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STRAIGHTENING EXPANSIONS OF GAS FROM VORTEX WITH LINEAR VELOCITY FIELD

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Abstract. In this paper we consider a submodel of the gas with a linear velocity field. It is described by a system of nonlinear differential equations with initial data. Several first integrals of the system are obtained. As a result the order of the system is reduced. An approximate solution of differential equations of the submodel is obtained for special initial data of the problem. Such a solution corresponds to world lines describing the radial expansion of the gas particles from the vortex. Trajectories of motion of gas particles are constructed.

Keywords: gas dynamics, submodel, approximate solution, radial expansion.

Introduction

Solution in the form of a linear field of velocities is fundamental for any equations of mechanics of continuous media. Such models were obtained in gas dynamics already by Dirichlet and Riemann [1], [2] in their study of dynamics of ellipsoidal figures of an ideal incompressible liquid. For a polytropic gas such a model was obtained by L.V. Ovsyannikov [3] and Dyson [4] in Lagrangian variables. Some integrals of the obtained system were found in this case. O.V. Lavrentyeva [5] considered the mathematical model of motion of an incompressible liquid ellipsoid with a free boundary, where the velocities of the liquid particles are linear functions of coordinates. She has also studied quality behaviour of the solution of such a model at large times. V.V. Pukhnachev in [6] considered plane motion of an ideal liquid with a linear field of velocities. He obtained the solution describing rotation of a liquid circle around the centre with constant angular velocity.

In the present paper we consider the model of gas motion with a linear field of velocities, described in [7], namely, SUBMODEL 1. The submodel is described by a system of ordinary differential equations with initial data. Several integrals of such a system and connections between these initial data are found. This allowed to reduce the number of parameters of the problem and lower the order of the system with the help of equivalence transformations. In a particular of initial data choice the obtained system was reduced to the Riccati equation. This allowed to find an approximate solution of the equations of the submodel. As a result we constructed world lines of gas particles for the given solution, which describe the gas particles expansion from the vortex.

1. Equations of SUBMODEL 1

In SUBMODEL 1 for the solution of equations of gas dynamics with a linear field of velociteis

$$\vec{u} = A(t)\vec{x} + \vec{u}_0(t), \tag{1.1}$$

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where $A = ||a_{ij}(t)||$ is matrix, $\vec{u}_0(t) = ||u_{01}, u_{02}, u_{03}||^T$, $\vec{x} = ||x^1, x^2, x^3||^T$ are vectors, we introduce supplementary variables by the formulae

$$B = A' + A^2, \quad \vec{v} = \vec{u}_0' + A\vec{u}_0, \quad \tau' = \tau \mathrm{tr} A, \tag{1.2}$$

and the equation of state has the form

$$p = \rho^\gamma h(\mathcal{S}) \pm \frac{a_0}{2} \ln \rho, \tag{1.3}$$

γ, a_0 are constants, $h(\mathcal{S})$ is the function of entropy $\mathcal{S}$. The density and the pressure are given by the formulae:

$$\rho = \frac{a_0 + \tau^{-\gamma}}{\vec{x} \cdot S\vec{x} + 2\vec{\xi} \cdot \vec{x} + \phi(t)}, \tag{1.4}$$

$$p = -\frac{a_0 + \tau^{-\gamma}}{2} \ln(\vec{x} \cdot S\vec{x} + 2\vec{\xi} \cdot \vec{x} + \phi) -$$

$$-\frac{a_0 + \tau^{-\gamma}}{\Delta} \omega^1 (v^2 \vec{s}_3 - v^3 \vec{s}_2 + \omega^1 \vec{v}) \int \frac{d\vec{x}}{\vec{x} \cdot S\vec{x} + 2\vec{\xi} \cdot \vec{x} + \phi} + p_0(t),$$

$$p_0' + (\ln \tau)' \gamma p_0 = (\ln \tau)' \gamma a_0 \ln(a_0 + \tau^{-\gamma}).$$

Then the entropy is determined from (1.3), $S = ||s_{ij}||$ is the symmetrical part of the matrix B. The vector $\vec{\xi}(t)$ is determined from the equation:

$$\Delta \vec{\xi}(t) = (s_{33} s_{22} - s_{23}^2) \vec{v} - \omega^1 (v^2 \vec{s}_3 - v^3 \vec{s}_2), \tag{1.5}$$

$$\Delta = (\omega^1)^2 + s_{33} s_{22} - s_{23}^2 \neq 0, \quad s_{33} s_{22} - s_{23}^2 \neq 0,$$

$\vec{s}_i$ is the column of the matrix S, v^j is a coordinate of the vector $\vec{v}$, ω^k is a coordinate of the vector $\vec{\omega}$, which determines the antisymmatrical part of the matrix B:

$$E < \vec{\omega} > = \left\| \begin{array}{ccc} 0 & -\omega^3 & \omega^2 \\ \omega^3 & 0 & -\omega^1 \\ -\omega^2 & \omega^1 & 0 \end{array} \right\|, \quad \omega^1 \neq 0,$$

that is

$$B = S + E < \vec{\omega} > . \tag{1.6}$$

The function $\phi(t)$ is determined by the relationship:

$$\Delta \phi(t) = (v^3)^2 s_{22} + (v^2)^2 s_{33} + 2 v^2 v^3 s_{23}. \tag{1.7}$$

Substituting of the solutions (1.1), (1.3), (1.4) in the equations of gas dynamics and taking into account the equalities (1.2), (1.5), (1.7) we obtain differential equations for determining of the matrix S, vectors $\vec{\omega}$, $\vec{v}$ [7]:

$$S' + SA + A^T S = (1 - \gamma + c_0(t)) (\ln \tau)' S, \quad \vec{\omega}' = A\vec{\omega} - \gamma (\ln \tau)' \vec{\omega}, \tag{1.8}$$

$$\vec{v}' + A^T \vec{v} + S\vec{u}_0 + \vec{\omega} \times \vec{u}_0 = \left((1-\gamma)\vec{v} + c_0(t) \vec{\xi} \right) (\ln \tau)', c_0(t) = \gamma \tau^\gamma \left(a_0 \tau^\gamma + 1 \right)^{-1},$$

and supplementary relationships

$$S\vec{\omega} = 0, \quad \vec{v} \cdot \vec{\omega} = 0, \tag{1.9}$$

which hold due to the equations of the submodel, if they are satisfied at the initial moment of time.

Thus, SUBMODEL 1 consists of 6 nonlinear first order differential equations (equations for A, $\vec{u}_0$, τ, S, $\vec{\omega}$, $\vec{v}$) to find 6 unknowns. The density, pressure and the equation of state are given. The submodel is completely determined.

We call differential equations for the matrix A, the function τ from (1.2) and the equation (1.8) for the matrix S and the vector $\vec{\omega}$ basic, since they are independent of the equations for the vectors $\vec{u}_0$ and $\vec{v}$. Let us introduce initial data for basic equations when $t=0$:

$$S(0) = S_0 = ||s^0_{ij}||, \quad \vec{\omega}(0) = \vec{\omega}_0 = ||\omega_{01}, \omega_{02}, \omega_{03}||^T, \quad \tau(0) = 1. \tag{1.10}$$

The expansion $A = S_A + E < \vec{\omega}_A >$, $S_A = S_A^T = ||s^A_{ij}||$, $\vec{\omega}_A = ||\omega^1_A, \omega^2_A, \omega^3_A||^T$ holds for the matrix A. Then the initial data for A when $t=0$ have the form:

$$\begin{gathered} S_A(0) = S_1 = ||s^1_{ij}||, \quad \vec{\omega}_A(0) = \vec{\omega}_1 = ||\omega_{11}, \omega_{12}, \omega_{13}||^T, \\ A(0) = S_1 + E < \vec{\omega}_1 > . \end{gathered} \tag{1.11}$$

The basic equations provide a nonlinear system of the 19th order with 18 parameters for initial data. To reduce the order of the system we find integrals of the system and reduce the number of parameters of the problem with the help of equivalent transformations.

