# TWO-DIMENSIONAL ALGEBRAS OF DYNAMIC SYMMETRIES OF ODES 

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#### Abstract

The efficiency of use of dynamic symmetries investigating integrability of differential equations is shown in this work. The generalization of S. Lie's classification of the second-order ordinary differential equations with respect to two-dimensional algebras of point symmetries is constructed. Integrable cases of the second order special type ODEs are indicated. It is noted that a part of the found integrability cases is related to Abel's type equations and apparently represents independent interest.


Keywords: dynamic symmetries, invariants, two-dimensional algebras, Riccati equation, Abel equation.

## Introduction

The notion of dynamic symmetries in presented, for example, in [1]. A differential equation in this case is replaced by the system of first-order differential equations:

$$
\begin{equation*}
y^{\prime \prime}=f\left(x, y, y^{\prime}\right) \Leftrightarrow \frac{d y}{d x}=z, \frac{d z}{d x}=f(x, y, z) . \tag{1}
\end{equation*}
$$

Then we consider the problem of an infinitesimal transformation

$$
\begin{equation*}
X=\xi(x, y, z) \frac{\partial}{\partial x}+\eta(x, y, z) \frac{\partial}{\partial y}+\mu(x, y, z) \frac{\partial}{\partial z} \tag{2}
\end{equation*}
$$

transforming a solution of the system (1) into a solution of the same system. For this purpose the operator (2) should satisfy the condition

$$
\begin{equation*}
[X, A]=\lambda(x, y, z) A \tag{3}
\end{equation*}
$$

where $A=\frac{\partial}{\partial x}+z \frac{\partial}{\partial y}+f(x, y, z) \frac{\partial}{\partial z}$.
It is important to note that the components of the operator (2) do not satisfy Lie's prolongation formula

$$
\mu=\frac{d \eta}{d x}-z \frac{d \xi}{d x}
$$

The components of the operator of a dynamic symmetry (2) are determined only by the condition (3).

It is known that the operator of point symmetry

$$
\begin{equation*}
\underset{2}{X}=\xi(x, y) \frac{\partial}{\partial x}+\eta(x, y) \frac{\partial}{\partial y}+\eta_{1}\left(x, y, y^{\prime}\right) \frac{\partial}{\partial y^{\prime}}+\eta_{2}\left(x, y, y^{\prime}, y^{\prime \prime}\right) \frac{\partial}{\partial y^{\prime \prime}}, \tag{4}
\end{equation*}
$$

which components satisfy the prolongation formula $\eta_{i}=\frac{d \eta_{i-1}}{d x}-y^{(i)} \frac{d \xi}{d x}$, have properties of prolongation invariants. If we know an invariant $v=v(x, y)$ and a differential invariant $u=$

[^0]$u\left(x, y, y^{\prime}\right)$ for the given operator (4), then the expression
$$
w=\frac{d u}{d v}=\frac{u_{x}^{\prime}+u_{y}^{\prime} y^{\prime}+u_{y^{\prime}}^{\prime} y^{\prime \prime}}{v_{x}^{\prime}+v_{y}^{\prime} y^{\prime}}
$$
is a second-order differential invariant of the operator (4).
In the paper [2] we have suggested that we should start the procedure of finding symmetries with invariants. Components of the corresponding operator in this case are written directly with the help of differentiating and arithmetic operations. If the functions $u=u\left(x, y, y^{\prime}\right)$, $v=v\left(x, y, y^{\prime}\right)$ are invariants of the operator (2), then it is natural to demand for the expression
$$
\frac{d u}{d v}=\frac{u_{x}^{\prime}+u_{y}^{\prime} y^{\prime}+u_{y^{\prime}}^{\prime} y^{\prime \prime}}{v_{x}^{\prime}+v_{y}^{\prime} y^{\prime}+v_{y^{\prime}}^{\prime} y^{\prime \prime}}
$$
to be also an invariant of a once prolonged dynamic symmetry (2).
It is more convenient to write components of the operator of dynamic symmetry in the form
\[

$$
\begin{equation*}
X=\xi\left(x, y, y^{\prime}\right) \frac{\partial}{\partial x}+\eta\left(x, y, y^{\prime}\right) \frac{\partial}{\partial y}+\mu\left(x, y, y^{\prime}\right) \frac{\partial}{\partial y^{\prime}}+\mu_{1}\left(x, y, y^{\prime}, y^{\prime \prime}\right) \frac{\partial}{\partial y^{\prime \prime}} \tag{5}
\end{equation*}
$$

