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ON AUTOMORPHIC SYSTEMS OF FINITE-DIMENSIONAL LIE GROUPS

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Abstract. It is shown in the present paper that any automorphic system for a finite-dimensional Lie group is a completely integrable system.

Keywords: Lie symmetries, automorphic systems, differential invariants.

Introduction

A system of differential equations is called automorphic with respect to a Lie group, if all its solutions are on the orbit of one of them. The statement that the diversity U' can be obtained from the diversity U by means of action of some transformation of the group if $U' \subset h(U, O)$, is used for justification of the structure of an automorphic system in the monograph [1, p. 329]. Generally speaking, this statement is incorrect.

It is shown in the present paper that the structure for the finite dimensional Lie groups described in [1] (with some refinements) actually determines the automorphic system. It is also shown that any automorphic system for the Lie group is always completely integrable. The approach suggested here significantly uses finite dimensionality of the Lie group and therefore is not applicable for infinite dimensional groups.

1. Automorphic systems

Definition. *A system of differential equations is called automorphic with respect to a group G, if any solution of this system is obtained from one fixed solution by means of action of transformations of the group G* [1, §25].

Further we use the following notation: $X = R^n$ is a space of independent variables, $Y = Y_0 = R^m$ is a space of dependent variables, $Z_k = X \times Y_0 \times \cdots \times Y_k$, $k = 0, 1 \ldots$, where $Y_k = R^m \otimes S^k R^n$, $k = 1, \ldots$ are prolonged spaces. The vectors of the spaces Y_k are denoted by $\underset{k}{y}$, and their components are denoted by y_α, where $\alpha = (\alpha_1, \ldots, \alpha_n)$ are multi-indexes and $|\alpha| = \alpha_1 + \cdots + \alpha_n = k$. Dimension of the space Z_k is denoted by ν_k, i.e.

$$\nu_k = \dim(Z_k) = n + m \binom{n+k}{n}.$$

The natural projection Z_{k+1} on Z_k is denoted by ρ_k^{k+1}:

$$\rho_k^{k+1}(x, y, \underset{1}{y}, \ldots, \underset{k}{y}, \underset{k+1}{y}) = (x, y, \underset{1}{y}, \ldots, \underset{k}{y}).$$

Operators of a total differentiation are denoted by D_i, i.e.

$$D_i = \partial_{x_i} + \sum_{|\alpha| \geqslant 0} y_{\alpha+\gamma_i} \partial_{y_\alpha} \quad i = 1, \ldots, n,$$

where $|\gamma_i| = 1$ and the component with the number i is equal to 1.

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We consider the Lie group $G^r(h)$ of the finite dimension r, generated by the mapping $h : Z \times B \to Z$, where $B = R^r$. The mapping h for every $k > 0$ explicitly expands on the space Z_k. Here this expansion is written in the form

$$\underset{k}{h} : Z_k \times B \to Z_k.$$

Starting with some k, the general rank of the group G^r on the space Z_k is equal to r. Here this value of k is denoted by k_1.

The mapping $u : X \to Y$ for every $k \geqslant 0$ determines the variety (¡¡graph of the mapping¿¿)

$$\underset{k}{U} = \left\{ y_\alpha = \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}}, \quad 0 \leqslant |\alpha| \leqslant k \right\} \tag{1}$$

in the prolonged space Z_k.

The result of the action of the group G^r on the variety (1) is called an orbit of the variety and is denoted by $\underset{k}{h}(\underset{k}{U}, O)$, where O is a neighbourhood 0 in the space B. Obviously there exists such $k_2 \geqslant k_1$ (k_2 depends of the mapping u), that when $k \geqslant k_2$ the orbit is a proper variety of the space Z_k. Dimension of the orbit d_k when $k \geqslant k_2$ satisfies the inequalities

$$\max\{n, r\} \leqslant d_k = \dim \underset{k}{h}(\underset{k}{U}, O) \leqslant n + r. \tag{2}$$