2. Integrals

We act on the vector $\vec{\omega}$ by the matrix equation (1.2), taking into account (1.9), (1.6) and the identity $E < \vec{\omega} > \vec{\omega} = \vec{\omega} \times \vec{\omega} = 0$ and obtain the equation

$$A'\vec{\omega} + A^2\vec{\omega} = 0.$$

From the equation (1.8) for the vector $\vec{\omega}$ we find $A\vec{\omega}$ and substitute it into the last equality. We obtain a linear homogeneous differential equation for the vector $A\vec{\omega}$ which solution has the form:

$$A\vec{\omega} = \vec{\sigma}_1 \tau^{-\gamma}, \tag{2.1}$$

where $\vec{\sigma}_1$ is a constant vector. The integral (2.1) allows to find the solution for the linear inhomogeneous differential equation (1.8) for the vector $\vec{\omega}$ in the form:

$$\vec{\omega} = (\vec{\sigma}_1 t + \vec{\sigma}_2)\, \tau^{-\gamma}, \tag{2.2}$$

where $\vec{\sigma}_2$ is a constant vector. Taking into account (2.2) the integral (2.1) can be rewritten in the form of the linear integral:

$$A\,(\vec{\sigma}_1 t + \vec{\sigma}_2) = \vec{\sigma}_1. \tag{2.3}$$

The constant vectors $\vec{\sigma}_1$, $\vec{\sigma}_2$ are determined by the initial data (1.10), (1.11). When $t=0$ we obtain from (2.2), (2.3):

$$\vec{\sigma}_2 = \vec{\omega}_0, \quad \vec{\sigma}_1 = S_1\vec{\omega}_0 + \vec{\omega}_1 \times \vec{\omega}_0. \tag{2.4}$$

Initial data when $t=0$ should satisfy the following relationship for the SUBMODEL 1:

$$S_0\vec{\omega}_0 = 0. \tag{2.5}$$

3. Equivalence transformations

The basic equations admit some equivalence transformations, conserving the structure of equations but varying the initial data. Let us use this fact to reduce the number of parameters of the problem with initial data.

The variable t is not included explicitly in the basic equations. Therefore they admit transformations of the shift $t \to t + t_0$. Then due to the choice of t_0 we can achieve $\vec{\sigma}_1 \cdot \vec{\sigma}_2 = 0$ in the integral (2.3) and obtain from (2.4) a supplementary relationship of initial data:

$$\vec{\omega}_0 \cdot S_1\vec{\omega}_0 = 0. \tag{3.1}$$

The basic equations, the integral (2.3) admit rotation given by the constant orthogonal matrix O: $A \to O^TAO$ ($\vec{\omega}_A \to O^T\vec{\omega}_A$, $S_A \to O^TS_AO$), $S \to O^TSO$, $\vec{\omega} \to O^T\vec{\omega}$. Due to the choice of the matrix O, we turn the vectors $\vec{\omega}$, $\vec{\omega}_A$ at the initial moment of time into the position:

$$\vec{\omega}_0 = ||\omega_0, 0, 0||^T, \quad \omega_0 \neq 0, \quad \vec{\omega}_1 = ||\omega_{11}, \omega_{12}, 0||^T. \tag{3.2}$$

Then, we obtain:

$$s^0_{11} = s^0_{12} = s^0_{13} = 0, \quad s^1_{11} = 0$$

using the connections of the initial data (2.5), (3.1).

The basic equations, the integral (2.3) admit the dilation: $t \to \delta^{-1}t$, $A \to \delta A$, $S \to \delta^2 S$, $\vec{\omega} \to \delta^2\vec{\omega}$. Due to the choice of parameter of δ the dilation, the value ω_0 from (3.2) can be made ± 1.

There are no other linear equivalence transformations.

The integrals (2.2) and (2.3):

$$\begin{gathered}
\omega_1 = \omega_0\tau^{-\gamma},\\
\omega_2 = \omega_0 t s^1_{12}\tau^{-\gamma},\\
\omega_3 = \omega_0 t\left(s^1_{13} - \omega_{12}\right)\tau^{-\gamma};\\
a_{11} + a_{12}ts^1_{12} + a_{13}t\left(s^1_{13} - \omega_{12}\right) = 0,\\
a_{21} + a_{22}ts^1_{12} + a_{23}t\left(s^1_{13} - \omega_{12}\right) = s^1_{12},\\
a_{31} + a_{32}ts^1_{12} + a_{33}t\left(s^1_{13} - \omega_{12}\right) = s^1_{13} - \omega_{12},
\end{gathered}$$

reduce the order of the basic system. Equivalence transformations have reduced the number of parameters of the initial problem from 15 to 10 significant parameters. Taking into account the found integrals, the solution of the basic equations is reduced to the solution of the system:

$$\begin{aligned}
&\vec{a}'_3 + \vec{a}_3\left(a_{33} - a_{13}t(s^1_{13} - \omega_{12})\right) + \vec{a}_2\left(a_{23} - a_{13}ts^1_{12}\right) =\\
&= \vec{s}_3 + \vec{\omega}\times\vec{k} - a_{13}\left(\vec{\omega}_1\times\vec{i} + \vec{s}^1_1\right),\\
&\vec{a}'_2 + \vec{a}_2\left(a_{22} - a_{12}ts^1_{12}\right) + \vec{a}_3\left(a_{32} - a_{12}t(s_{13} - \omega_{12})\right) =\\
&= \vec{s}_2 + \vec{\omega}\times\vec{j} - a_{12}\left(\vec{\omega}_1\times\vec{i} + \vec{s}^1_1\right),\\
&s'_{ij} + \vec{s}_j\cdot\vec{a}_i + \vec{s}_i\cdot\vec{a}_j = f(\tau)(\ln\tau)'s_{ij}, \quad i,j = 1,2,3,\\
&\tau' = \tau\,\mathrm{tr}A,
\end{aligned} \tag{3.3}$$

where $\vec{i},\vec{j},\vec{k}$ is the Cartesian basis; $f(\tau) = 1 - \gamma + \gamma\tau^\gamma(a_0\tau^\gamma + 1)^{-1}$, $A = ||\vec{a}_1,\vec{a}_2,\vec{a}_3||$, $S = ||\vec{s}_1,\vec{s}_2,\vec{s}_3||$.

The initial data:

$$\begin{gathered}
A(0) = \begin{pmatrix} 0 & s^1_{12} & s^1_{13} + \omega_{12}\\ s^1_{12} & s^1_{22} & s^1_{23} - \omega_{11}\\ s^1_{13} - \omega_{12} & s^1_{23} + \omega_{11} & s^1_{33}\end{pmatrix},\\
S_0 = \begin{pmatrix} 0 & 0 & 0\\ 0 & s^0_{22} & s^0_{23}\\ 0 & s^0_{23} & s^0_{33}\end{pmatrix}, \tau(0) = 1.
\end{gathered} \tag{3.4}$$

We haven't found further integrals of the system (3.3) different from (2.2), (2.3),therefore, it seems impossible to solve the system (3.3) with arbitrary significant parameters analytically. Therefore, let us consider the system (3.3) with special values of initial data.

4. A plane model

The system (3.3) is written for the matrices A and S of the third order. The system (3.3) has supplementary integrals for special initial data.

Definition 1. *If the matrices A and S have the form*

$$A = \begin{pmatrix} 0 & 0\\ 0 & \bar{A}\end{pmatrix}, \quad S = \begin{pmatrix} 0 & 0\\ 0 & \bar{S}\end{pmatrix}, \quad \bar{A} = \begin{pmatrix} a_{22} & a_{23}\\ a_{32} & a_{33}\end{pmatrix}, \quad \bar{S} = \begin{pmatrix} s_{22} & s_{23}\\ s_{23} & s_{33}\end{pmatrix},$$

we say that they determine a plane (two-dimensional) case of the system (3.3).

Theorem 1. *If the initial data for the matrix A are chosen in the form*

$$S_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & s_{22}^1 & s_{23}^1 \\ 0 & s_{23}^1 & s_{33}^1 \end{pmatrix}, \quad \vec{\omega}_1 = ||\omega_{11}, 0, 0||^T, \quad \omega_{11} \neq 0,$$

then the system (3.3) *is reduced to a plane case.*

Proof. The conditions of the Theorem 1 mean that in the initial data (3.4) it is sufficient to set $s_{12}^1 = s_{13}^1 = 0$, $\omega_{12} = 0$. Then we obtain $a_{11} = a_{21} = a_{31} = 0$, $\omega_2 = \omega_3 = 0$ from the integrals (2.2), (2.3) using (2.4). It results from (1.9) that $s_{11} = s_{12} = s_{13} = 0$. It remains to show that $a_{12} = a_{13} = 0$. For these elements we write the Cauchy problem from (3.3):

$$a_{12}' + a_{12}a_{22} + a_{13}a_{32} = 0, \quad a_{13}' + a_{12}a_{23} + a_{13}a_{33} = 0, \quad a_{13}(0) = a_{12}(0) = 0.$$

The zero solution is the solution of the latter problem, and using the uniqueness of the solution of the Cauchy problem it is unique in case of any functions $a_{22}(t), a_{23}(t), a_{32}(t), a_{33}(t)$. Consequently, $a_{12}(t) = a_{13}(t) = 0$, which was to be proved.

