta(\tau)}d\tau\right)\right)t^{\gamma/2}\left(\frac{\gamma(4-\gamma)}{16}t^{-2} + W(t)\right), & t > x > 0, \\ 0, & 0 < t < x. \end{cases}$$

It follows from condition (22) that for each $b > 0$ the operator $A_\pm$ is bounded in the space $C[b, +\infty)$ and its norm tends to zero as $b \to +\infty$. Hence,

$$\widetilde{e}_\pm(x) \sim 1,\ x \to +\infty, \tag{29}$$

that implies (26) for $\nu = 0$. In order to obtain (26) as $\nu = 1$, we differentiate (28) and substitute there (29). □

Theorem 2. *Let $\beta > 0$ and the function W satisfy estimate* (22) *and condition $e_\pm(0) \neq 0$. Then the eigenvalues $\mu_k(\beta)$ of the operator M_β (taken in an appropriate order) satisfy expansion* (24)*, where δ_β are calculated by formulae* (48)*,* (36)*, and* (26)*.*

Remark 2. *If the function W is real, the condition $e_\pm(0) \neq 0$ holds true, since $e_-(x) = \overline{e_+(x)}$ and $W(e_-, e_+) = 2i \neq 0$.*

A perturbation $W(x)$ not satisfying the condition $e_\pm(0) \neq 0$ can be constructed sufficiently easily. Let $\tilde{e}_+ = e_+|_{W\equiv 0}$ and $b > 0 :\ \tilde{e}_+(b) \neq 0$. Further, let $\varphi(x) = x\psi(x)$, where $\psi(x)$ is an arbitrary twice continuously differentiable function on $[0, b]$ having no zeroes and satisfying the conditions

$$\psi'(0) = 0,\ \psi(b) = \frac{\tilde{e}_+(b)}{b},\ \psi'(b) = \frac{b\cdot\tilde{e}'_+(b) - \tilde{e}_+(b)}{b^2}.$$

We let

$$W(x) = \begin{cases} 0, & x \geq b, \\ \frac{\varphi''(x)}{\varphi(x)} - q_\beta(x), & 0 \leq x < b. \end{cases}$$

Then $e_+(x) = \varphi(x)$ as $x \in [0, b]$, so that $e_+(0) = 0$.

We split the proof of Theorem 2 into two lemmata.

According to Lemma 4, the equation

$$-y'' + (-q_\beta + \varepsilon W)y = 0. \tag{30}$$

has two solutions $e_\pm(\varepsilon, x)$ obeying estimates (26) uniform w.r.t. ε in each compact set $K \subset \mathbb{C}$.

Lemma 5. *For each fixed $x \geq 0$ $\quad e_\pm(\varepsilon, x)$ are entire functions w.r.t. ε.*

Proof. The functions $e_\pm(\varepsilon, x)$ satisfy the equation

$$e_\pm(\varepsilon, x) = e_\pm(0, x) + \frac{\varepsilon}{2i}\int_x^{+\infty} \left(e_+(0,x)e_-(0,t) - e_-(0,x)e_+(0,t)\right) W(t) e_\pm(\varepsilon, t)dt.$$

Letting $\widetilde{e}_\pm(\varepsilon, x) = e_\pm(\varepsilon, x)(1+x)^{-\gamma/4}$, we thus obtain

$$\widetilde{e}_\pm(\varepsilon, \cdot) = \widetilde{e}_\pm(0, \cdot) + \varepsilon A_\pm \widetilde{e}_\pm(\varepsilon, \cdot),$$

where the operator A acts by the formula

$$Af = \frac{1}{2i}\int_x^{+\infty} \left(\widetilde{e}_+(0,x)\widetilde{e}_-(0,t) - \widetilde{e}_-(0,x)\widetilde{e}_+(0,t)\right)\left(1 + t^{\gamma/2}\right) W(t) f(t)dt.$$

It is clear that A is a Volterra operator in the space $C[0, +\infty)$, so that

$$\widetilde{e}_\pm(\varepsilon, \cdot) = \sum_{k=0}^{\infty} \varepsilon^k A^k[\widetilde{e}_\pm(0, \cdot)].$$

It yields the statement of the lemma. $\square$

Let $\varphi_0(\varepsilon, x)$ be a solution to equation (30) satisfying the initial conditions

$$\varphi(\varepsilon, 0) = 0,\ \ \varphi'(\varepsilon, 0) = 1. \tag{31}$$

We have

$$\varphi_0(\varepsilon, x) = \frac{e_-(\varepsilon, 0)e_+(\varepsilon, x) - e_+(\varepsilon, 0)e_-(\varepsilon, x)}{2i}. \tag{32}$$

Since $e_\pm(1, x) = e_\pm(x)$, by the assumption of Theorem 2 $e_\pm(1, 0) \neq 0$. In view of the inequality $e_\pm(0, 0) \neq 0$ (cf. Remark 2) and by Lemma 5 there exists a curve l connecting the points 0 and 1 such that $e_\pm(\varepsilon, 0) \neq 0 \ \forall\ \varepsilon \in l$.

Denote by $\varphi(\varepsilon, x, \lambda)$ the solution to the equation

$$-y'' + (-q_\beta + \varepsilon W)y = \lambda y \tag{33}$$

satisfying initial conditions (31).