It is obvious that for all $k \geqslant 0$

$$\rho_k^{k+1} \left(\underset{k+1}{h}(\underset{k+1}{U}, O) \right) = \underset{k}{h}(\underset{k}{U}, O). \tag{3}$$

When $k \geqslant k_2$ there exist such mappings $\psi_k : Z_k \to R^{s_k}$ that

$$\underset{k}{h}(\underset{k}{U}, O) = \{ z_k \in Z_k : \psi_k(z_k) = 0 \},$$

where $s_k = \nu_k - d_k$. Due to (3) the mappings ψ_k can be chosen so that the set of the mappings ψ_{k+1} for every k is an expansion of the set of the mappings ψ_k.

The relationships

$$\psi_k(z) = 0, \quad z \in Z_k \tag{4}$$

give a system of differential equations of the order k on m functions of n variables, and the mapping u is the solution of this system.

The system

$$p(\psi_k)(z) = \{ \psi_k(\rho_k^{k+1}(z)), (D_1\psi_k)(z), \dots, (D_n\psi_k)(z) \} = 0, \quad z \in Z_{k+1}$$

is called the first prolongation of the system (4). Accordingly the variety

$$p(\underset{k}{h}(\underset{k}{U}, O)) = \{ z \in Z_{k+1} : p(\psi_k)(z) = 0 \}$$

is called the first prolongation of the orbit $\underset{k}{h}(\underset{k}{U}, O)$.

Lemma 1. *The following relationship holds for every $k \geqslant k_2$:*

$$p(\underset{k}{h}(\underset{k}{U}, O)) \supseteq \underset{k+1}{h}(\underset{k+1}{U}, O). \tag{5}$$

Proof. Prolongation of any system of differential equations admits every symmetry of the initial system [2], i.e. the prolonged system $p(\psi_k(z)) = 0$ admits the group G^r, and the mapping u is its solution. Therefore the orbit of the solution u in the space Z_{k+1} belongs to the prolongation of the orbit u from the space Z_k. ∎

Lemma 2. *The following relationship holds for every $k \geqslant k_2$:*

$$\rho_k^{k+1}(p(\underset{k}{h}(\underset{k}{U}, O))) = \underset{k}{h}(\underset{k}{U}, O).$$

Proof. On the one hand, the obvious embedding holds

$$\rho_k^{k+1}(p(\underset{k}{h}(\underset{k}{U},O)) \subseteq \underset{k}{h}(\underset{k}{U},O).$$

On the other hand, applying the projection ρ_k^{k+1} to both sides of the relationship (5) invokingo the equality (3) provides the reverse embedding. ∎

Restriction of the Pfaff system

$$dy_\alpha - \sum_{j=1}^{n} y_{\alpha+\gamma_j} dx_j = 0, \quad 0 \leqslant |\alpha| < k$$

on the variety $\underset{k}{h}(\underset{k}{U},O))$ provides the Pfaff system with the given independent variables x. This system is equivalent to the system of differential equations (4). The opposite also holds: every Pfaff system with given independent variables corresponds to an equivalent system of differential equations. The equivalence in this case is understood as the fact that there is a reciprocal implicit relationship between integral diversities of the system of exterior equations and solutions of the system of differential equations. In terms of this equivalence a system of differential equations is called here completely integrable, if the Pfaff system equivalent to it is also completely integrable. Further forms of the Pfaff system equivalent to the system (4) are denoted by $\omega(\psi_k)$.