Let us rewrite the system (3.3) for a plane case due to expansion of the matrix $A = S_A + E < \vec{\omega}_A >$:

$$\begin{aligned}
&\omega_A' + \omega_A(\ln\tau)' = \omega_0\tau^{-\gamma}, \\
&(s_{22}^A)' + (s_{22}^A)^2 + (s_{23}^A)^2 - (\omega_A)^2 = s_{22}, \\
&(s_{23}^A)' + s_{23}^A(\ln\tau)' = s_{23}, \\
&(s_{33}^A)' + (s_{33}^A)^2 + (s_{23}^A)^2 - (\omega_A)^2 = s_{33}, \\
&s_{22}' + 2(s_{22}s_{22}^A + s_{23}s_{23}^A + s_{23}\omega_A) = f(\tau)(\ln\tau)'s_{22}, \\
&s_{23}' + s_{23}(\ln\tau)' + s_{23}^A(s_{22} + s_{33}) + \omega_A(s_{33} - s_{22}) = f(\tau)(\ln\tau)'s_{23}, \\
&s_{33}' + 2(s_{23}s_{23}^A + s_{33}s_{33}^A - s_{23}\omega_A) = f(\tau)(\ln\tau)'s_{33}, \\
&(\ln\tau)' = s_{22}^A + s_{33}^A,
\end{aligned} \tag{4.1}$$

with the initial data:

$$\begin{aligned}
\omega_A(0) = \omega_{11},\ s_{22}^A(0) = s_{22}^0,\ s_{33}^A(0) = s_{33}^0,\ s_{23}^A(0) = s_{23}^0, \\
s_{22}(0) = s_{22}^1,\ s_{23}(0) = s_{23}^1,\ s_{33}(0) = s_{33}^1,\ \tau(0) = 1.
\end{aligned} \tag{4.2}$$

We pass from the variables ω_A, s_{22}^A, s_{23}^A, s_{33}^A, s_{22}, s_{23}, s_{33}, τ of the system (4.1) of the 8th order to the variables ω_A, s_{23}^A, s_{23}, $\mathrm{tr}S_A$, τ, $\mathrm{tr}S$, $|S|$, $|S_A|$, where $|S|$ is a determinant of the matrix S:

$$\begin{aligned}
&(\tau\omega_A)' = \omega_0\tau^{1-\gamma}, \\
&(s_{23}^A\tau)' = s_{23}\tau, \\
&s_{23}' + s_{23}^A\mathrm{tr}S - \omega_A(s_{22} - s_{33}) = (f(\tau) - 1)(\ln\tau)'s_{23}, \\
&\tau''\tau^{-1} = \mathrm{tr}S + 2|S_A| + 2\omega_A^2, \\
&(\tau|S_A|)' = \omega_A^2\tau' + \tau(G - 2s_{23}^A s_{23}), \\
&|S|'\tau = 2|S|\tau'(f(\tau) - 1), \\
&(\mathrm{tr}S)' = f(\tau)\mathrm{tr}S\tau'\tau^{-1} - 2(F + 2s_{23}^A s_{23}), \\
&\tau' = \tau\mathrm{tr}S_A,
\end{aligned} \tag{4.3}$$

where $G = s_{33}^A s_{22} + s_{22}^A s_{33}$, $F = s_{22}s_{22}^A + s_{33}s_{33}^A$ (the quantities F, G satisfy the relation of relationship $F + G = \mathrm{tr}S\mathrm{tr}S_A$),

$$\begin{aligned} s_{22} &= \frac{1}{2}\mathrm{tr}S + \sqrt{\frac{1}{4}(\mathrm{tr}S)^2 - s_{23}^2 - |S|}, \quad s_{33} = \mathrm{tr}S - s_{22}, \\ s_{22}^A &= \frac{1}{2}\mathrm{tr}S_A \pm \sqrt{\frac{1}{4}(\mathrm{tr}S_A)^2 - (s_{23}^A)^2 - |S_A|}, \quad s_{33}^A = \mathrm{tr}S_A - s_{22}^A. \end{aligned} \tag{4.4}$$

The sign $+$ is chosen in the first equation because the sign $-$ transforms to the sign $+$ after the transformation $s_{22} \leftrightarrow s_{33}$.

The system (4.3) has the integral

$$|S| = \frac{|S_0|(a_0\tau^\gamma + 1)^{2/a_0}}{(a_0 + 1)^{2/a_0}\tau^{2\gamma}}, \quad |S_0| = s_{22}^0 s_{33}^0 - (s_{23}^0)^2. \tag{4.5}$$

Since (4.3) does not contain t explicitly, we make the substitution

$$\tau' = \lambda(\tau) \neq 0, \tag{4.6}$$

λ is some function from the variable τ. Then $dt = \lambda^{-1}d\tau$. Therefore, the order of the system (4.3) is reduced by 2:

$$\begin{aligned} &(\tau\omega_A)_\tau = \omega_0\tau^{1-\gamma}\lambda^{-1}, \\ &(\tau s_{23}^A)_\tau = s_{23}\tau\lambda^{-1}, \\ &\lambda(s_{23})_\tau + s_{23}^A\mathrm{tr}S - \omega_A(s_{22} - s_{33}) = (f(\tau) - 1)\lambda\tau^{-1}s_{23}, \\ &\lambda\lambda_\tau\tau^{-1} = \mathrm{tr}S + 2|S_A| + 2\omega_A^2, \\ &(\tau|S_A|)_\tau = \omega_A^2 + \tau\lambda^{-1}(G - 2s_{23}^A s_{23}), \\ &\tau(\mathrm{tr}S)_\tau = f(\tau)\mathrm{tr}S - 2\tau\lambda^{-1}(F + 2s_{23}s_{23}^A). \end{aligned} \tag{4.7}$$

Here, $\omega_A \neq 0$ as a result of the first equation.

5. A particular solution of the plane model

Let in the system (4.7)

$$s_{23}^A = 0.$$

Then the system (4.7) has one more integral. It results from the first and the second equations (4.7) that

$$s_{23} = 0, \quad s_{22} = s_{33}, \quad \omega_A \neq 0,$$

$F = G = s_{22}\lambda\tau^{-1}$. Then the 6th equation is integrable:

$$s_{22} = \frac{s_{22}^0(a_0\tau^\gamma + 1)^{1/a_0}}{(a_0 + 1)^{1/a_0}\tau^\gamma}.$$

The system (4.7) takes the form:

$$(\tau\omega_A)_\tau = \omega_0\tau^{1-\gamma}\lambda^{-1}, \quad \lambda\lambda_\tau = 2(s_{22} + |S_A| + \omega_A^2)\tau, \quad (\tau|S_A|)_\tau = \omega_A^2 + s_{22}. \tag{5.1}$$

It results from the equations for λ, $|S_A|$ that $(\lambda^2)_\tau = 4(\tau^2|S_A|)_\tau$. We obtain the integral of the system (5.1):

$$\lambda^2 = 4\tau^2|S_A| + k, \quad k = (s_{22}^1 - s_{33}^1)^2. \tag{5.2}$$

It determines $|S_A|$. The equations (5.1) have been reduced to two equations:

$$\begin{aligned} &\lambda c_\tau = \omega_0\tau^{1-\gamma}, \\ &2\tau\lambda\lambda_\tau = \lambda^2 - k + 4c^2 + N_0(a_0\tau^\gamma + 1)^{1/a_0}\tau^{2-\gamma}, \\ &c = \tau\omega_A, \quad N_0 = 4s_{22}^0(a_0 + 1)^{-1/a_0} \end{aligned} \tag{5.3}$$

with the initial data

$$c(1) = \omega_{11}, \quad \lambda(1) = s^1_{22} + s^1_{33}. \tag{5.4}$$

Let us find approximate solutions for the equations (5.3).