Lemma 6. *Under the assumption of Theorem 2 for $\Omega(r, M) \ni \lambda \to 0$*

$$\varphi(\varepsilon, |\lambda|^{-1/2}, \lambda) = \Delta|\lambda|^{-\gamma/8}\left[\sin\left(\frac{2\sqrt{\beta}}{2-\gamma}|\lambda|^{-(2-\gamma)/4} + \delta(\varepsilon)\right) + O\left(\lambda^{(2-\gamma)/4}\right)\right], \tag{34}$$

$$\frac{\partial}{\partial x}\varphi(\varepsilon, |\lambda|^{-1/2}, \lambda) = \Delta\sqrt{\beta}|\lambda|^{\ \gamma/8}\left[\cos\left(\frac{2\sqrt{\beta}}{2-\gamma}|\lambda|^{-(2-\gamma)/4} + \delta(\varepsilon)\right) + O\left(\lambda^{(2-\gamma)/4}\right)\right], \tag{35}$$

where the estimate for the error terms in uniform in $\arg\lambda$, $\varepsilon \in l$,

$$\Delta = \sqrt{e_-(\varepsilon, 0)e_+(\varepsilon, 0)}, \quad \delta(\varepsilon) = \ln\sqrt{\frac{e_-(\varepsilon, 0)}{e_+(\varepsilon, 0)}}, \tag{36}$$

the branches of $\sqrt{z}$, $\ln z$ are fixed so that they are positive as $z > 1$.

Proof. The function $\varphi(\varepsilon, x, \lambda)$ is the solution to the equation

$$\varphi(\varepsilon, x, \lambda) = \varphi_0(\varepsilon, x) - \frac{\lambda}{2i}\int_0^x \left(e_+(\varepsilon, x)e_-(\varepsilon, t) - e_-(\varepsilon, x)e_+(\varepsilon, t)\right)\varphi(\varepsilon, t, \lambda)dt, \tag{37}$$

which by the change $\widetilde{\varphi}(\varepsilon, x, \lambda) = \varphi(\varepsilon, x, \lambda)(1+x)^{-\gamma/4}$, $\widetilde{\varphi}_0(\varepsilon, x) = \varphi_0(\varepsilon, x)(1+x)^{-\gamma/4}$, $\widetilde{e}_\pm(\varepsilon, x) = e_\pm(\varepsilon, x)(1+x)^{-\gamma/4}$ is reduced to the equation

$$\widetilde{\varphi} = \widetilde{\varphi}_0 + B(\varepsilon, \lambda)\widetilde{\varphi},$$

where $B(\varepsilon, \lambda)$ acts by the formula

$$B(\varepsilon, \lambda)f = -\frac{\lambda}{2i}\int_0^x \left(e_+(\varepsilon, x)\widetilde{e}_-(\varepsilon, t) - \widetilde{e}_-(\varepsilon, x)\widetilde{e}_+(\varepsilon, t)\right)t^{\gamma/2}\varphi(t)dt.$$

Since by estimates (26)

$$\sup_{x\le 0,\ \varepsilon\in l} |\widetilde{e}_\pm(\varepsilon, x)| \le c_0 < \infty$$

the norm of the operator $B(\varepsilon, \lambda)$ in the space $C\left[0, |\lambda|^{-1/2}\right]$ satisfies the estimate

$$\|B(\varepsilon, \lambda)\| = O\left(|\lambda|^{(2-\gamma)/4}\right),\ \lambda \to 0,$$

uniformly in $\varepsilon \in l$. Together with estimates (26) and by (32) it follows (34). In order to obtain (35), one should differentiate (37) and employ the obtained estimate for $\varphi(\varepsilon, x, \lambda)$. The proof is complete. □

Now we construct the solution to equation (33) belonging to $L^2(|\lambda|^{-1/2}, +\infty)$.

We introduce the notations. Let

$$\Omega(r, M) = \{\lambda = \mu + i\nu :\ -r < \mu < 0,\ |\nu| \le M|\mu|^{(2+\gamma)/(2\gamma)}\},$$

where $r > 0, M > 0$. Further, let

$$a_\lambda = \left(-\frac{\lambda}{\beta}\right)^{1/\gamma},\ Q(x, \lambda) = \int_{a_\lambda}^x \sqrt{-\lambda - q_\beta(t)}dt,\ P(x, \lambda) = \int_x^{a_\lambda} \sqrt{\lambda + q_\beta(t)}dt.$$

Lemma 7. *Under condition* (22) *equation* (33) *has a solution*$v(\varepsilon, x, \lambda)$ *satisfying the following estimates,*

a) for a fixed $\lambda \notin [0, +\infty)$ *and* $x \to +\infty$

$$v(\varepsilon, x, \lambda) \sim \frac{1}{2}(-\lambda - q_\beta(x))^{-1/4}\exp\left(-Q(x, \lambda)\right); \tag{38}$$

b) as $\Omega(r, M) \ni \lambda \to 0$

$$v(\varepsilon, |\lambda|^{-1/2}, \lambda) \sim (\lambda + q_\beta(|\lambda|^{-1/2}))^{-1/4}\left[\sin\left(P(|\lambda|^{-1/2}, \lambda) + \pi/4\right) + o(1)\right], \tag{39}$$

$$\frac{\partial}{\partial x}v(\varepsilon, |\lambda|^{-1/2}, \lambda) \sim (\lambda + q_\beta(|\lambda|^{-1/2}))^{1/4}\left[-\cos\left(P(|\lambda|^{-1/2}, \lambda) + \pi/4\right) + o(1)\right]. \tag{40}$$

The proof is the same as that of Lemma 6. Let us just describe how to choose sample solutions and the corresponding integral equation. In the domain $D_0 = \{\lambda < 0, x > a_\lambda\}$ we consider a positive function

$$\xi(x, \lambda) = \left(\frac{3}{2}Q(x, \lambda)\right)^{2/3}$$

and continue it by the analyticity. It is easy to check that as $\lambda < 0$ and $0 < x < a_\lambda$

$$\xi(x, \lambda) = -\left(\frac{3}{2}P(x, \lambda)\right)^{2/3}.$$

We let

$$v_1(x, \lambda) = \xi'^{-1/2}Bi(\xi(x, \lambda)),\quad v_2 = \xi'^{-1/2}Ai(\xi(x, \lambda)), \tag{41}$$

where $Ai(\xi)$, $Bi(\xi)$ are the Airy functions [25, Ch. IV, Sec. 1].