Lemma 3. *There exists $k_3 \geqslant k_2$ such that for every $k \geqslant k_3$ the system (4) is completely integrable.*

Proof. Due to the first of the inequalities (2) there is such k_3 that

$$d_{k_3-1} = d_{k_3} < \nu_{k_3-1}.$$

It follows that

$$\operatorname{rank} \frac{\partial \psi_{k_3}}{\partial \underset{k_3}{y}} = \nu_{k_3} - \nu_{k_3-1} = \dim Y_{k_3},$$

i.e. the system of differential equations $\psi_{k_3} = 0$ can be solved with respect to all higher derivatives. Therefore the system $p(\psi_{k_3}) = 0$ is also solvable with respect to all higher derivatives and, consequently,

$$\dim p(\underset{k_3}{h}(\underset{k_3}{U},O)) = \dim \underset{k_3}{h}(\underset{k_3}{U},O) = d_{k_3}. \tag{6}$$

Therefore, due to Lemma 1, $d_k = d_{k_3}$ for all $k > k_3$.

It follows from Lemma 2 that for $k \geqslant k_3$ the ideal generated by the system $\omega(\psi_k) = 0$ is closed with respect to the operation of the exterior differentiation. Hence, the statement of the Lemma results from the Frobenius theorem [3]. ∎

Lemma 4. *The system (4) is automorphic for every $k \geqslant k_3$.*

Proof. The system of exterior equations $\omega(\psi_k)$ with $k \geqslant k_3$ is completely integrable, i.e. the unique integral manifold [3] passes through every point of the orbit $\underset{k}{h}(\underset{k}{U},O)$. On the other hand, for any point $z' \in \underset{k}{h}(\underset{k}{U},O)$ there exist a transformation $g \in G^r$ and a point $z \in \underset{k}{U}$ such that the transformation g maps the point z into the point z'. Therefore, this unique integral manifold coincides with the image of the manifold $\underset{k}{U}$ under the action of the transformation g. ∎

Remark 1. *It results from the equation (6), in particular, that the system $\psi_{k+1}(z) = 0$ for every $k \geqslant k_3$ is the first prolongation of the system $\psi_k(z) = 0$. Consequently, all the systems $\psi_k(z) = 0$ are expansions of the system $\psi_{k_3}(z) = 0$ when $k > k_3$. The system $\psi_{k_3}(z) = 0$, as Example 1 (variant 7) shows, is not always the prolongation of the system $\psi_{k_3-1}(z) = 0$.*

Remark 2. *Lemmas 1 and 2 also hold for the infinite dimensional Lie groups, if there exists the finite* k_2. *If the mapping* u *is the solution of the system of differential equations, admitting a group, then the finite* k_2 *exists and does not exceed the order of the system.*

It results from the proof of Lemma 3 that if $d_{k-1} = d_k$ *for some* k, *then in the case of the infinite dimensional Lie group the system (4) is also completely integrable and automorphic for this and higher values ó of* k.

2. Construction of automorphic systems

To construct automorphic systems one uses, according to [1, §25], the theorem on representation of a non-singular invariant manifold. Due to this Theorem one can choose the invariant of the group G^r of the corresponding dimension as ψ_{k_3}. This invariant is expressed via the universal invariant of the group $\underset{k_3}{J}$ of the order k_3, i.e.

$$\psi_{k_3}(z) = \Psi(\underset{k_3}{J}(z)) = 0. \tag{7}$$

The requirement of complete integrability of this system imposes conditions on the mapping Ψ. These conditions give a system of differential equations on the mapping Ψ, which is called the ¡¡resolvent system¿¿ [1, §25].

The algorithm of construction of all the automorphic systems of the given group consists in investigating all possible dimensions of orbits, determined by the inequalities (2). One and the same dimension of the orbit, at least from the point of view of the inequalities (2), can be obtained for different values k_3. The number of these different values is finite. Therefore, up to the operation of prolongation, there is a finite number of different automorphic systems. But not all the variants admitted by the inequalities (2) are realized.

It is stated in the monograph [1, §25, s. 4] that for given n, m and r the type of the automorphic system is completely determined by one parameter: rank, defect or dimension of the orbit (these three values quantities are uniquely expressed via each other). As shown in Example 1 (variants 6 and 7), apart from defect the value k_3, i.e. the order of the automorphic system, is also important.