6. Approximate solutions

The first equation (5.3) admits the dilation $\tau = T\tau_1$, $c = T^{1-\gamma/2}c_1$, $\lambda = T^{1-\gamma/2}\lambda_1$. If we apply the dilation to the second equation (5.3), we obtain

$$2\tau_1\lambda_1(\lambda_1)_{\tau_1} = \lambda_1^2 - kT^{\gamma-2} + 4c_1^2 + N_0(a_0T^\gamma\tau_1^\gamma + 1)^{1/a_0}\tau_1^{2-\gamma}.$$

We set

$$T^2 = k, \quad T^\gamma = \varepsilon, \tag{6.1}$$

where ε is a small parameter, γ is a fixed constant, k is a small parameter. Let us expand λ_1 and c_1 in the series in the degrees ε:

$$\lambda_1 = \lambda_0 + \varepsilon\lambda_{01} + \ldots, \quad c_1 = c_0 + \varepsilon c_{01} + \ldots. \tag{6.2}$$

When $\varepsilon = 0$ we obtain the equation of zero approximation:

$$\lambda_0 c_{0\tau_1} = \omega_0\tau_1^{1-\gamma}, \quad \tau_1(\lambda_0^2)_{\tau_1} = \lambda_0^2 + 4c_0^2 + N_0\tau_1^{2-\gamma},$$

which, obviously, admits the dilation. Introducing the change of variables with the help of the invariants of dilation [8]:

$$\lambda_0 = \mu\tau_1^{1-\gamma/2}, \quad c_0 = 2^{-1}g\tau_1^{1-\gamma/2}, \quad s = \ln\tau_1, \tag{6.3}$$

we obtain the autonomous system

$$g_s + g(1-\gamma/2) = 2\omega_0\mu^{-1}, \quad (\mu^2)_s + (1-\gamma)\mu^2 = g^2 + N_0. \tag{6.4}$$

This results in the Abel equation:

$$\frac{d\mu}{dg} = \frac{N_0 + g^2 + (\gamma-1)\mu^2}{4\omega_0 + g\mu(\gamma-2)}. \tag{6.5}$$

The equation (6.5) admits the discrete symmetries: $\mu \to -\mu$, $g \to -g$; $\omega_0 \to -\omega_0$, $\mu \to -\mu$. Consequently, integral curves of the equation (6.5) can be constructed in the semiplane $g \geq 0$ with $\omega_0 = +1$. We consider further the simple case $\gamma = 2$, $N_0 = 1$. The equation (6.5) takes the form:

$$4\frac{d\mu}{dg} = 1 + g^2 + \mu^2. \tag{6.6}$$

Proposition 1. *Any integral curve of the equation* (6.6) *has, as $\mu \to \infty$, its own asymptote $g = g_0$, where g_0 is a constant, $0 < g_0 < \infty$ and it is represented by a convergent series*

$$g = g_0 - \frac{4}{\mu} + \frac{4(1+g_0^2)}{3\mu^3} + O(\mu^{-4}) \tag{6.7}$$

или

$$\frac{1}{\mu} = -\frac{(g-g_0)}{4} - \frac{(1+g_0^2)}{192}(g-g_0)^3 - O\left((g-g_0)^4\right). \tag{6.8}$$

Proof: Due to the inequality $1 + g^2 + \mu^2 > 1 + \mu^2$, the solution of the equation $4\frac{d\overline{\mu}}{dg} = 1 + \overline{\mu}^2$, which is equal to $\overline{\mu} = \operatorname{tg}\left(\frac{g}{4} + \frac{C}{4}\right)$, $0 < C < 2\pi$ is some constant, provides a lower bound of the solution of the equation (6.6) ($\mu > \overline{\mu}$ in case of equal initial data). Since $\overline{\mu} \to \infty$ when $g \to 2\pi - C$, then $\mu \to \infty$ when $g \to g_0 \leqslant 2\pi$. Therefore, the solution $\mu^{-1}(g)$ can be expanded into a series by integral degrees of $g - g_0$. Since the right-hand side of the equation (6.6) is an analytical function, then according to the Cauchy-Kovalevskaya theorem on existence and uniqueness the function μ is expandable into a convergent power series.

Let us show that the equation (6.6) has the solution in the form of the series (6.7). For this purpose we substitute the solution of the form

$$g = g_0 + \frac{g_1}{\mu} + \frac{g_2}{\mu^2} + \frac{g_3}{\mu^3} + \dots$$

into (6.6)

$$-4 = \left(1 + \mu^2 + g_0^2 + 2g_0 \sum_{j\geq 1} \frac{g_j}{\mu^j} + \sum_{k\geq 2} \frac{1}{\mu^k} \sum_{j\geq 1}^{k-1} g_j g_{k-j}\right) \sum_{j\geq 1} \frac{j g_j}{\mu^{j+1}}.$$

Equating coefficients of the series of different degrees μ we obtain the following expressions for the coefficients of the series:

$$g_1 = -4, \quad g_2 = 0, \quad g_3 = \frac{4(1+g_0^2)}{3}, \dots .$$

When $k > 3$

$$(k+1)g_{k+1} + (1+g_0^2)(k-1)g_{k-1} + 2g_0 \sum_{i\geq 1}^{k-2} g_i g_{k-1-i}(k-1-i) +$$

$$+ \sum_{i=2}^{k-2} \left(\sum_{j\geq 1}^{i-1} g_j g_{i-j} \right) (k-i-1) g_{k-i-1} = 0.$$

Coefficients of the series (6.7) are determined via the previous coefficients.

Inversion of the series (6.6) provides the series (6.7).

Let us construct a picture of integral curves of the equation (6.6) (Figure 1).

Since the right-hand side of the equation (6.6) is positive, the integral curves increase in the semiplane $g \geq 0$. Let us find the second derivative:

$$\frac{d^2\mu}{dg^2} = 2\left(g + \frac{\mu}{4}\left(1 + g^2 + \mu^2\right)\right).$$

All the points of inflection lie on the curve $4g + \mu\left(1 + g^2 + \mu^2\right) = 0$. This curve has a point of minimum $g = \sqrt{(1+\sqrt{17})/2}$, $\mu = -2/g$. A part of an integral curve lying above the line of inflections is convex downwards; the one lying below is convex upwards. When $g = 0$: $4\dfrac{d\mu}{dg} = 1 + \mu^2$, the higher is $|\mu|$, the closer is the slope angle of the tangent to $\pi/2$ in the points of the straight line $g = 0$. When $\mu = 0$: $4\dfrac{d\mu}{dg} = 1 + g^2$, the higher is $|g|$, the closer is slope angle of the tangent to $\pi/2$ in the points of the straight line $\mu = 0$. When $\mu = g = 0$ the tangent of the slope angle of the integral curve is equal to $1/4$. Integral curves passing through the points $(0, \mu_0)$, $(0, -\mu_0)$ prolong each other after the reflection with respect to the origin of coordinates.

Every integral curve has an asymptote $g = g_0$, g_0 is a constant: $\mu = F(g, g_0)$. Let us choose initial data for this curve on the axis μ, i.e. in the point $(0, \mu_0)$. Then there is a functional relationship between μ_0 and g_0:

$$\mu_0 = F(0, g_0).$$

Let us choose an integral curve corresponding to zero initial data $\mu(0) = 0$:

$$\mu = F(g). \tag{6.9}$$

The further solution is searched for this curve. The numerical calculations provide $g_0 \simeq 3,65$.

For the further finding unknown functions it is necessary to determine initial data of the problem (6.4) when $t = 0$. Since the function $\mu(\tau_1)$ is the zero approximation for $\lambda(\tau_1)$, then

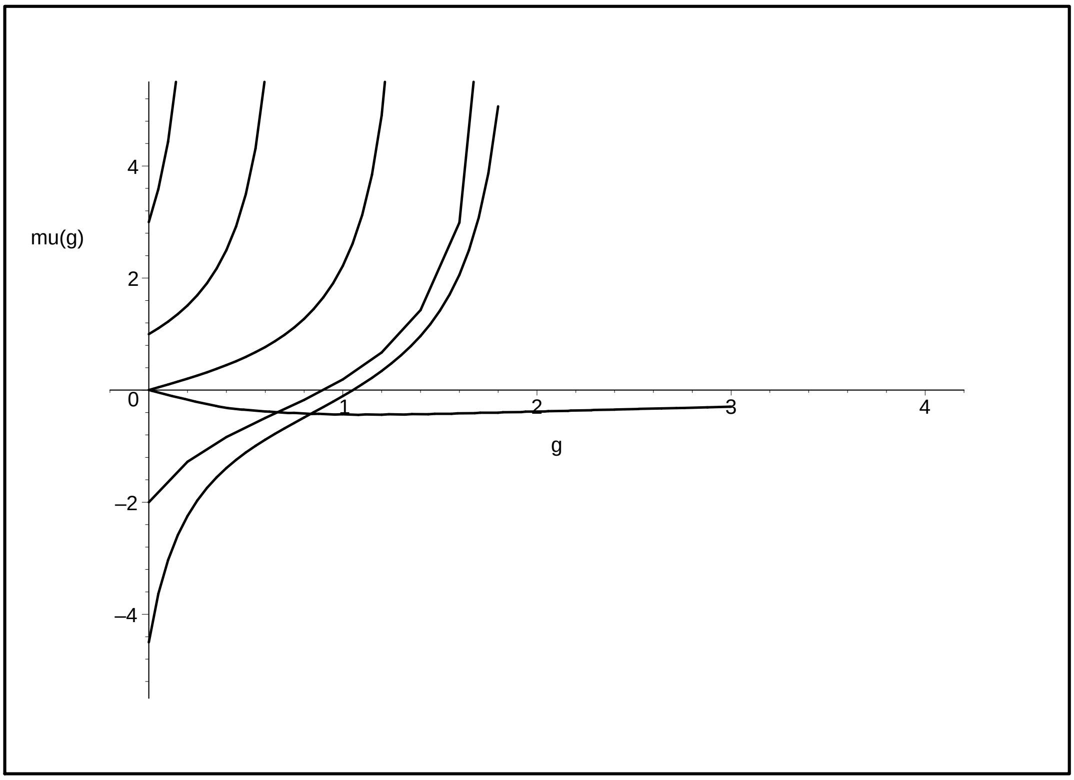

Рис. 1. Integral curves of the equation (6.6)

$\mu(1/\sqrt{\varepsilon}) \sim s^1_{22} + s^1_{33} = \alpha$, and since $g(\tau_1)$ is the zero approximation for $c(\tau)$, then $g(1/\sqrt{\varepsilon}) \sim 2\omega_{11} = \beta$ (see (5.4), (6.2), (6.3)).