Asymptotic formulae for Airy functions imply the following relations,

as $\operatorname{Re} Q \to +\infty$

$$v_k(x,\lambda) \sim \frac{1}{k}(-\lambda - q_\beta(x))^{-1/4} \exp\left((-1)^{k-1}Q(x,\lambda)\right)\left[1 + O\left(Q^{-1}(x,\lambda)\right)\right], \; k = 1,2,$$

as $\operatorname{Re} P \to +\infty$

$$\begin{aligned} v_1(x,\lambda) &\sim (\lambda + q_\beta(x))^{-1/4}\left[\cos\left(P(x,\lambda) + \pi/4\right) + O\left(P^{-1}(x,\lambda)\right)\right], \\ v_2(x,\lambda) &\sim (\lambda + q_\beta(x))^{-1/4}\left[\sin\left(P(x,\lambda) + \pi/4\right) + O\left(P^{-1}(x,\lambda)\right)\right], \\ v_1'(x,\lambda) &\sim (\lambda + q_\beta(x))^{1/4}\left[\sin\left(P(x,\lambda) + \pi/4\right) + O\left(P^{-1}(x,\lambda)\right)\right], \\ v_2'(x,\lambda) &\sim (\lambda + q_\beta(x))^{1/4}\left[-\cos\left(P(x,\lambda) + \pi/4\right) + O\left(P^{-1}(x,\lambda)\right)\right]. \end{aligned}$$

Consider the equation

$$v(\varepsilon,x,\lambda) = v_2(x,\lambda) - \varepsilon\int_x^{+\infty}\left(v_1(x,\lambda)v_2(t,\lambda) - v_2(x,\lambda)v_1(t,\lambda)\right)W(t)v(\varepsilon,t,\lambda)dt. \tag{42}$$

According to (41), $W(v_1,v_2) = -1$, so, $v(\varepsilon,x,\lambda)$ solves equation (33). Arguing as in the proof of Lemma 6, we obtain estimates (38)–(40).

Remark 3. *As one can see in the proof, to satisfy estimate* (38) *instead of* (22)*, it is sufficient to suppose* $W \in L^1(0,+\infty)$.

Proof of Theorem 2. Let $M(\beta,\varepsilon) = L_\beta + \varepsilon W$, $\varepsilon \in \mathbb{C}$. It follows from estimate (38) that for each $\lambda \notin [0,+\infty)$ $\quad v(\varepsilon,\cdot,\lambda) \in L^2[|\lambda|^{-1/2},+\infty)$, so, λ is an eigenvalue of the operator $M(\beta,\varepsilon)$ if and only if

$$\Phi(\varepsilon,\lambda) := \langle\varphi(\varepsilon,x,\lambda), v(\varepsilon,x,\lambda)\rangle|_{x=|\lambda|^{-1/2}} = 0, \tag{43}$$

where

$$\langle f,g\rangle(x) = f(x)g'(x) - f'(x)g(x). \tag{44}$$

Substituting here asymptotic formulae (34), (34) and (39), (40), we obtain

$$\Phi(\varepsilon,\lambda) = -\Delta\sqrt{\beta}\Phi_0(\varepsilon,\lambda) + o(1), \; \Omega(r,M) \ni \lambda \to 0, \tag{45}$$

where

$$\Phi_0(\varepsilon,\lambda) = \sin\left(\int_0^{a_\lambda}\sqrt{\lambda + q_\beta(t)}dt + \pi/4 + \delta(\varepsilon)\right), \tag{46}$$

and the estimate for the error term is uniform in $\varepsilon \in l$.

We denote by $\lambda_k(\beta,\varepsilon)(k = 1,2,\dots)$ the negative roots of the function $\Phi_0(\varepsilon,\lambda)$ taken in the ascending order. We have (see (10))

$$\lambda_k(\beta,\varepsilon) \sim \lambda_k(\beta)\left(1 - \frac{2\gamma}{2-\gamma}\delta(\varepsilon)k^{-1}\right), \quad k \to +\infty. \tag{47}$$

Due to relations (43)–(47) by Rouché theorem we conclude that for each $\sigma > 0$ there exists $K_\sigma \in \mathbb{N}$ such that for all $k \in \mathbb{N}$: $k \geq K_\sigma$ and $\varepsilon \in l$, there exists exactly one simple (of algebraic multiplicity 1) eigenvalue of the operator $M(\beta,\varepsilon)$ in the disk $B_k(\varepsilon,\sigma) = \{|\lambda - \lambda_k(\beta,\varepsilon)| \leq \sigma|\lambda_k(\beta)|k^{-1}\}$. We call it σ-property.

We denote by $\mu_n(\beta,\varepsilon)$, $n = 1,2,\dots$, the eigenvalues of the operator $M(\beta,\varepsilon)$ taken in the order of modules descending counting multiplicities. It follows from (10), (11), and the definition of $B_k(\varepsilon,\sigma)$ that there exists $\sigma_0 > 0$, $K_0 \in \mathbb{N}$ such that for all $0 < \sigma < \sigma_0$ and $k \leq K_0$ the disks $B_k(\varepsilon,\sigma)$ are mutually disjoint. We choose $0 < \sigma < \sigma_0$ so that $K_\sigma \geq K_0$. Let us show that for all $k \geq K_\sigma$ $\quad \mu_k(\beta,\varepsilon) \in B_k(\varepsilon,\sigma)$ for all $\varepsilon \in l$.