The system (7) is written, as a rule, in the solved form with respect to the part of invariants. The equation (3) allows to write the system (7) in the form

$$J'' = \varphi(J'), \tag{8}$$

$$J''' = \psi(J'), \tag{9}$$

where the order of the invariants J' and J'' is lower than k_3, and all the invariants of the order k_3 are denoted via J'''. Dividing the invariants into J' and J'' is not always unique and ¡¡the branching¿¿ of the process is possible. ¡¡The branching¿¿ is also possible for further calculations.

Apart from the inequalities (2) there is one more restriction on the dimension of the orbit, which is connected with consideration of orbits of ¡¡graphs¿¿ of the mappings $u : X \to Y$. Therefore the equations of orbits should not impose restrictions on variables of the space X, i.e. the following inequality should hold

$$\nu_k - \operatorname{rank}\left(\partial_v \underset{k}{J}\right) \leqslant d_k, \quad \text{where} \quad v = (y, \underset{1}{y}, \ldots, \underset{k}{y}), \quad k = k_3 - 1. \tag{10}$$

The set of invariants J'' should be chosen so that $\partial_v J'' \geqslant \nu_{k3-1} - d_{k3-1}$.

If u is not an arbitrary mapping but a solution of a system of differential equations E admitting a group G^r, the following condition is imposed on the functions φ and ψ from (8), (9): the system E should be a differential-algebraic corollary of the equations (8), (9). If the order of the system E does not exceed k_3, then it is expedient to include the system E, written via invariants of the group, into the system (8), (9).

3. Invariant and partially invariant solutions

Invariant and partially invariant solutions [1, §19, §22] are solutions of corresponding automorphic systems. For invariant solutions $k_3 = 1$.

Invariant solutions exist when the inequality (10) holds. Moreover the equations (8) are solvable with respect to the variables y, and the equations (9), due to Lemmas 2 and 3, simply provide expressions for the functions ψ. Therefore there is no need to calculate invariants of the first order and the resolvent system is obtained after substitution of expressions of the variables y into the initial system of differential equations. That is the technology of construction of invariant solutions with the use of the notion of automorphic systems does not differ from that described in [1, §19].

To construct partially invariant solution with the use of automorphic systems one needs differential invariants of the first or, possibly, a higher order. This complicates the algorithm as compared to the one described in [1, §22]. But there is no need to use the notion of ¡¡redundant¿¿ functions.

4. On <<simple>> solutions

When $r \geqslant n$, the minimal possible dimension of the orbit is equal to r. In case of this minimal dimension $s_{k_3} = \nu_{k_3} - r$, i.e. it coincides with the dimension of the space of invariants . Therefore the set of invariants J' from (8), (9) is empty, and the functions φ, ψ are constants. The resolvent system in this case is the system of algebraic equations for these constants. For the case $r = n$ such automorphic systems provide invariant solutions, which are called in the paper [4] ¡¡simple¿¿. By analogy, solutions of such automorphic systems can be also called ¡¡simple¿¿ even when $r > n$. We use this term up to the end of this section.

If H is a subgroup of the group G^r, then every ¡¡simple¿¿ solution with respect to the subgroup H is a ¡¡simple¿¿ solution with respect to the group G^r. Indeed, every differential invariant of the group G^r is a differential invariant of the subgroup H, and the subgroup H also has other invariants. Therefore the system (8), (9) for the subgroup H is an expansion of a similar system for the group G^r.