Determine the values of the variable τ at the boundary points of the chosen curve. The function $g(\tau_1)$ satisfies the equation (6.4):

$$\mu g_\mu d\mu = 2\tau_1^{-1} d\tau_1 = 2\tau^{-1} d\tau. \tag{6.10}$$

Integrating (6.10) in τ from 1 to τ we obtain:

$$2\ln\tau = \int_\alpha^\mu \frac{\mu d\mu}{1+g^2+\mu^2} \geq \int_\alpha^\mu \frac{\mu d\mu}{1+g_0^2+\mu^2} = \frac{1}{2}\ln|\mu^2+g_0^2+1|\ |_\alpha^\mu\ .$$

Hence $\tau \to \infty$ as $\mu \to \infty$.

If we integrate (6.10) in τ from τ to 1, we obtain:

$$\tau = \exp\left(-\frac{1}{2}\int_g^\beta F(g)dg\right) \to \tau_0(\omega_{11}),$$

when $g \to 0$, τ_0 is a finite number, contained within the interval $0 < \tau_0 < 1$. Validity of the following proposition results from the written above.

Proposition 2. *When the point (g,μ) moves along the curve (6.9) from the point (0, 0) to the point (g_0,∞) the quantity τ varies from $\tau_0 > 0$ to ∞.*

Let us determine the functions $A(\tau)$, $\vec{u}_0(\tau)$. The function $g = G(\tau)$ is determined from the first equation (6.4) implicitly by the relationship

$$\int_{g_1}^g F(g)dg = 2\ln\frac{\tau}{\tau_1}, \quad \mu = F(G(\tau)) = M(\tau) \sim \lambda(\tau), \tag{6.11}$$

where $g_1 = g(\tau_1)$, $\tau_1 \in (\tau_0;\infty)$.

Dependence of the functions M and G on the variable τ is presented on Figure 2 and Figure 3.

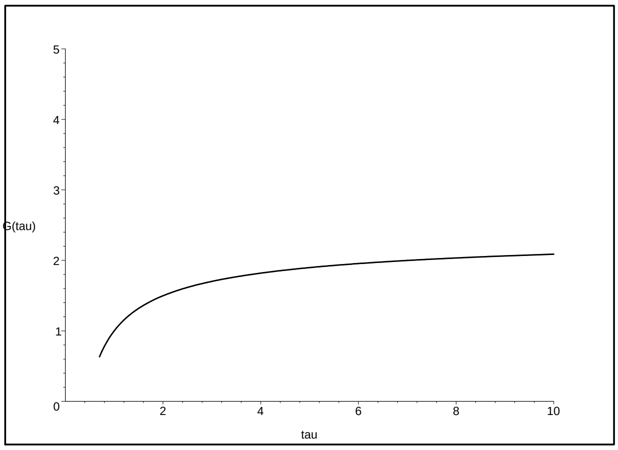

FIGURE 2. Graph of the function $G(\tau)$

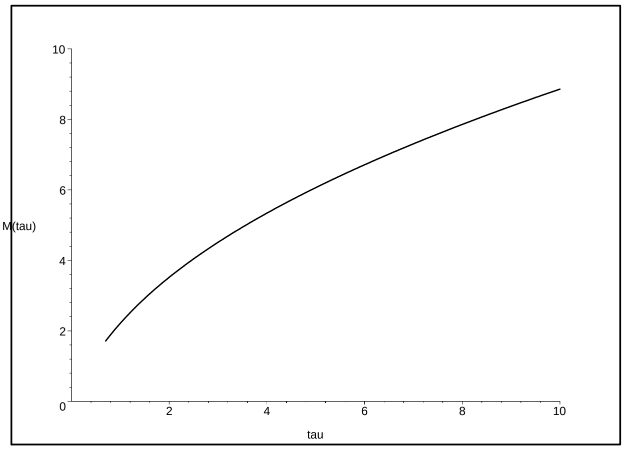

FIGURE 3. Graph of the function $M(\tau)$

It results from (5.3), (6.2), (6.3) that

$$\omega_A \sim \frac{G(\tau)}{2\tau}.$$

It results from (4.4), (5.2), (6.1) that $s_{22}^A = \frac{1}{2}(\lambda + \sqrt{\varepsilon})\tau^{-1}$, $s_{33}^A = \frac{1}{2}(\lambda - \sqrt{\varepsilon})\tau^{-1}$, where the sign $\pm$ in (4.4) is replaced by $+$ due to symmetry of the function $\mu = F(g)$ with respect to the origin of coordinates. The elements of the matrix A from (1.1) are determined by

$$-A^T = A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & s_{22}^A & -\omega_A \\ 0 & \omega_A & s_{33}^A \end{pmatrix} = \frac{D}{2\tau}, \quad D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & M(\tau) + \sqrt{\varepsilon} & -G(\tau) \\ 0 & G(\tau) & M(\tau) - \sqrt{\varepsilon} \end{pmatrix}. \tag{6.12}$$

The equations (1.2), (1.8) hold for the vector $\vec{u}_0$. After substitution of (1.2) into (1.8) we obtain:

$$\vec{u}_0'' + 2A'\vec{u}_0 = \left(\frac{2\tau^2}{a_0\tau^2 + 1}\vec{\xi} - \vec{u}_0' - A\vec{u}_0\right)\frac{\tau'}{\tau}.$$

Let us proceed to differentiating by τ according to the formula (4.6) $\tau' = \lambda(\tau) \sim \mu(\tau)$:

$$\mu\vec{u}_{0\tau\tau} + \left(\mu_\tau + \frac{\mu}{\tau}\right)\vec{u}_{0\tau} + \left(2A_\tau + \frac{1}{\tau}A\right)\vec{u}_0 = \frac{2\tau}{a_0\tau^2 + 1}\vec{\xi}, \tag{6.13}$$

where due to (1.5)

$$\vec{\xi}(1 + 16(a_0\tau^2 + 1)^{-2/a_0}) = \mu\vec{u}_{0\tau} + A\vec{u}_0 - 4(a_0\tau^2 + 1)^{-1/a_0}\left\| \begin{matrix} 0 \\ \mu(\vec{u}_{02})_\tau + (A\vec{u}_0)_2 \\ -\mu(\vec{u}_{03})_\tau - (A\vec{u}_0)_3 \end{matrix} \right\|,$$

$$2A_\tau + \frac{1}{\tau}A = \frac{1}{\tau^2}\left(\frac{2}{F(G)}H - \frac{1}{2}D\right), \quad H(G) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & F_g(G) & -1 \\ 0 & 1 & F_g(G) \end{pmatrix}.$$

We expand the variable $\tau = \sqrt{\varepsilon}\tau_1$ for the approximate solution. The equation (6.13) takes the form:

$$\mu\vec{u}_{0\tau_1\tau_1} + \left(\mu_{\tau_1} + \frac{\mu}{\tau_1}\right)\vec{u}_{0\tau_1} + \left[\frac{2}{\tau_1^2 F(G)}H(G) - \frac{1}{2\tau_1^2}D(\tau_1)\right]\vec{u}_0 = 0 \tag{6.14}$$

with precision to ε.

Initial data at $\tau = 1$ or $\tau_1 = \varepsilon^{-1/2}$ can be taken in the form

$$\vec{u}_0 = \vec{u}_{00}, \quad \vec{u}_{0\tau} = \vec{u}_{01}.$$

If initial data is equal to zero, then the solution is equal to zero $\vec{u}_0 = 0$.

7. The time of existence of the solution

Dependence of τ on the time t is determined from the solution of the problem:

$$\tau_t' = F(g), \quad \tau(0) = 1, \tag{7.1}$$

where the right-hand part of the equation is determined by the formula (6.11).