\]

and determine them, solving the system of equations

$$
\begin{equation*}
X u=0, \quad X v=0, \quad X \frac{d u}{d v}=0 \tag{6}
\end{equation*}
$$

Therefore, components of a dynamic symmetry are determined up to a functional factor.
To find symmetries of ODE of the second order $F\left(x, y, y^{\prime}, y^{\prime \prime}\right)=0$ we can apply the criterion of invariance

$$
\begin{equation*}
\left.X F\right|_{F=0}=0 . \tag{7}
\end{equation*}
$$

In the general case the criterion of invariance (7), as the criterion of invariance (3) does not lead to a system of differential equations. The advantage of the criterion (7) is that it allows to speak about dynamic symmetry (5) satisfying the equations (6) with precision to the functional factor.

Let us note that all point symmetries can be considered as a particular case of dynamic symmetries when

$$
u=\frac{\frac{\partial \beta}{\partial x}+\frac{\partial \beta}{\partial y} y^{\prime}}{\frac{\partial \alpha}{\partial x}+\frac{\partial \alpha}{\partial y} y^{\prime}}, \quad v=\beta(x, y)
$$

where $\alpha(x, y), \quad \beta(x, y)$ are arbitrary functions.
If we desire to conserve the possibility to reduce the criterion (7) to the system of differential equations, it is natural to put forward a problem of dynamic symmetries as an expansion of the set of point symmetries limiting ourselves to functions of two variables.

In the paper [3] we considered the problem of dynamic symmetries generated by three functions of two variables, which contain a great number of point symmetries. It was noted that the operator of the point symmetry

$$
X=\xi(x, y) \frac{\partial}{\partial x}+\eta(x, y) \frac{\partial}{\partial y}
$$

having the invariant $v=\tau(x, y)$ takes the form

$$
X=\chi(v, y) \frac{\partial}{\partial y}
$$

in the variables $(v, y)$. If we represent the function $\chi(v, y)$ in the form $\chi(v, y)=\frac{1}{\alpha_{y}^{\prime}}$, we can write the first differential invariant in the form $u=\alpha_{v}^{\prime}+\alpha_{y}^{\prime} \frac{d y}{d v}$. Therefore, having taken the first differential invariant in the form $u=\alpha+\beta \frac{d y}{d v}$, we can expand the set of point symmetries with the help of three functions $\tau(x, y), \alpha(v, y), \beta(v, y)$.

Practical finding dynamic symmetries of the considered type was suggested in several steps:
A. Using a point change of variables $t=\tau(x, y), y=y$, we map the equation

$$
F\left(t, y, \frac{d y}{d t}, \frac{d^{2} y}{d t^{2}}\right)=0
$$

to the equation

$$
\Phi\left(x, y, \frac{d y}{d x}, \frac{d^{2} y}{d x^{2}}\right)=0
$$

B. Having the invariants

$$
v=x, \quad u=\alpha(x, y)+\beta(x, y) \frac{d y}{d x}
$$

We construct an operator of dynamic symmetry by solving the system (6).
C. We equate in the criterion of invariance

$$
\left.X \Phi\right|_{\Phi=0}=0
$$

coefficients for $y^{\prime}$ to zero, we write the determining system of differential equations.
D. We find solutions of the determining system of equations.

In the papers [2, 3] we showed efficiency of application of dynamic symmetries for finding explicit solutions on the example of the equation

$$
\begin{equation*}
y^{\prime \prime}=y^{\prime}+f(y) \tag{8}
\end{equation*}
$$

. With the help of the found solutions we managed to write exact solutions of the Kolmagorov-Petrovsky-Piskunov diffusion equation, the Semenov (Fitz-Hugh-Nagumo) equation, used in the theory of chain chemical reactions.