5. Example 1

Equations of one-dimensional dynamics of polytropic gas

$$u_t + uu_x + \rho^{-1}p_x = 0, \quad \rho_t + u\rho_x + \rho u_x = 0, \quad p_t + up_x + \gamma p u_x = 0, \tag{11}$$

admit the group with the Lie algebra

$$\partial_t, \quad \partial_x, \quad t\partial_x + \partial_u, \quad t\partial_t + x\partial_x, \quad t\partial_t - u\partial_u + 2\rho\partial_\rho, \quad p\partial_p + \rho\partial_\rho.$$

Differential invariants of the first order can be chosen in the form

$$J_1 = \frac{\rho(u_t + uu_x)}{p_x}, \quad J_2 = \frac{\rho_t + u\rho_x}{\rho u_x}, \quad J_3 = \frac{p_t + up_x}{pu_x},$$

$$J_4 = \frac{p_x}{u_x\sqrt{\rho p}}, \quad J_5 = \frac{\rho_x\sqrt{p}}{u_x\sqrt{\rho^3}}.$$

The given set of invariants forms a basis, i.e. any invariant can be obtained from this set by means of algebraic operations and actions of the operators of the invariant differentiation

$$\Lambda_1 = \frac{1}{u_x}D_t + \frac{u}{u_x}D_x, \quad \Lambda_2 = \frac{u_t + uu_x}{u_x^2}D_x.$$

Below we use the following set of differential invariants of the second order:

$$J_{21} = \frac{p_{tt} + 2up_{tx} + u^2 p_{xx}}{pu_x^2}, \quad J_{22} = \frac{(p_{tx} + up_{xx})p_x}{\rho p u_x^3}, \quad J_{23} = \frac{p_{xx} p_x^2}{\rho^2 p u_x^4},$$
$$J_{24} = \frac{(\rho_{tt} + 2u\rho_{tx} + u^2 \rho_{xx})p_x^2}{\rho^2 p u_x^4}, \quad J_{25} = \frac{(\rho_{tx} + u\rho_{xx})p_x^3}{\rho^3 p u_x^5}, \quad J_{26} = \frac{\rho_{xx} p_x^4}{\rho^4 p u_x^6},$$
$$J_{27} = \frac{(u_{tt} + 2uu_{tx} + u^2 u_{xx})\rho}{p_x u_x}, \quad J_{28} = \frac{u_{tx} + uu_{xx}}{u_x^2}, \quad J_{29} = \frac{u_{xx} p_x}{\rho u_x^3}.$$

The system (11) is written in the space of invariants in the form

$$J_1 = -1, \quad J_2 = -1, \quad J_3 = -\gamma. \tag{12}$$

The inequalities (2) admit the following variants of automorphic systems for the equations (11):

	d_1	d_2	d_3	d_4	k_3	δ
1	6	6	6	6	2	4
2	6	7	7	7	3	5
3	6	7	8	8	4	6
4	6	8	8	8	3	6

	d_1	d_2	d_3	d_4	k_3	δ
5	7	7	7	7	2	5
6	7	8	8	8	3	6
7	8	8	8	8	2	6

Here $\delta = d_{k_3} - n$ is the defect of invariance. Further we construct the system (8), (9) for every variant from the table and investigate the resolvent system for some of them.

The dimension $d_1 = 6$, i.e. $s_1 = \nu_1 - d_1 = 5$ for variants 1-4 coincides with the number of the invariants of the first order. Therefore all invariants of the first order should be equal to constants, i.e. the equations (12) should be supplemented by the equations

$$J_4 = c_4, \quad J_5 = c_5, \tag{13}$$

where c_4, c_5 are some constants.

Since the complete set of invariants of a higher order can be obtained by the action of operators of invariant differentiation on the invariants of the first order, the invariants of a higher order should be equal to zero. Therefore, the arbitrary way in construction of the system (8), (9) for variants 1-4 does not exceed two constants.