We make several notations. Let $\varepsilon = \varepsilon(t)$, $0 \leq t \leq 1$ be a parametrization of the curve l. For the points $\varepsilon_1 = \varepsilon(t_1)$ and $\varepsilon_2 = \varepsilon(t_2)$ we write $\varepsilon_1 \prec \varepsilon_2$ if $t_1 < t_2$. If $a \prec b$, by l_{ab} we denote the arc $\{\varepsilon \in l: \; a \prec \varepsilon \prec b\}$.

Suppose that for some $m \geq K_0$ and $\delta \in l$ $\mu_m(\beta,\delta) \notin B_m(\delta,\sigma)$. Lemma 5 and formulae (36) yield that the function $\lambda_m(\beta,\varepsilon)$ is continuous w.r.t. ε on the curve l. This is why the family of the circles $\Gamma_m(\varepsilon,\sigma) = \{|\lambda - \lambda_m(\beta,\varepsilon)| = \sigma|\lambda_m(\beta)|m^{-1}\}$ moves continuously as ε moves along the curve l. The function$\mu_m(\beta,\varepsilon)$ is also continuous on l (that implied by the analyticity of the function $\Phi(\beta,\varepsilon)$ on l). This is why in the arc $l_{0\delta}$ there exists a point ξ such that $\mu_m(\beta,\xi)$ lies in the circle $\Gamma_m(\xi,\sigma)$ and $\mu_m(\beta,\varepsilon)$ lies outside $B_m(\varepsilon,\sigma)$ for all $\varepsilon \succ \xi$. By σ-property, on $\Gamma_m(\xi,\sigma)$ there exists at least one eigenvalue of the operator $M(\beta,\xi)$ not coinciding with $\mu_m(\beta,\xi)$. Then the disk $B_m(\xi,\sigma_1)$, where $\sigma_1 > \sigma$, contains at least two eigenvalues of the operator $M(\beta,\xi)$ that contradicts σ-property.

Thus, $\mu_k(\beta,\varepsilon) \sim \lambda_k(\beta,\varepsilon)$, $k \to +\infty$, uniformly in $\varepsilon \in l$. By the identities $\mu_k(\beta,1) = \mu_k(\beta)$ and

$$\lambda_k(\beta,1) = C\,[k - 1/4 + \delta_\beta]^{-\frac{2\gamma}{2-\gamma}},$$

where

$$\delta_\beta = -\frac{\delta(1)}{\pi}, \tag{48}$$

it implies (24). The proof is complete. □

Remark 4. *In work* [27] *by the same method the quantum defect of Dirac operator on the semi-axis was calculated.*

3.2. Case $0 < \arg\beta < \frac{2-\gamma}{2}\pi$**.** Since the cases $-\frac{2-\gamma}{2}\pi < \arg < 0$ and $0 < \arg\beta < \frac{2-\gamma}{2}\pi$ are equal in rights, we consider only the case $0 < \arg\beta < \frac{2-\gamma}{2}\pi$. Let $\omega_\beta = -\frac{\arg\beta}{2-\gamma}$, $U_\beta = \{z : \omega_\beta < \arg z < 0\}$, $U_\beta(R) = U_\beta \cap \{|z| < R\}$, Ω be the domain bounded by a rectifiable Jordan curve γ, $p > 1$, and $E_p(\Omega)$ be the Smirnov class [28, c. 203], i.e., the set of functions $f(z)$ analytic in the domain Ω and such that for some sequence of rectifiable curves γ_n contracting to γ

$$\int_{\gamma_n} |f(z)|^p|dz| < C,$$

where C is independent of n.

If $f \in L^p_{loc}[0,+\infty)$, $p > 1$, we shall say that f possesses an analytic continuation $\widetilde{f}(z)$ into the angle U_β if $\forall\ R > 0$ $\widetilde{f}(z) \in E_p(U_\beta(R))$ and for a.e. $x > 0$ the angular boundary value of the function $\widetilde{f}$ at the point x coincides with $f(x)$.

Theorem 3. *Let*

a) a function $W \in L^2_{loc}(0,+\infty)$ possesses an analytic continuation $\widetilde{W}(z)$ into the angle U_β so that $\widetilde{W}(z) \to 0$, $z \to \infty$ uniformly in $\omega_\beta \leq \arg z \leq 0$ (on rays $\arg z = 0$ and $\arg z = \omega_\beta$ the limit is understood in a.e. sense);

b) the function $\widehat{W}(x) = \widetilde{W}(xe^{i\omega_\beta})$ satisfies the estimate

$$\int_0^\infty (1 + x^{\gamma/2})|\widehat{W}(x)|dx < \infty, \tag{49}$$

c) $\widehat{e}_\pm(0) \neq 0$, where $\widehat{e}_\pm$ are obtained from $e_\pm$ by replacing $-q_\beta(x) + W(x)$ to $-|\beta|x^{-\gamma} + e^{2\omega_\beta i}\widehat{W}(x)$ in (25).

Then for the eigenvalues $\mu_k(\beta)$ of the operator M_β (under an appropriate ordering) the expansion (24) *holds true, where δ_β are calculated by the formulae* (48), (36), *and* (26) *for $W(x) = e^{2\omega_\beta i}\widehat{W}(x)$.*

Proof. Since $W \in L^2_{loc}(0,+\infty)$ and $W \to 0$, $x \to +\infty$, the operator $W(L_0+1)^{-1}$ (W is the operator of multiplication by the function $W(x)$) is the uniform limit of Hilbert-Schmidt operators (see the proof of Lemma 3) and is thus compact. Therefore, $\sigma_{\text{ess}}(M_\beta) = [0,+\infty)$. Arguing then as in the proof of Item 1) of Theorem 1, we obtain

$$\mu_k(\beta) = e^{2\omega_\beta i}r_k(\beta),$$

where $\{r_k(\beta)\}_{k=1}^{\infty}$ are the eigenvalues of the operator $L_{|\beta|} + e^{2\omega_\beta i}\widehat{W}$ having expansion (24) by the conditions b), c), and Theorem 2. The proof is complete. $\square$