Unfortunately, the suggested approach was time-consuming. Therefore in the present paper we consider two-dimensional algebras, composed by twice prolonged operator of point symmetry (4) and the operator of dynamic symmetry (5) with invariants

$$
v=x, \quad u=\alpha(x, y)+\beta(x, y) \frac{d y}{d x}, \quad u_{1}=\alpha_{x}^{\prime}+y^{\prime}\left(\alpha_{y}^{\prime}+\beta_{x}^{\prime}\right)+y^{\prime 2} \beta_{y}^{\prime}+y^{\prime \prime} \beta
$$

## 1. Two-dimensional algebras of dynamic symmetries of ODEs

Sophus Lie [4, 5] presents a classification of ODEs of the second order on the basis of classification of two-dimensional algebras of operators of transformation on the plane:
I. $X_{1}=\frac{\partial}{\partial x}, \quad X_{2}=\frac{\partial}{\partial y}, \quad\left[X_{1}, X_{2}\right]=0, \quad X_{1} \vee X_{2} \neq 0, \quad y^{\prime \prime}=f\left(y^{\prime}\right) ;$
II. $X_{1}=\frac{\partial}{\partial y}, \quad X_{2}=x \frac{\partial}{\partial y}, \quad\left[X_{1}, X_{2}\right]=0, \quad X_{1} \vee X_{2}=0, \quad y^{\prime \prime}=f(x)$;
III. $X_{1}=\frac{\partial}{\partial y}, \quad X_{2}=x \frac{\partial}{\partial x}+y \frac{\partial}{\partial y}, \quad\left[X_{1}, X_{2}\right]=X_{1}, \quad X_{1} \vee X_{2} \neq 0, \quad y^{\prime \prime}=\frac{1}{x} f\left(y^{\prime}\right)$;
IV. $X_{1}=\frac{\partial}{\partial y}, \quad X_{2}=y \frac{\partial}{\partial y}, \quad\left[X_{1}, X_{2}\right]=X_{1}, \quad X_{1} \vee X_{2}=0, \quad y^{\prime \prime}=f(x) y^{\prime}$.

Let us formulate an obvious observation, which simplifies considerstion of two-dimensional algebras od dynamic symmetries.

Let a differential equation

$$
\begin{equation*}
y^{\prime \prime}=f\left(x, y, y^{\prime}\right) \tag{9}
\end{equation*}
$$

admit the point symmetry (4). In this case the finite one-parameter transformation generated by the operator (4), has the following properties:
I. It transforms solution of the equation (9) into solutions of the same equation.
II. It transforms hypersurface determined by the equation (9) into itself.

If we multiply the operator (4) by an arbitrary functional factor $\varphi\left(x, y, y^{\prime}, y^{\prime \prime}\right)$, we obtain the operator

$$
\begin{equation*}
\underset{2}{\hat{X}}=\varphi\left(x, y, y^{\prime}, y^{\prime \prime}\right)\left(\xi(x, y) \frac{\partial}{\partial x}+\eta(x, y) \frac{\partial}{\partial y}+\eta_{1}\left(x, y, y^{\prime}\right) \frac{\partial}{\partial y^{\prime}}+\eta_{2}\left(x, y, y^{\prime}, y^{\prime \prime}\right) \frac{\partial}{\partial y^{\prime \prime}}\right) . \tag{10}
\end{equation*}
$$

A finite one-parameter transformation generated by the operator (10) do not already have the first property. But is conserves the second property since the operators (4) and (10) have one and the same set of invariants.

On solving the problem of finding ODEs of the second order the basic role is played by invariants, therefore in this case we can consider two-dimensional algebras with respect to a functional field but not numerical one. Transition to the functional field significantly simplifies the problem, since it allows to proceed from considering complete systems of differential equations to Jacobian equations.

Let us consider two-dimensional algebras consisting of the operator (4) and the operator of dynamic symmetry (5).