The system (12), (13) is compatible if $c_5 = (c_4^2 - \gamma + 1)/c_4$, and the first prolongation of these equations is completely integrable under the condition $c_4^2 \neq \gamma$. Under these conditions and when $\gamma \neq 1$ the solution of the system (12), (13) has the form

$$u = ax + u_1(t), \quad \rho = \rho_1 p^\alpha, \quad p = (p_1(t) + 0.5(1-\alpha)c_4 a\sqrt{\rho_1}x)^{2/(1-\alpha)},$$

where

$$\alpha = (c_4^2 - \gamma + 1)/c_4^2, \quad a = 1/(c_6 + 0.5(1+\gamma)t), \quad \rho_1 = c_7(c_6 + 0.5(1+\gamma)t)^{2/(1+\gamma)},$$

c_6, c_7 are constants, and the functions $u_1(t)$ and $p_1(t)$ satisfy the linear system of ordinary differential equations

$$\frac{du_1}{dt} = -a\left(u_1 + \frac{c_4}{\sqrt{\rho_1}}p_1\right), \quad \frac{dp_1}{dt} = -\frac{\gamma - 1}{2c_4^2}a(c_4\sqrt{\rho_1}u_1 + \gamma p_1).$$

If $c_4^2 = \gamma$, the system (12), (13) is involutive, and none of its prolongations is completely integrable. Hence, in particular, there are no automorphic systems of variants 2, 3, 4 for the system (11).

For variant 5 the equations (8), (9) consist of the equations (12) and one of the following systems of equations

$$J_4 = \varphi(J_5), \quad J_{2i} = \psi_i(J_5), \quad i = 1, \ldots, 9,$$
$$J_5 = c_5, \quad J_{2i} = \psi_i(J_4), \quad i = 1, \ldots, 9.$$

For variant 6 the equations (8), (9) consist of the equations (12) and one of the following system of equations

$$J_4 = \varphi_0(J_5), \quad J_{2i} = \varphi_i(J_5, J_{26}), \quad i \in \{1, \dots, 9\}, \ i \neq 6, \quad \Lambda_2 J_{26} = \psi(J_5, J_{26}),$$
$$J_5 = c_5, \quad J_{2i} = \varphi_i(J_4, J_{23}), \quad i \in \{1, \dots, 9\}, \ i \neq 3, \quad \Lambda_2 J_{23} = \psi(J_4, J_{23}). \tag{14}$$

The condition of complete integrability of the system (12), (14) gives the following equations for determining the functions φ_i:

$$\varphi_{21} = (\gamma^2 J_4^2 + \gamma J_{23} + \gamma J_4^2 + J_4^4)/J_4^2, \quad \varphi_{22} = -J_4^2(\gamma + 1), \quad \varphi_{24} = J_{23} + 2J_4^2,$$
$$\varphi_{25} = \varphi_{26} = \varphi_{29} = 0, \quad \varphi_{27} = \gamma + 1, \quad \varphi_{28} = -(J_{23} + J_4^2)/J_4^2,$$

$c_5 = 0$ and $\psi = \theta(J_{23}/J_4^4)J_4^6$, where θ is an arbitrary function of one argument.

For variant 7 the equations (8) coincide with the equations (12), and the equations (9) are written in the form

$$J_{2i} = \psi_i(J_4, J_5), \quad i = 1, \dots, 9. \tag{15}$$

Six of the functions ψ_i are determined from the condition of complete integrability of the system (12), (15) by the equations

$$\begin{aligned}
\psi_1 &= (\gamma^2 J_4^2 - \gamma J_4^3 J_5 + \gamma J_4^2 + \gamma\psi_3 + J_4^4)J_4^{-2},\\
\psi_2 &= -(\gamma J_4^2 + \gamma\psi_9 + J_4^2),\\
\psi_4 &= 2J_4^2 + \psi_3,\\
\psi_5 &= -J_4^2(2J_4 J_5 + \psi_9),\\
\psi_7 &= (\gamma J_4^2 + \gamma\psi_9 + J_4^2)J_4^{-2},\\
\psi_8 &= (J_4^3 J_5 - J_4^2 - \psi_3)J_4^{-2},
\end{aligned}$$