Let us describe the behaviour of τ from t near the points $\tau = \tau_0$, $\tau = 1$, $\tau = \infty$.

In the neighbourhood of the point $\tau = \tau_0$ the function $F(0) = 0$ (see Figure 1). In this point the function F is expanded into the series $F = \dfrac{g}{4} + O(g^3)$. We obtain from (6.11)

$$2\ln\frac{\tau}{\tau_0} = \frac{g^2}{8} + O(g^4).$$

Let us introduce the small parameter $\dfrac{\tau}{\tau_0} - 1 = \delta$. Then the approximate equality:

$$g \sim 4\delta^{1/2} = 4\left(\frac{\tau}{\tau_0} - 1\right)^{1/2}$$

results from the last equation. Then the approximate equality

$$t - T_0 \sim \int_{\tau_0}^{\tau} \frac{4d\tau}{g(\tau)} \sim 2\tau_0\left(\frac{\tau}{\tau_0} - 1\right)^{1/2}, \tag{7.2}$$

where T_0 is the beginning of the time reading, results from the differential equation (7.1).

In the neighbourhood of the point $\tau = 1$ the function $F(\beta) = \alpha$. Let us expand $F(g)$ into series in the point (β, α), taking only two first terms of the series: $F(g) \sim ng + m$, $4n = 1 + \alpha^2 + \beta^2$, $m = \alpha - \beta n$. We obtain from (6.11) that $2\ln\tau \sim \dfrac{n}{2}(g^2 - \beta^2) + m(g - \beta)$, then $F(g) = \alpha + \dfrac{2n}{\alpha}(\tau - 1) + O\left((\tau - 1)^2\right)$.

The equation (7.1) provides an approximate solution in the neighbourhood of $t = 0$ (or $\tau = 1$)

$$t = \frac{\alpha}{2n}\ln\left|1 + \frac{2n(\tau - 1)}{\alpha^2}\right| \sim \frac{\tau - 1}{\alpha}. \tag{7.3}$$

When $\tau \to \infty$, $g \to g_0$ it results from (6.8) and (6.11) that

$$-2^{-1}\ln\frac{\tau}{\tau_1} = \int_{g_1}^{g}\left(\frac{1}{g - g_0} - \frac{1 + g_0^2}{48}(g - g_0) + O(\delta^2)\right)dg = \ln\delta - \ln C + O(\delta^2),$$

where $g_0 - g = \delta$ is a small parameter, C is some constant.

Consequently,

$$g_0 - g \sim \frac{c}{\tau^{1/2}}, \tag{7.4}$$

where c is some constant.

In this case the differential equation (7.1) takes the form:

$$dt \sim \frac{c}{\tau^{1/2}}d\tau \implies t - t_1 \sim 2c(\tau^{1/2} - \tau_1^{1/2}) \to \infty \quad \text{when} \quad \tau \to \infty. \tag{7.5}$$

The formulae (7.2), (7.3), (7.5) describe the behaviour of the function $t(\tau)$ in the neighbourhood of the point $\tau = \tau_0$, $\tau = 1$, $\tau \to \infty$.

The numerical calculations for the chosen $\alpha = 0,3411$, $\beta = 1$ showed that $\tau_0 = 0,7$ and the graph of the function $t(\tau)$ is presented on Figure 4.

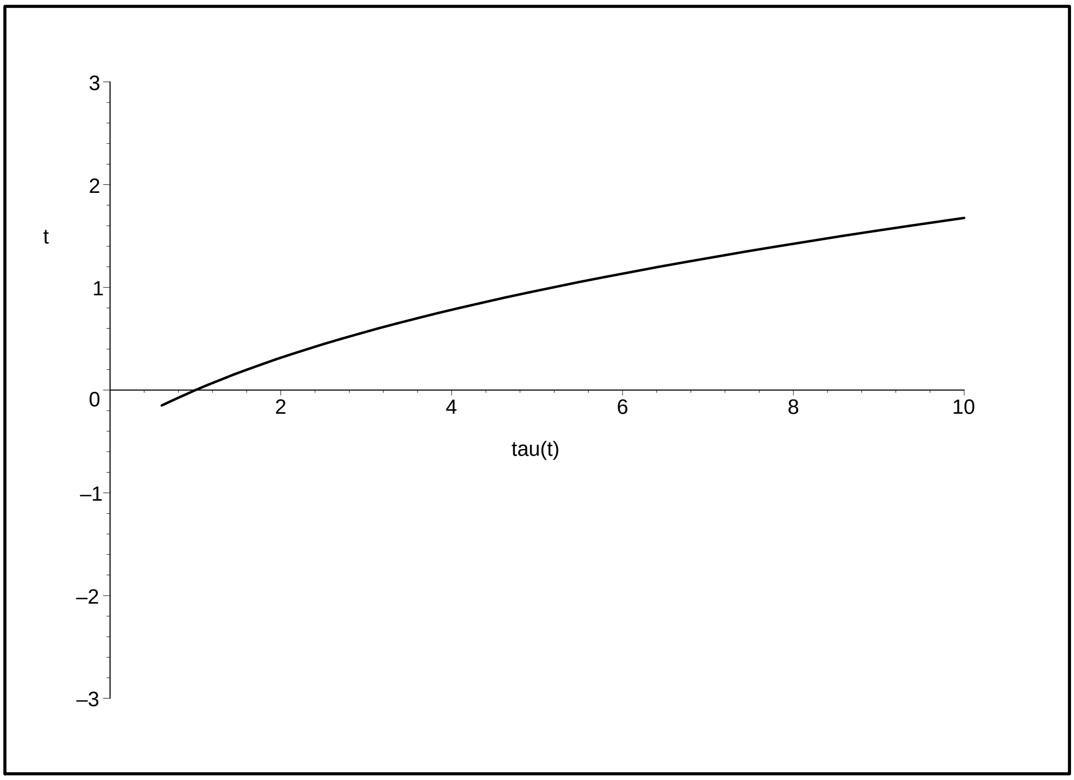

FIGURE 4. Graph of the function $\tau(t)$

8. World lines of particles

World lines are given by the equation [9]

$$\frac{d\vec{x}}{dt} = A\vec{x} + \vec{u}_0,$$

where the matrix A is defined by the formula (6.12), the vector $\vec{u}_0$ is given by the formula (6.14). Let us proceed to differentiating in τ:

$$M(\tau)\frac{d\vec{x}}{d\tau} \sim A\vec{x} + \vec{u}_0.$$

In the coordinate form we have:

$$\begin{aligned} M(\tau)x_\tau &= u_{01}, \\ M(\tau)y_\tau &= \frac{1}{2\tau}\Big((M(\tau) + \sqrt{\varepsilon})y - G(\tau)z\Big) + u_{02}, \\ M(\tau)z_\tau &= \frac{1}{2\tau}\Big(G(\tau)y + (M(\tau) - \sqrt{\varepsilon})z\Big) + u_{03}, \end{aligned} \tag{8.1}$$

where $M(\tau)$ and $G(\tau)$ are calculated by the formulae (6.11). Solving the system (8.1) numerically when $\epsilon = 0,1$, $\tau \in [0,7;10]$, $G(1) = 1$, $M(1) = 0,3411$, we obtain world lines. Trajectories of particles are presented on Figure 5. Every particle of gas moves along by its own trajectory. Particles belonging to one trajectory at the initial moment of time move along by it. The particle velocity makes a turn during its motion. It is not clear from Figure 5 how trajectories behave when $\tau \to \infty$. Let us clarify it with the help of expansion into the series of functions $G(\tau)$, $M(\tau)$, $x(\tau)$, $y(\tau)$ when $\tau \to \infty$.