Remark 5. *In order to obtain expansion* (24) *for complex* β*, we have to impose much stricter restrictions on* W *(analyticity in the angle* U_β*) in comparison with the case* $\beta > 0$*. In view of this fact a question on how necessary this condition appears. In Sec. 5 we shall show that up to some reservations this condition is necessary.*

4. Compactly supported perturbations

It was shown in work [18] that under the assumption of Theorem 3 the Weyl function (cf. (62)) of operator M_β possesses a meromorphic continuation into the angle

$$Y_\beta = \{2\pi < \arg\lambda < 2(\pi + \arg\beta/(2-\gamma))\}, \tag{50}$$

and its poles in this angle form a bounded set and can accumulate to the ray $\arg\lambda = 2(\pi + \arg\beta/(2-\gamma))$ only. In this section we formulate two statements which in some sense justify the necessity of the analyticity of the perturbation W in order to ensure the mentioned properties of the Weyl function and in this way hint the choice of the property P appearing in Problem 1.

Theorem 4. *Let* W *be compactly supported* ($\mathrm{supp} W \subset [0,b]$) *and in some semi-neighborhood of the point* b *it can be represented as*

$$W(x) = (b-x)^n V(x),$$

where $n \geq 0$*,* $V(b-0)$ *is well-defined, finite, and is non-zero.*

Then the Weyl function of the operator M_β *possesses a meromorphic continuation into the angle* Y_β*, which has an unbounded sequence of poles in a vicinity of the ray* $\arg\lambda = 2\pi$*,*

$$\lambda_k \sim \left(\frac{\pi k}{b} + i\frac{n+2}{2b}\ln k + O(1)\right)^2, \quad k \to +\infty. \tag{51}$$

Proof. The existence of the meromorphic extension for the Weyl function of the operator M_β into the angle Y_β follows from the arguments of work [18] (see Theorem 2 and Remark 4). Formula (51) is proven exactly in the same way as in [29] (see Theorem 3). $\square$

The next result being an analogue of well-known Ambartsumian theorem seems to be completely unexpected; the possibility of recovering the perturbation just by a part of spectrum has an exceptional character and can be realized very rarely.

Theorem 5. *Let the function* W *be a compactly supported and summable on its support. If* $\sigma_{\mathrm{disc}}(M_\beta) = \sigma_{\mathrm{disc}}(L_\beta)$*, then* $W = 0$ *a.e. on* $(0, +\infty)$*.*

Proof. We introduce the notations. Let $S(x,\lambda)$ and $C(x,\lambda)$ be the solutions to the equation

$$-y'' + (-q_\beta + W)y = \lambda y \tag{52}$$

satisfying the conditions

$$S(0,\lambda) = 0,\ S'(0,\lambda) = 1, \quad C(b,\lambda) = 1,\ C'(b,\lambda) = 0,$$

and let

$$S_0(x,\lambda) = S(x,\lambda)|_{W\equiv 0},\ C_0(x,\lambda) = C(x,\lambda)|_{W\equiv 0}. \tag{53}$$

By $v_0(x,\lambda)$ we denote the solution to equation (19) satisfying asymptotic relation (20) for all $\lambda \notin [0,+\infty)$.

Let $b > 0$ be such that supp $W \subset [0,b]$. Then the eigenvalues of the operator M_β are the roots of the equation (cf. (44))

$$\langle S, v_0\rangle(b) = 0. \tag{54}$$

Since $W(x) \equiv 0$ as $x > b$,

$$S(x,\lambda) = a_1(\lambda)S_0(x,\lambda) + a_2(\lambda)C_0(x,\lambda), \tag{55}$$

where $a_1(\lambda) = \langle C_0, S\rangle(b)$, $a_2(\lambda) = -\langle S_0, S\rangle(b)$. Substituting (55) into (54) and taking into consideration that $\langle S_0, v_0\rangle(b) = \langle S_0, v_0\rangle(0) = -v_0(0,\lambda)$, $\langle C_0, v_0\rangle(b) = v_0'(b,\lambda)$, for the eigenvalues of the operator M_β we get

$$-a_1(\lambda)v_0(0,\lambda) + a_2(\lambda)v_0'(b,\lambda) = 0. \tag{56}$$

By the hypothesis of the theorem, $\sigma_{\rm disc}(M_\beta) = \sigma_{\rm disc}(L_\beta) = \{\lambda_k\}_1^\infty$, where $\lambda_k \to 0,\ k \to \infty$. Then $\forall\ k \in \mathbb{N},\ \ v_0(0,\lambda_k) = 0$. Let us show that $v_0'(b,\lambda_k) \neq 0$ for sufficiently large k. Indeed, otherwise the spectrum of the problem

$$\begin{aligned} -y'' - q_\beta y &= \lambda y,\ 0 < x < b, \\ y(0) = y'(b) &= 0 \end{aligned}$$

accumulates at zero that is impossible due to the discreteness of the spectrum of this problem.

Then it follows from (53) that $a_2(\lambda_k) = 0$ starting from some index. But $a_2(\lambda)$ is entire function and hence $a_2(\lambda) \equiv 0$, so (55) becomes

$$S(x,\lambda) = a_1(\lambda)S_0(x,\lambda),\ x \geq b.$$

The entire function $a_1(\lambda)$ has no zeroes (if $a_1(\lambda_0) = 0$, then $S(x,\lambda_0) \equiv 0$), so $a_1(\lambda) = e^{P(\lambda)}$, where $P(\lambda)$ is entire. We have $\ln|a_1(\lambda)| = O(\lambda^{1/2})$, $\lambda \to \infty$, and hence $P(\lambda) = \mathrm{const}$, i.e., $a_1(\lambda) \equiv \mathrm{const}$.