We require that the operator (5) in some new system of coordnates

$$
\begin{equation*}
t=t(x, y), \quad p=p(x, y), \quad \dot{p}=\frac{d p}{d t}, \quad \ddot{p}=\frac{d^{2} p}{d t^{2}} \tag{11}
\end{equation*}
$$

has the invariants $v=t, u=\alpha(t, p)+\beta(t, p) \frac{d p}{d t}$.
Let us note that the transformation (11) is pointwise, consequently, the most general form of the operator of point symmetry (4) does not change. Solving the system (6) and using old notation $x, y, y^{\prime}, y^{\prime \prime}$, we arrive to the operators

$$
\underset{2}{X}=\xi(x, y) \frac{\partial}{\partial x}+\eta(x, y) \frac{\partial}{\partial y}+\eta_{1}\left(x, y, y^{\prime}\right) \frac{\partial}{\partial y^{\prime}}+\eta_{2}\left(x, y, y^{\prime}, y^{\prime \prime}\right) \frac{\partial}{\partial y^{\prime \prime}},
$$

where

$$
\begin{aligned}
\eta_{1}= & \eta_{x}^{\prime}+y^{\prime}\left(\eta_{y}^{\prime}-\xi_{x}^{\prime}\right)-\xi_{y}^{\prime} y^{\prime 2} \\
\eta_{2}= & \eta_{x x}^{\prime \prime}+y^{\prime}\left(2 \eta_{x y}^{\prime \prime}-\xi_{x x}^{\prime \prime}\right)+y^{\prime 2}\left(\eta_{y y}^{\prime \prime}-2 \xi_{x y}^{\prime \prime}\right)-y^{\prime 3} \xi_{y y}^{\prime \prime}+y^{\prime \prime}\left(\eta_{y}^{\prime}-2 \xi_{x}^{\prime}-3 \xi_{y}^{\prime} y^{\prime}\right) \\
\tilde{X} & =\varphi\left(x, y, y^{\prime}, y^{\prime \prime}\right)\left(\frac{\partial}{\partial y}+\mu\left(x, y, y^{\prime}\right) \frac{\partial}{\partial y^{\prime}}+\mu_{1}\left(x, y, y^{\prime}, y^{\prime \prime}\right) \frac{\partial}{\partial y^{\prime \prime}}\right) \\
\mu & =-\left(\alpha_{y}^{\prime}+\beta_{y}^{\prime} y^{\prime}\right) \beta^{-1} \\
\mu_{1} & =\left(\alpha_{y}^{\prime 2}+\alpha_{y}^{\prime} \beta_{x}^{\prime}-\alpha_{x y}^{\prime \prime} \beta+y^{\prime}\left(\beta_{x}^{\prime} \beta_{y}^{\prime}+3 \alpha_{y}^{\prime} \beta_{y}^{\prime}-\alpha_{y y}^{\prime \prime} \beta-\beta_{x y}^{\prime \prime} \beta\right)+\right. \\
& \left.+y^{\prime 2}\left(2{\beta_{y}^{\prime}}^{2}-\beta_{y y}^{\prime \prime} \beta\right)-y^{\prime \prime} \beta_{y}^{\prime} \beta\right) \beta^{-2} .
\end{aligned}
$$

To determine ODEs of the second order admitting the presented operators it is necessary to solve the system of linear partial differential equations

$$
\left\{\begin{array}{c}
\xi(x, y) \frac{\partial \vartheta}{\partial x}+\eta(x, y) \frac{\partial \vartheta}{\partial y}+\eta_{1}\left(x, y, y^{\prime}\right) \frac{\partial \vartheta}{\partial y^{\prime}}+\eta_{2}\left(x, y, y^{\prime}, y^{\prime \prime}\right) \frac{\partial \vartheta}{\partial y^{\prime \prime}}=0  \tag{12}\\
\frac{\partial \vartheta}{\partial y}+\mu\left(x, y, y^{\prime}\right) \frac{\partial \vartheta}{\partial y^{\prime}}+\mu_{1}\left(x, y, y^{\prime}, y^{\prime \prime}\right) \frac{\partial \vartheta}{\partial y^{\prime \prime}}=0
\end{array}\right.
$$

The system of differential equations (12) is complete, since it is supposed that the operators form a two-dimensional algebra.