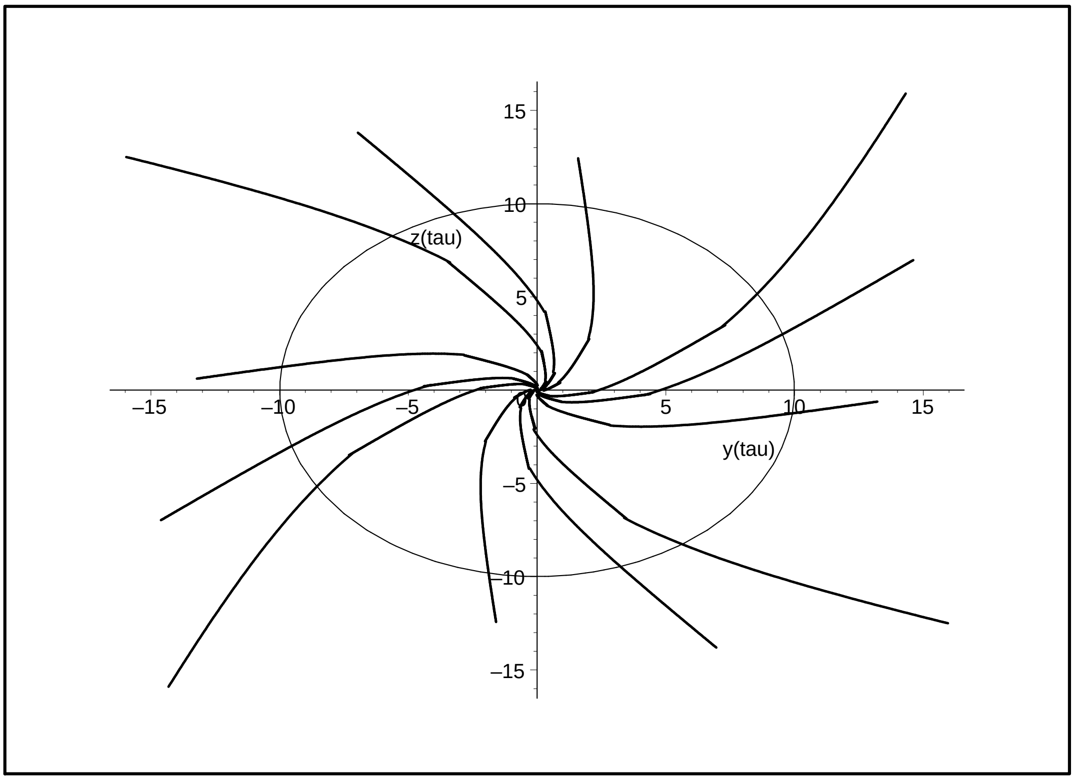

FIGURE 5. Particles trajectory

The system (8.1) splits in polar coordinates $y = r\cos\varphi$, $z = r\sin\varphi$ under zero initial conditions for the equation (6.14):

$$\begin{aligned} &2\tau r^{-1} r_\tau = 1 + \sqrt{\varepsilon} M^{-1}\cos 2\varphi, \\ &2\tau M \varphi_\tau = G - \sqrt{\varepsilon}\sin 2\varphi. \end{aligned} \tag{8.2}$$

The last equation is the Riccati equation after the substitution $\nu = \operatorname{tg}\varphi$.

The solution with respect to φ is periodical with the period π. Therefore, the initial data $\varphi(1) = \varphi_0$ can be taken within the interval $\varphi_0 \in (0;\pi)$. The initial data $r(1) = r_0$ are determined by an arbitrary constant r_0. When $r_0 = 0$ we have the solution $r = 0$. The particle is not moving.

An approximate solution for φ is as follows

$$\varphi = \varphi_1 + \sqrt{\varepsilon}\varphi_2 + O(\varepsilon), \tag{8.3}$$

with the initial data

$$\varphi_0 = \varphi_{01} + \sqrt{\varepsilon}\varphi_{02},$$

where

$$\begin{gathered} 2\tau M \varphi_{1\tau} = G, \quad 2\tau M\varphi_{2\tau} = -\sin 2\varphi_1 \quad \Longrightarrow \\ \Longrightarrow \varphi_1 = \widetilde{\varphi}_{01} + \int \frac{G}{2\tau M} d\tau, \quad \varphi_2 = \widetilde{\varphi}_{02} - \int \frac{\sin 2\varphi_1}{2\tau M} d\tau, \end{gathered}$$

where $\widetilde{\varphi}_{01}$, $\widetilde{\varphi}_{02}$ are constants matching the limits of integrating.

The approximate solution (8.2) for r has the form

$$r = r_0\sqrt{\tau}\exp\left\{\frac{\sqrt{\varepsilon}}{2}\int \frac{\cos 2\varphi}{M\tau} d\tau\right\} = r_0\sqrt{\tau}\left(1 + \frac{\sqrt{\varepsilon}}{2}\int\frac{\cos 2\varphi_1}{M\tau} d\tau\right) + O(\varepsilon). \tag{8.4}$$

We determine the function $M(\tau)$ when $\tau \to \infty$ by the formulae (6.11), (6.8) and (7.4):

$$M(\tau) = \frac{4\tau^{1/2}}{c}.$$

We find the function φ by the formulae (8.3):

$$\varphi \sim \overline{\varphi}_0 - \frac{c}{4\tau^{1/2}}\left(g_0 - \sqrt{\epsilon}\sin 2\overline{\varphi}_{01}\right),$$

where $\overline{\varphi}_0 = \widetilde{\varphi}_{01} + \sqrt{\epsilon}\widetilde{\varphi}_{02} + \dfrac{c}{4\tau_1^{1/2}}(g_0 - \sqrt{\epsilon}\sin 2\overline{\varphi}_{01})$. We find the function $r(\tau)$ by the formula (8.4):

$$r = r_0\left(\sqrt{\tau} - \frac{c\sqrt{\epsilon}}{4}\left(\cos 2\overline{\varphi}_{01} - \frac{cg_0}{4\sqrt{\tau}}\sin 2\overline{\varphi}_{01}\right)\right) + O(\varepsilon\sqrt{\tau}) + O(\sqrt{\varepsilon}\tau^{-1/2}),$$

where $\overline{\varphi}_{01} = \widetilde{\varphi}_{01} + \dfrac{cg_0}{4\tau_1^{1/2}}$. Thus, $\varphi \to \overline{\varphi}_0$, $r \to \infty$ when $\tau \to \infty$.

We find from the formulae for $\varphi(\tau)$ and $r(\tau)$ an expression for $r = r(\varphi)$:

$$r \sim r_1(\overline{\varphi}_0 - \varphi)^{-1} + r_2 + O(\overline{\varphi}_0 - \varphi),$$

where $r_1 = r_0 k$, $r_2 = -r_0 c k^{-1}\sqrt{\epsilon}\sin 2\overline{\varphi}_{01}/16$, $k = c(g_0 - \sqrt{\epsilon}\sin 2\overline{\varphi}_{01})/4 > 0$.

Let us determine the angle ψ between the tangent to the line $r = r(\varphi)$ in some point and a radius-vector of this point when $\tau \to \infty$. The angle is calculated by the formula:

$$\operatorname{ctg}\psi = \frac{d\ln r}{d\varphi} \sim \frac{r_1}{(\overline{\varphi}_0 - \varphi)(r_1 + r_2(\overline{\varphi}_0 - \varphi))} \to \infty \quad \Rightarrow \quad \psi \sim \overline{\varphi}_0 - \varphi.$$

It means that when $\tau \to \infty$ the tangent approaches the radius-vector: $\psi \to 0$. The slope angle of the tangent to the axis y has the limit: $\varphi + \psi \to \overline{\varphi}_0$.

Let us clarify existence of asymptotes of the trajectory. For this purpose we determine how the value of y_1 varies when $r \to \infty$ (see Figure 6):

$$y_1 = r\cos\varphi - r\sin\varphi\operatorname{ctg}(\varphi + \psi) = r\frac{\sin\psi}{\sin(\varphi + \psi)} \sim r_1(\overline{\varphi}_0 - \varphi)^{-1}\frac{(\overline{\varphi}_0 - \varphi)}{\sin\overline{\varphi}_0} = \frac{r_1}{\sin\overline{\varphi}_0} < \infty.$$

Consequently, the tangent has eliminating position, i.e. any trajectory has an asymptote. This fact completely corresponds to Figure 5.

Therefore, the solution described in the paper defines a radial expansion of gas from vortex.

Remark. If $u_{01} \neq 0$ in (8.1), then we have a non-vanishing component of the velocity on the axis x. We obtain an unfolding vortex column.

The trajectories of particles illustrated in Figure 5 are constructed in the case when initial data for the functions $G(\tau)$ and $M(\tau)$ have the form $G(1) = 1$, $M(1) = 1$. These initial data correspond to zero initial data for the equation (6.6). Let us construct trajectories of particles in the case when the initial data of the equation (6.6) have the form $\mu(0) = 1$, $\mu(0) = 3.5$, $\mu(0) = -2$, $\mu(0) = -4$ (Figures 7, 8, 9, 10 respectively).

Figures 7, 8 show that the higher is the value of the function $\mu(g)$ at zero, the closer to straight lines particles trajectories become. In case of negative values of the function $\mu(g)$ at zero we obtain a vortex (Figures 9, 10).