and the rest of them satisfy the system of quasi-linear equations

$$\begin{aligned}
&J_4^3 A\psi_9 + B\psi_3 + 2J_4(\gamma J_4^4\psi_9 + \gamma J_4^2\psi_9^2 + 2J_4^6 J_5^2 + 3J_4^5 J_5\psi_9 +\\
&+J_4^4\psi_3 + J_4^2\psi_9\psi_3 - J_4^2\psi_6 - 2\psi_3^2) = 0,\\
&\gamma B\psi_9 + J_4 A\psi_3 + 2J_4(2\gamma J_4^4\psi_9 + \gamma J_4^3 J_5\psi_9 + 3\gamma J_4^2\psi_9^2 + 2\gamma J_4^2\psi_3 +\\
&+\gamma\psi_9\psi_3 + J_4^4\psi_9 + 4J_4^3 J_5\psi_3 - 2J_4^2\psi_3 - 4\psi_3^2) = 0,\\
&J_4 B\psi_9 + A\psi_6 + 2(3\gamma J_4^2\psi_6 + 4\gamma\psi_9\psi_6 + 4J_4^5 J_5\psi_9 + 3J_4^4\psi_9^2 +\\
&+6J_4^3 J_5\psi_6 - J_4^2\psi_9\psi_3 - 3J_4^2\psi_6 - 6\psi_6\psi_3) = 0,
\end{aligned}$$

where

$$A = J_4\alpha_1\frac{\partial}{\partial J_4} + \alpha_2\frac{\partial}{\partial J_5}, \quad B = J_4^2\alpha_3\frac{\partial}{\partial J_4} + \alpha_4\frac{\partial}{\partial J_5},$$
$$\begin{aligned}
\alpha_1 &= -\gamma J_4^2 - 2\gamma\psi_9 - 2J_4^3 J_5 + J_4^2 + 2\psi_3,\\
\alpha_2 &= -\gamma J_4^2 J_5 - 2J_4^3 J_5^2 + J_4^2 J_5 - 2J_4\psi_9 + 2J_5\psi_3,\\
\alpha_3 &= -J_4^4 - J_4^3 J_5 - 2J_4^2\psi_9 + 2\psi_3,\\
\alpha_4 &= J_4^5 J_5 - 3J_4^4 J_5^2 - 2J_4^3 J_5\psi_9 + 2\psi_6.
\end{aligned}$$

6. Example 2

The example of the present section demonstrates that the automorphic system of the infinite dimensional Lie group is not necessarily completely integrable.

Group foliation for the Karman-Guderley equation

$$-\varphi_x\varphi_{xx} + \varphi_{yy} + \varphi_{zz} = 0 \tag{16}$$

was constructed in the paper [5] with respect to the finite dimensional group with the infinitesimal operator

$f(y,z)\partial_\varphi$, where $f(y,z)$ is an arbitrary harmonic function. This operator determines the transformation $\varphi \to \varphi + f(y,z)$.

Group foliation of the equation (16) is given, as shown in the paper [5], by the automorphic system

$$\varphi_x = a, \quad \varphi_{yy} + \varphi_{zz} = aa_x \tag{17}$$

and the resolvent equation

$$-aa_{xx} - a_x^2 + a_{yy} + a_{zz} = 0.$$

The difference of any two solutions of the system (17) is a harmonic function of variables y, z and does not depend on the variable x. Therefore the system (17) is in fact automorphic. The system (17), where a satisfies the resolvent equation, is involutive but not completely integrable.

Conclusion

If the system of differential equations E admits the finite dimensional Lie group G^r, then any solution of the system E is the solution of some automorphic system of the group G^r. An automorphic system is relatively simply integrable, but the resolvent system can be significantly more complex than the initial system E. Example 1 (variant 7) demonstrates it. An exception is provided by automorphic systems of a minimal defect. In this case the resolvent system is a system of algebraic equations for the totality of constants.

The fact that an automorphic system should be completely integrable allows to write representation of the system of a given type (or several representations like in variants 5 and 6 of Example 1) immediately. There is no need to write all the equations of the system (9) for definite calculations. To carry out length calculations we used the system of analytical calculations ¡¡Reduce 3.8¿¿ (http://reduce-algebra.sourceforge.net).

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