Introducing the variables

$$
x=x, \quad y=y, \quad u=\alpha(x, y)+\beta(x, y) y^{\prime}, \quad u_{1}=\alpha_{x}^{\prime}+y^{\prime}\left(\alpha_{y}^{\prime}+\beta_{x}^{\prime}\right)+y^{\prime 2} \beta_{y}^{\prime}+y^{\prime \prime} \beta,
$$

we obtain a system of the form

$$
\left\{\begin{array}{c}
\xi(x, y) \frac{\partial \vartheta}{\partial x}+\eta(x, y) \frac{\partial \vartheta}{\partial y}+\tilde{\eta}_{1}(x, y, u) \frac{\partial \vartheta}{\partial u}+\tilde{\eta}_{2}\left(x, y, u, u_{1}\right) \frac{\partial \vartheta}{\partial u_{1}}=0  \tag{13}\\
\frac{\partial \vartheta}{\partial y}=0
\end{array}\right.
$$

where the transformed components $\tilde{\eta}_{1}(x, y, u), \quad \tilde{\eta}_{2}\left(x, y, u, u_{1}\right)$ polynomially depend on new variables $u, u_{1}$.

The system (13) is reduced to the Jacobi type, for example, if $\xi(x, y) \neq 0$, then we come to the system

$$
\left\{\begin{array}{c}
\frac{\partial \vartheta}{\partial x}+\frac{\tilde{\eta}_{1}(x, y, u)}{\xi(x, y)} \frac{\partial \vartheta}{\partial u}+\frac{\tilde{\eta}_{2}\left(x, y, u, u_{1}\right)}{\xi(x, y)} \frac{\partial \vartheta}{\partial u_{1}}=0  \tag{14}\\
\frac{\partial \vartheta}{\partial y}=0
\end{array}\right.
$$

The fact that the system (14) is the Jacobi type yields that

$$
\begin{equation*}
\frac{\partial}{\partial y}\left(\frac{\tilde{\eta}_{1}(x, y, u)}{\xi(x, y)}\right)=0, \quad \frac{\partial}{\partial y}\left(\frac{\tilde{\eta}_{2}\left(x, y, u, u_{1}\right)}{\xi(x, y)}\right)=0 \tag{15}
\end{equation*}
$$

Equating coefficients for $u, u_{1}$ in the system (15) to zero, we obtain a overdetermined system of differential equations with respect to the functions $\xi(x, y), \eta(x, y), \alpha(x, y), \beta(x, y)$.

The analysis of the overdetermined system up to point transformations allows to specify an ordinary differential equation of the form

$$
\begin{equation*}
y^{\prime \prime}+y^{\prime} \frac{d \lambda}{d y}=f\left(y^{\prime}+\lambda(y)\right) \tag{16}
\end{equation*}
$$

alongside with four classes written by S. Lie.
Investigation of the system (13) in the general case allows to conclude that the considered class of subalgebras contains all the equations specified by Sophus Lie and the equation (16). There are no other equations of the second order in this class.
2. Solvable cases of equations $y^{\prime \prime}+y^{\prime} \frac{d \lambda}{d y}=f\left(y^{\prime}+\lambda(y)\right)$

First of all we should note that if we take the function

$$
\lambda(y)=\lambda_{1}+\lambda_{2} y
$$

in the equation (16) we obtain the equation

$$
y^{\prime \prime}+y^{\prime} \lambda_{2}=f\left(y^{\prime}+\lambda_{1}+\lambda_{2} y\right)
$$

admitting two-dimensional algebra of point symmetries

$$
X_{1}=\frac{\partial}{\partial x}, \quad X_{2}=\exp \left(-\lambda_{2} x\right) \frac{\partial}{\partial y}
$$

Therefore, in case of the linear function $\lambda(y)$ we arrive to the differential equation, belonging to III type of the S. Lie classification. Using the change of the variables

$$
x=\frac{\ln t}{\lambda_{2}}, \quad y=\frac{u}{t}, y^{\prime}=\lambda_{2}\left(\dot{u}-\frac{u}{t}\right), \quad y^{\prime \prime}=\lambda_{2}^{2}\left(\ddot{u} t-\dot{u}+\frac{u}{t}\right),
$$