Further,

$$a_1(\lambda) = \langle C_0, S\rangle(0) + \int_0^b (C_0S'' - C_0''S)dx = 1 + \int_0^b WC_0S dx.$$

On the other hand (see, for instance, [30, c. 13]), as $\lambda \to +\infty$,

$$C_0(x,\lambda) \sim \cos\sqrt{\lambda}x + O(\lambda^{-1/2}), \tag{57}$$

$$S(x,\lambda) \sim \frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}} + O(\lambda^{-1}) \tag{58}$$

uniformly in $x \in [0,b]$. It yields $a_1(\lambda) = 1 + O(\lambda^{-1/2})$, $\lambda \to +\infty$, therefore, $a_1(\lambda) \equiv 1$, i.e.,

$$S(x,\lambda) \equiv S_0(x,\lambda),\ x \geq b,\ \lambda \in \mathbb{C}. \tag{59}$$

Further,

$$C(x,\lambda) = b(\lambda)C_0(x,\lambda),\ x \geq b, \tag{60}$$

where $b(\lambda) = -\langle S_0, C\rangle(b) = -S_0(\lambda)$. Let us find $b(\lambda)$. Since

$$v_0(x,\lambda) = C(x,\lambda) + m_\beta^0(\lambda)S_0(x,\lambda),$$

it follows from (59) and (60) that

$$v(x,\lambda) = b(\lambda))v_0(x,\lambda),\ x \geq b,$$

hence, $m_\beta(\lambda) = b(\lambda)m_\beta^0(\lambda)$ or

$$b(\lambda) = \frac{m_\beta(\lambda)}{m_\beta^0(\lambda)}.$$

For large λ uniformly in $\arg\lambda \in [0,2\pi]$ we have [30, Ch I, Sec. 12]

$$\begin{aligned} m_\beta(\lambda) &\sim i\sqrt{\lambda} + O(1), \\ m_\beta^0(\lambda) &\sim i\sqrt{\lambda} + O(1). \end{aligned}$$

Therefore, the entire function $b(\lambda)$ is bounded on $\mathbb{C}$, and hence $b(\lambda) \equiv 1$. Thus, $m_\beta(\lambda) \equiv m_\beta^0(\lambda)$ that by the unique solvability of the inverse problem [31, c. 202] implies $M_\beta = L_\beta$, i.e., $W = 0$ a.e. $\square$

5. Main result

We introduce the notations. Let U_β be an angle introduced in Subsec. 3.2. We shall say that a locally summable on $[0,+\infty)$ function f possesses a meromorphic continuation $\widetilde{f}$ in the angle U_β, if

a) $\forall\ R>0$ the function $\widetilde{f}$ in the domain $U_\beta(R)$ has a finite number of poles $z_1,\dots,z_n$ so that the function

$$\widetilde{f}-\sum_{k=1}^{n}G_k(z)$$

belongs to $E_1(U_\beta(R))$, where $G_k(z)$ is the principal part of Laurent series for $\widetilde{f}$ at the point z_k;

b) for a.e. $x\in(0,+\infty)$ the angular boundary value of the function $\widetilde{f}$ at the point x equals $f(x)$.

Further, we shall say that the pole z_0 of the function $f(z)$ satisfy the monodromy-free condition, if in some neighborhood U of the point z_0 the expansion

$$f(z)=\frac{m(m+1)}{(z-z_0)^2}+\sum_{k=0}^{m-1}f_k(z-z_0)^{2k}+(z-z_0)^{2m}r_m(z) \tag{61}$$

holds true, where $m\in\mathbb{N}$, $r_m(z)$ is analytic in U.

Remark 6. *It is known* [32] *that condition* (61) *is necessary and sufficient for all the solutions to the equations $-y''+fy=\lambda y$ to be uniquely defined in the neighborhood U of the point z_0 for all values of the parameter λ. Following* [33], *we call it the monodromy-free condition.*

Let $W\in L^1(0,+\infty)$. According to Remark 3, equation (52) has a solution $v(x,\lambda)$ satisfying estimate (38) for $\lambda\notin[0,+\infty)$. It is known (see, for instance, [18] or [31, Ch. 2]) that for each fixed $x\geq 0$ the functions $v_\beta(x,\lambda)$ and $v'_\beta(x,\lambda)$ are analytic in $\mathbb{C}\setminus[0,+\infty)$ and continuous up to upper and lower sides of the cut w.r.t. $\lambda>0$ and the zeroes of $v_\beta(0,\lambda)$ form a bounded set Λ. Therefore,

$$m_\beta(\lambda)=\frac{v'(0,\lambda)}{v(0,\lambda)}, \tag{62}$$

which is the Weyl function of the operator $M_\beta(\lambda)$, is meromorphic in $\mathbb{C}\setminus[0,+\infty)$, its poles form a bounded set, can accumulate to the ray $[0,+\infty)$ only, and $M_\beta(\lambda)$ is continuous in $\{\lambda\neq 0:\ 0\leq\arg\lambda\leq 2\pi\}\setminus\Lambda$.

Now we are in the position to formulate the main result.Let $0<\arg\beta<\frac{2-\gamma}{2}\pi$ (the case $-\frac{2-\gamma}{2}\pi<\arg\beta<0$ is similar). Consider the operator $M_\beta=L_\beta+W$, where $W\in L^1(0,+\infty)$.