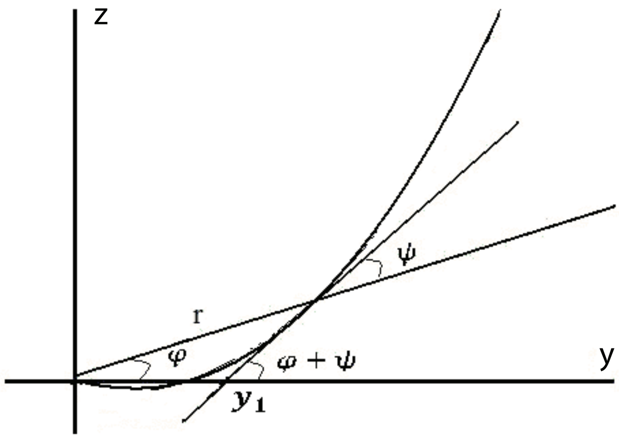

FIGURE 6. Position of the tangent to the trajectory

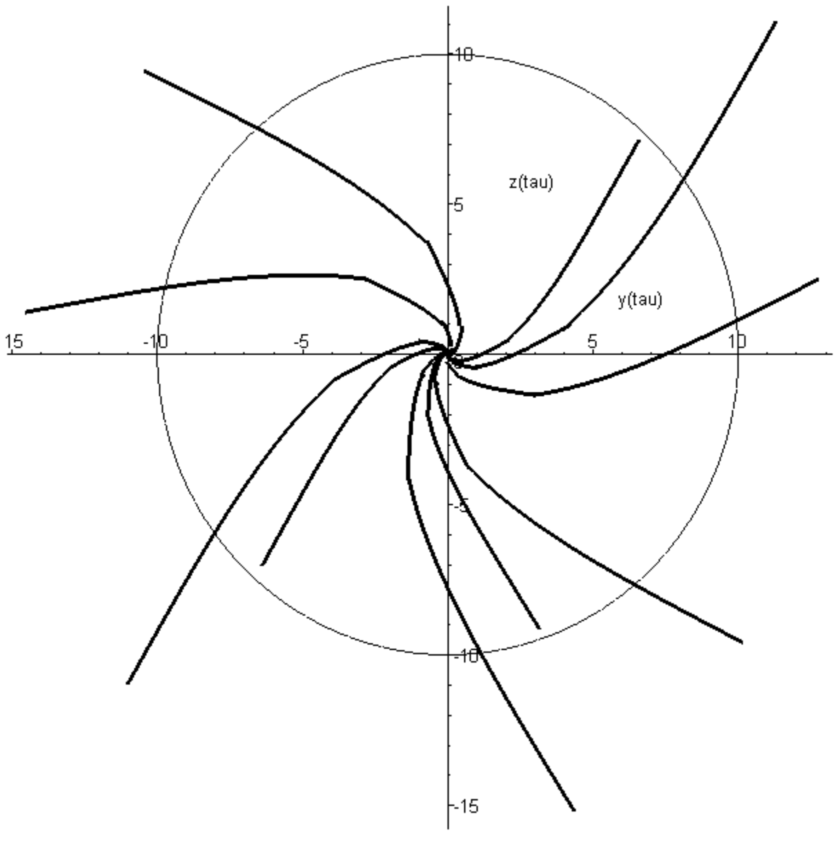

FIGURE 7. Particles trajectory when $\mu(0) = 1$

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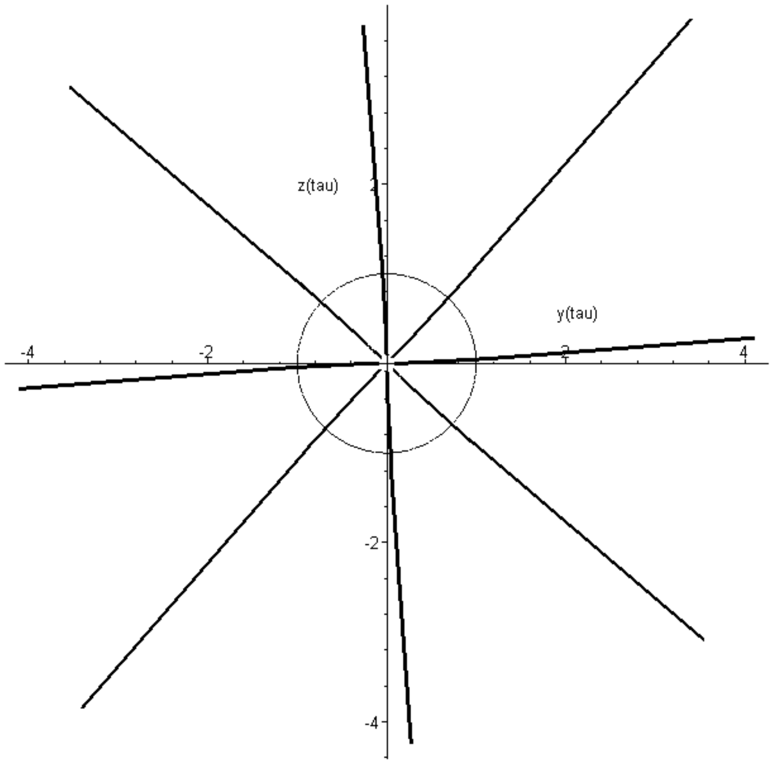

FIGURE 8. Particles trajectory when $\mu(0) = 3.5$

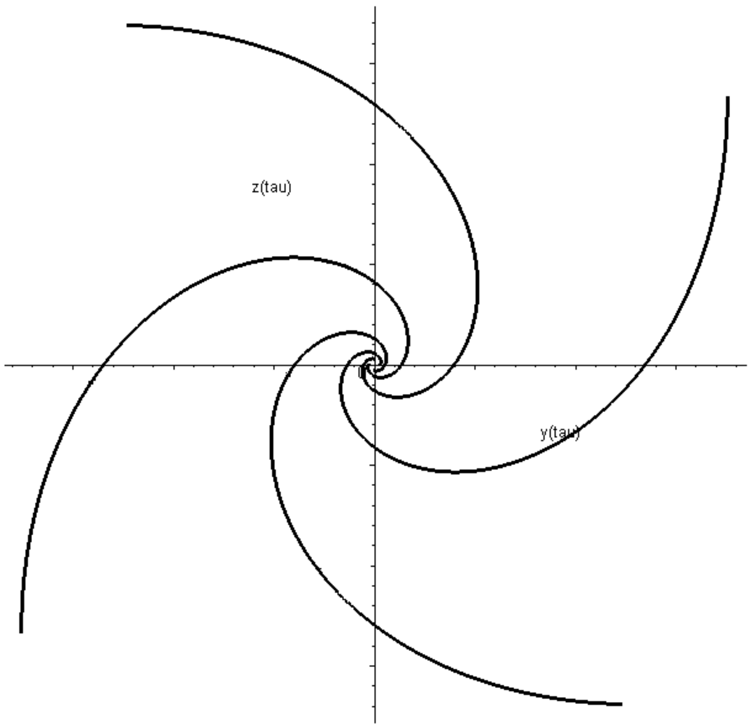

FIGURE 9. Particles trajectory when $\mu(0) = -2$

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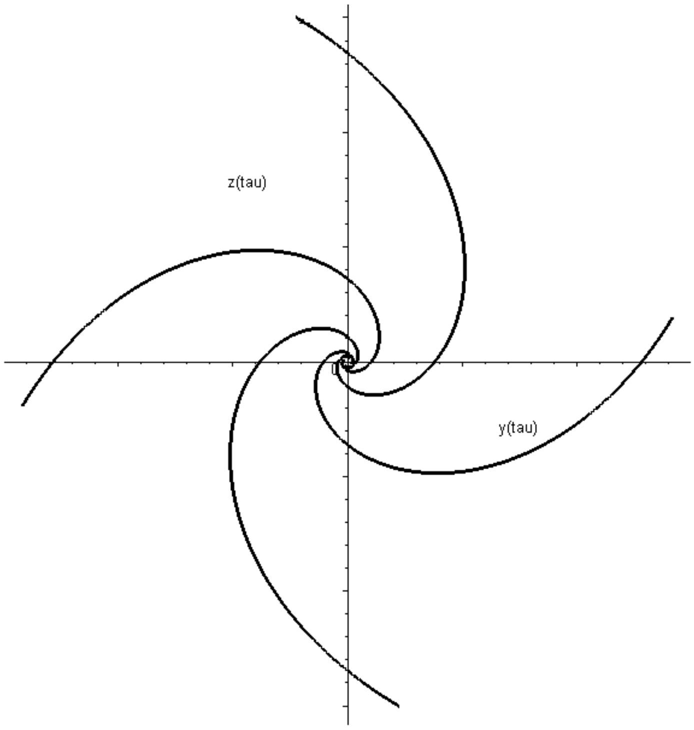

FIGURE 10. Particles trajectory when $\mu(0) = -4$

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