we come to the equation

$$
\lambda_{2}^{2} \ddot{u} t=f\left(\lambda_{2} \dot{u}+\lambda_{1}\right)
$$

specified by S. Lie.
In the case of quadric dependence

$$
\lambda(y)=\lambda_{1}+\lambda_{2} y+\lambda_{3} y^{2}
$$

the equation

$$
y^{\prime \prime}+y^{\prime} \lambda_{2}+2 y^{\prime} y \lambda_{3}=f\left(y^{\prime}+\lambda_{1}+\lambda_{2} y+\lambda_{3} y^{2}\right)
$$

does not admit two-dimensional algebra of point symmetries.
It is easy to write the first integral of the equation (16). Denoting $\frac{d \Phi(v)}{d v}=\frac{1}{f(v)}$, we obtain an integral in the form

$$
\begin{equation*}
\Phi\left(y^{\prime}+\lambda(y)\right)=x+C_{1} . \tag{17}
\end{equation*}
$$

up to the functional factor. Finding the functional factor with the help of the criterion (3) is equivalent to the problem of integrating the equation (16). Therefore it is impossible to apply a dynamic group to obtain the general solution.

It is obvious that solution of the equation (17) is equivalent to solution of the equation

$$
\begin{equation*}
\frac{d y}{d v}=\frac{d \Phi}{d v}(v-\lambda(y)) . \tag{18}
\end{equation*}
$$

In the case of quadric dependence $\lambda(y)=\lambda_{1}+\lambda_{2} y+\lambda_{3} y^{2}$ the equation (18) becomes a Riccati equation. Rewriting the equation (18) in the independent variable $t=\Phi(v)$, we obtain the equation

$$
\frac{d y}{d t}=\Phi^{-1}(t)-\lambda_{1}-\lambda_{2} y-\lambda_{3} y^{2} .
$$

The standard transformation reduces the Riccati equation to the canonic form

$$
\begin{equation*}
\frac{d u}{d t}+u^{2}=\Phi^{-1}(t) \tag{19}
\end{equation*}
$$

The following Theorem is known with respect to integrability of the equation (19)
Theorem 1. [6] The canonic Riccati equation (19) is integrable in quadratures if and only if $\Phi^{-1}(t)$ can be presented in the form

$$
\Phi^{-1}(t)=\frac{r_{1}}{E^{2}}+\frac{E^{\prime \prime}}{2 E}-\frac{E^{\prime 2}}{4 E^{2}},
$$

where $r_{1}=$ const,$\quad E=E(t)$. There are three types of solutions here:

$$
\begin{aligned}
& \text { A. } r_{1}=0, \quad u=\frac{E^{\prime}}{2 E}+\frac{1}{E\left(\int \frac{d t}{E}+C_{1}\right)} ; \\
& \text { B. } \quad r_{1}=a^{2}, \quad u=\frac{E^{\prime}}{2 E}+a \frac{\exp \left(2 a \int \frac{d t}{E}+2 a C_{1}\right)+1}{E\left(\exp \left(2 a \int \frac{d t}{E}+2 a C_{1}\right)-1\right)} ; \\
& \text { C. } \quad r_{1}=-a^{2}, \quad u=\frac{E^{\prime}}{2 E}-a \frac{\tan \left(a \int \frac{d t}{E}+a C_{1}\right)}{E} .
\end{aligned}
$$

If the function $\lambda(y)=\lambda_{1}+\lambda_{2} \exp \left(\lambda_{3} y\right)$, then the equation (18) is easily linearised by the substitution $u=\exp \left(-\lambda_{3} y\right)$ and, consequently, is integrable in quadratures.

Let us present some types of the equation (18) admitting point symmetries of the form
$X=\xi(v) \frac{\partial}{\partial v}+\left(\eta_{1}(v)+\eta_{2}(v) y\right) \frac{\partial}{\partial y} \quad$ or $\quad X=\left(\xi_{1}(y)+\xi_{2}(y) v\right) \frac{\partial}{\partial v}+\eta(y) \frac{\partial}{\partial y}$.