Theorem 6. *Suppose the function W has a meromorphic continuation $\widetilde{W}(z)$ to an angle U_β so that*

(a) each pole of the function $\widetilde{W}(z)$ satisfies monodromy-free condition;

(b) the function $\widehat{W}(x):=e^{2i\omega_\beta}\widetilde{W}(e^{i\omega_\beta}x)$, $x>0$, is summable on $(0,+\infty)$;

(c) there exists an infinite set $\Lambda'\subset\{\lambda\neq 0:\ -2\omega_\beta\leq\arg\lambda\leq 2\pi\}$ having at least one finite limit point $\lambda_0\neq 0$ such that for all $\lambda\in\Lambda'$

$$v'(0,\lambda)\widehat{v}\left(0,\lambda e^{2i\omega_\beta}\right)-e^{-i\omega_\beta}v(0,\lambda)\widehat{v}'\left(0,\lambda e^{2i\omega_\beta}\right)=0, \tag{63}$$

where $\widehat{v}(x,\mu)$ is the solution to the equation

$$-v''+(-|\beta|x^{-\gamma}+\widehat{W})v=\mu v \tag{64}$$

satisfying the estimate (38).

Then $m_\beta(\lambda)$, which is the Weyl function for the operator M_β, has a meromorphic continuation $\widetilde{m}_\beta(\lambda)$ from the domain $\mathbb{C}\setminus[0,+\infty)$ into the agnle Y_β (see (50)*) such that*

$$\widehat{m}_\beta(\mu):=e^{i\omega_\beta}\widetilde{m}_\beta\left(e^{-2i\omega_\beta}\mu\right) \tag{65}$$

is the Weyl function of the operator $L_{|\beta|}+\widehat{W}$.

Vice versa, if $m_\beta(\lambda)$ *has a meromorphic continuation* $\widetilde{m}_\beta(\lambda)$ *into the angle* Y_β *so that* (65) *is the Weyl function of the operator* $L_{|\beta|}+V$ *with some* $V\in L^1(0,+\infty)$, *then* W *has a meromorphic continuation* $\widetilde{W}(z)$ *into the angle* U_β, *at that, (a)-(b) hold true, and* $\widehat{W}(x)\equiv V(x)$.

Proof. According to (62) and the definition of $\widehat{v}(x,\mu)$, the Weyl function of the operator $L_{|\beta|}+\widehat{W}$ reads as

$$\widehat{m}_\beta(\mu)=\frac{\widehat{v}'(0,\mu)}{\widehat{v}(0,\mu)}. \tag{66}$$

Further, by the above the functions $v_\beta(0,\mu)$ and $v_\beta'(0,\mu)$ are analytic in $\mathbb{C}\setminus[0,+\infty)$ and continuous up to upper and lower sides of the cut along $\mu>0$ and the zeroes of $v_\beta(0,\mu)$ form a bounded set in M. Therefore, the left hand side of (63) is analytic in the angle $\{\lambda\neq 0:\ -2\omega_\beta<\arg\lambda<2\pi\}$, continuous up to its sides except the origin. This is why identity (63) holds true for all $\lambda\in\{\lambda\neq 0:\ -2\omega_\beta\le\arg\lambda\le 2\pi\}$, i.e.,

$$m_\beta(\lambda)=e^{-i\omega_\beta}\widehat{m}_\beta\left(e^{2i\omega_\beta}\lambda\right),\ -2\omega_\beta\le\arg\lambda\le 2\pi,\ \lambda\notin\Lambda\cup\Lambda'', \tag{67}$$

where $\Lambda''=e^{-2i\omega_\beta}M$. But the right hand side is defined also for $\lambda\in Y_\beta\setminus\Lambda''$ that gives an analytic continuation of $m_\beta(\lambda)$ into the domain $Y_\beta\setminus\Lambda''$. Identity (65) follows from (66) and (67).

Let us prove the inverse statement. Suppose $m_\beta(\lambda)$ has a meromorphic continuation $\widetilde{m}_\beta(\lambda)$ into the angle Y_β so that (65) is the Weyl function of the operator $L_{|\beta|}+V$ for some $V\in L^1(0,+\infty)$. We introduce a family of polygonal lines $\Gamma_a=[a,0]\cup[0,ae^{i\omega_\beta}]$, $a>0$, with a parametrization

$$z=\begin{cases} a(1-2t), & 0\le t\le 1/2,\\ ae^{i\omega_\beta}(2t-1), & 1/2<t\le 1.\end{cases}$$

We then let

$$\widetilde{W}(z)=\begin{cases} W(z), & z>0,\\ e^{-2i\omega_\beta}V\left(e^{-i\omega_\beta}z\right), & z=e^{-i\omega_\beta}r,\ r>0,\end{cases}$$

and introduce the family of Strum-Liouville operator T_a, $a>0$ defined as follows

$$\begin{aligned} T_a y &= -y''(z)+(-q_\beta(z)+\widetilde{W}(z))y(z),\ z\in\Gamma_a,\\ D(T_a) &= \{y:\ y,y'\in AC(\Gamma_a),\ -y''+(-q_\beta+\widetilde{W})y\in L^2(\Gamma_a),\ y(a)=y(ae^{i\omega_\beta})=0\},\end{aligned}$$

where the prime indicates the differentiation along Γ_a.

Let $\varphi_a(z,\lambda)$ be the solution to the equation

$$-y''(z)+(-q_\beta(z)+\widetilde{W}(z)y(z)=\lambda^2y(z) \tag{68}$$

satisfying initial conditions

$$\varphi_a(a,\lambda)=0,\ \left.\frac{d}{dz}\varphi_a(z,\lambda)\right|_{z=a}=1.$$

We let $\Phi_a(\lambda)=\varphi_a\left(ae^{i\omega_\beta}\right)$. Then λ^2 is an eigenvalue of the operator T_a if and only if

$$\Phi_a(\lambda)=0. \tag{69}$$

The function $\Phi_a(\lambda)$ is even. By $\{\lambda_n\}_1^\infty$ we denote the roots to equation (69) lying in the upper half-plane and taken in the order of ascending modules counting algebraic multiplicities. Lemma 2 of work [34] yields that except a finite number all λ_n lie in the angle $\{-\omega_\beta\le\arg\lambda\le\pi\}$.