1. $\quad \lambda(y)=\frac{2\left(\lambda_{1}+\lambda_{2} y+\lambda_{3} y^{2}\right) \lambda_{3}}{\lambda_{2} \lambda_{5}-c+2 \lambda_{5} \lambda_{3} y}, \quad \frac{d \Phi}{d v}=\frac{\lambda_{5}^{3}}{\left(1-\lambda_{5} b k\left(\lambda_{5}^{2} v-c\right)^{2}\right)\left(\lambda_{5}^{2} v-c\right) \lambda_{3}}$,

$$
\xi(v)=-\eta_{2}(v) \frac{\left(\lambda_{5}^{2} v-c\right)}{\lambda_{5}^{2}}, \quad \eta_{1}(v)=\eta_{2}(v) \frac{\left(\lambda_{2} \lambda_{5}-c\right)}{2 \lambda_{3} \lambda_{5}}, \quad \eta_{2}=-\frac{\left(1-\lambda_{5} b k\left(\lambda_{5}^{2} v-c\right)^{2}\right)}{\left(\lambda_{5}^{2} v-c\right)^{2}} ;
$$

2. $\quad \lambda(y)=-b \ln (c-a b+y)+\lambda_{1}, \quad \frac{d \Phi}{d v}=-\frac{\phi_{1}}{b} \exp \left(-\frac{v}{b}\right)$,

$$
\xi(v)=1, \quad \eta_{1}(v)=a-\frac{c}{b}, \quad \eta_{2}=-\frac{1}{b}
$$

3. $\lambda(y)=-\frac{\left(15 p_{10}\left(\lambda_{1} y+\lambda_{2}\right)\right)^{-\frac{1}{3}}+p_{9}}{5 p_{10}}, \quad \frac{d \Phi}{d v}=\frac{3125 p_{1} p_{10}^{4}}{\left(5 p_{10} v+p_{9}\right)^{5}}$,

$$
\xi_{1}(y)=-\frac{p_{9} \lambda_{1}}{15 p_{10}}, \quad \xi_{2}(y)=-\frac{\lambda_{1}}{3}, \quad \eta(y)=\lambda_{1} y+\lambda_{2}
$$

4. $\lambda(y)=-\frac{4 p_{9} \sqrt{-p_{10}\left(\lambda_{2}+\lambda_{1} y\right)} \pm \sqrt{2}}{16 p_{10} \sqrt{-p_{10}\left(\lambda_{2}+\lambda_{1} y\right)}}, \quad \frac{d \Phi}{d v}=\frac{256 p_{2} p_{10}^{3}}{\left(p_{9}+4 p_{10} v\right)^{4}}$,

$$
\xi_{1}(y)=-\frac{p_{9}}{8 p_{10}}, \quad \xi_{2}(y)=-\frac{1}{2}, \quad \eta(y)=\frac{\lambda_{1} y+\lambda_{2}}{\lambda_{1}}
$$

5. $\lambda(y)=\frac{b \exp \left(b \lambda_{1} p_{3} \exp \left(-\frac{y}{p_{3}}\right)\right)+\lambda_{2} a \exp \left(a \lambda_{1} p_{3} \exp \left(-\frac{y}{p_{3}}\right)\right)}{\exp \left(b \lambda_{1} p_{3} \exp \left(-\frac{y}{p_{3}}\right)\right)+\lambda_{2} \exp \left(a \lambda_{1} p_{3} \exp \left(-\frac{y}{p_{3}}\right)\right)}-\frac{\exp \left(\frac{y}{p_{3}}\right)}{\lambda_{1} p_{3}}$, $\frac{d \Phi}{d v}=\frac{p_{3}}{(b-v)(a-v)}$,

$$
\xi_{1}(y)=\left(p_{3} \frac{d \lambda}{d y}-\lambda(y)\right) \xi_{2}(y), \quad \xi_{2}(y)=\frac{\exp \left(\frac{2 y}{p_{3}}\right)}{p_{3}(\lambda(y)-a)(\lambda(y)-b)}, \quad \eta(y)=p_{3} \xi_{2}(y)
$$

6. $\lambda(y)=\frac{a p_{3} \lambda_{1} \exp \left(-\frac{y}{p_{3}}\right)-\exp \left(\frac{y}{p_{3}}\right)-a p_{3} \lambda_{2}}{p_{3}\left(\lambda_{1} \exp \left(-\frac{y}{p_{3}}\right)-\lambda_{2}\right)}, \quad \frac{d \Phi}{d v}=\frac{p_{3}}{(v-a)^{2}}$,
$\xi_{1}(y)=\left(p_{3} \frac{d \lambda}{d y}-\lambda(y)\right) \xi_{2}(y), \quad \xi_{2}(y)=\frac{\exp \left(\frac{2 y}{p_{3}}\right)}{p_{3}(\lambda(y)-a)^{2}}, \quad \eta(y)=p_{3} \xi_{2}(y) ;$
7. $\lambda(y)=\frac{\exp \left(\frac{y}{p_{3}}\right)}{\lambda_{1} p_{3}}+\frac{\left(a-b \lambda_{2}\right) \cos \left(b \lambda_{1} p_{3} \exp \left(-\frac{y}{p_{3}}\right)\right)+\left(b+a \lambda_{2}\right) \sin \left(b \lambda_{1} p_{3} \exp \left(-\frac{y}{p_{3}}\right)\right)}{\cos \left(b \lambda_{1} p_{3} \exp \left(-\frac{y}{p_{3}}\right)\right)+\lambda_{2} \sin \left(b \lambda_{1} p_{3} \exp \left(-\frac{y}{p_{3}}\right)\right)}$, $\frac{d \Phi}{d v}=\frac{p_{3}}{v^{2}-2 a v+a^{2}+b^{2}}$,
$\xi_{1}(y)=\left(p_{3} \frac{d \lambda}{d y}-\lambda(y)\right) \xi_{2}(y), \quad \xi_{2}(y)=\frac{\exp \left(\frac{2 y}{p_{3}}\right)}{p_{3}\left(\lambda(y)^{2}-2 a \lambda(y)+a^{2}+b^{2}\right)}, \quad \eta(y)=p_{3} \xi_{2}(y) ;$
8. $\quad \lambda(y)=-\frac{-\left(3 p_{3} p_{6}\left(\lambda_{1} y+\lambda_{2}\right)\right)^{\frac{1}{3}}-p_{3} p_{5}+p_{2} p_{6}}{p_{3} p_{6}}, \quad \frac{d \Phi}{d v}=\frac{p_{6} p_{2}+p_{3} p_{6} v-p_{3} p_{5}}{p_{6}^{2}}$, $\xi_{1}(y)=\frac{\lambda_{1}\left(p_{2} p_{6}-p_{3} p_{5}\right)}{3 p_{3} p_{6}}, \quad \xi_{2}(y)=\frac{\lambda_{1}}{3}, \quad \eta(y)=\lambda_{1} y+\lambda_{2} ;$
9. $\left.\quad \lambda(y)=\left(\lambda_{1} y+\lambda_{2}\right)\right)^{\frac{1}{5}}+\lambda_{3}, \quad \frac{d \Phi}{d v}=\frac{\left(3 p_{1}+p_{2} v\right)^{3}}{27 p_{3}}$,

$$
\xi_{1}(y)=\frac{3 p_{1}}{p_{2}}, \quad \xi_{2}(y)=1, \quad \eta(y)=\frac{5\left(\lambda_{1} y+\lambda_{2}\right)}{\lambda_{1}}
$$

10. $\lambda(y)=-\lambda_{2}+\lambda_{1}\left(4 \lambda_{3} y+\lambda_{4}\right)^{\frac{1}{4}}, \quad \frac{d \Phi}{d v}=a\left(v+\lambda_{2}\right)^{2}$,

$$
\xi_{1}(y)=\lambda_{2} \lambda_{3}, \quad \xi_{2}(y)=\lambda_{3}, \quad \eta(y)=4 \lambda_{3} y+\lambda_{4}
$$

11. $\lambda(y)=\lambda_{1}+\lambda_{2} \sqrt{y+\lambda_{3}}, \quad \frac{d \Phi}{d v}=a$,

$$
\xi_{1}(y)=-\lambda_{1}, \quad \xi_{2}(y)=1, \quad \eta(y)=2\left(y+\lambda_{3}\right) .
$$

In conclusion we note that the cases $1,5,6,7,11$ are integrable cases of the Abel equation and are, obviously, of interest of their own.

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