Lemma 8. *If* (65) *is the Weyl function of the operator* $L_{|\beta|}+V$, *then*

$$\lambda_n\sim\frac{\pi n}{a(e^{i\omega_\beta}-1)},\ n\to+\infty. \tag{70}$$

Proof. Let

$$\alpha = \arg\lambda,\ B_\alpha = \begin{cases} 0, & -\omega_\beta \le \alpha \le \pi, \\ a, & \pi < \alpha \le \pi - \omega_\beta, \\ b, & 0 \le \alpha < -\omega_\beta. \end{cases}$$

Denote by $\psi(z,\lambda)$ the solution to equation (68) satisfying conditions $\psi(B_\alpha) = 1$, $\psi'(B_\alpha,\lambda) = -i\lambda$. We let

$$e_\pm(z,\lambda) = (\lambda^2 + p_\beta)^{-1/4} \exp\left(\pm i \int_0^x \sqrt{\lambda^2 + p_\beta} dt\right),$$

where $p_\beta = q_\beta \cdot (1 - \chi_r)$, χ_r is the characteristic function of the disk $|z| < r$, $r > a$. Then ψ satisfies the equation

$$\psi(z,\lambda) = \sqrt{\lambda} e_(z,\lambda) + \frac{1}{2i} \int_{B_\alpha}^z (e_-(z,\lambda)e_+(t,\lambda) - e_+(z,\lambda)e_-(t,\lambda)) V_\beta(t,\lambda)\psi(t,\lambda)dt,$$

where $V_\beta = \chi_r q_\beta - \widetilde{W} + \frac{d^2}{dt^2}\left((p_\beta + \lambda^2)^{-1/4}\right)(p_\beta + \lambda^2)^{1/4}$. For the function $\widetilde{\psi} = \psi/(\sqrt{\lambda}e_-)$ it implies

$$\widetilde{\psi} = 1 + A(\lambda)\widetilde{\psi},$$

where

$$A(\lambda)f = \frac{1}{2i} \int_{B_\alpha}^z \left(1 - \exp\left(2i \int_t^z \sqrt{\lambda^2 + p_\beta} dt\right)\right) \left(\lambda^2 + p_\beta\right)^{-1/2} V_\beta(t,\lambda) f(t) dt.$$

It is easy to check the operator $A(\lambda)$ is bounded in the space $C(\Gamma)$, and its norm in this space can be estimated by $O(\lambda^{-1})$, $l \to \infty$, uniformly in $0 \le \arg\lambda \le \pi - \omega$. It yields

$$\psi(z,\lambda) \sim \sqrt{\lambda} e_(z,\lambda)\left(1 + O\left(\lambda^{-1}\right)\right),\ \lambda \to \infty, \tag{71}$$

uniformly in $z \in \Gamma$, $0 \le \arg\lambda \le \pi - \omega$.

Then

$$\varphi_a(z,\lambda) = \psi(a,\lambda)\psi(z,\lambda) \int_a^z \psi^{-2}(t,\lambda)dt$$

for sufficiently large λ and $-\omega_\beta \le \alpha \le \pi$. Therefore,

$$\Phi_a(\lambda) \sim \lambda^{-1} e^{-\lambda a(1+e^{i\omega})} F_a(\lambda), \tag{72}$$

where

$$F_a(\lambda) = \int_a^{ae^{i\omega}} \psi^{-2}(t,\lambda)dt. \tag{73}$$

According to (71), for sufficiently large λ in the angle $\{0 \le \arg\lambda \le \pi - \omega\}$

$$\psi(z,\lambda) \sim \sqrt{\lambda} e_(z,\lambda)(1 + O\left(\lambda^{-1}\right),\ \Gamma \ni z \to \infty.$$

Hence,

$$v_\beta(x,\lambda) = C_0\psi(x,\lambda) \int_x^{+\infty} \psi^{-2}(t,\lambda)dt,\ \widehat{v}_\beta(x,\lambda) = C_1\psi(xe^{i\omega_\beta},\lambda) \int_x^{+\infty} \psi^{-2}(te^{i\omega_\beta},\lambda)dt, \tag{74}$$

where $C_{0,1} = \text{const}$.

The assumption of the lemma implies that the functions v_β and $\widehat{v}_\beta$ satisfy identity (63) (for all λ for which the functions v_β and $\widehat{v}_\beta$ are well-defined) that by identities (72) yield

$$\int_\Gamma \psi^{-2}(t,\lambda)dt = 0,$$

and due to (73) it implies

$$F_a(\lambda) = \int_a^{+\infty} \psi^{-2}(t,\lambda)dt - \int_{ae^{i\omega}}^{\infty e^{i\omega}} \psi^{-2}(t,\lambda)dt.$$

Substituting here (71), we get

$$\begin{aligned} F_a(\lambda) &\sim \frac{1}{2i\lambda}\left(e^{2i\lambda a}(1+O\left(\lambda^{-1}\right) - e^{2i\lambda a e^{i\omega}}(1+O\left(\lambda^{-1}\right)\right) = \\ &= \frac{1}{2i\lambda}e^{2i\lambda a}\left(1-e^{2i\lambda a(1-e^{i\omega})}\left(1+O\left(\lambda^{-1}\right)\right)\right),\ \lambda\to\infty, \end{aligned}$$

uniformly in $-\omega_\beta \le \arg\lambda \le \pi$. Together with (72) it implies the statement of the lemma. □

Now it remains to apply Theorem 2 from [11], according to which relation (70) follow a) — c). The proof is complete. □

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