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THE PROBLEM OF DEFLECTION SHAPES OF A SIMPLY SUPPORTED PLATE UNDER A LONGITUDINAL STRAIN

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Abstract. In this paper the approximate study of the problem of the bifurcation behaviour of an elastic plate with the change of the longitudinal compressive strain is carried out. A new scheme which allows to determine the critical values of the strain at which the plate takes a stable curvilinear equilibrium is constructed. The scheme also leads to an asymptotic formula which describes the nonlinear deflections of the plate when passing through the critical strain.

Keywords: Deflection of the plate, approximate study, critical strain, bifurcation points, asymptotic formulas, state of balance.

1. Formulation of the problem

Let $\Omega = \{(x,y) : 0 \leqslant x \leqslant a, 0 \leqslant y \leqslant b\}$ be a rectangular closed domain on the plane $\mathbb{R}^2$. We consider the problem of deflections of a rectangular plate P with the length a and the width b. According to the theory of flexible plates [1] differential equations, combining the function v of tensions (the Airy function) in the middle surface and the function of the deflection w for a freely supported by the contour plate, have the form

$$L_1 \equiv d \cdot \Delta^2 w - h \cdot L(w,v) = 0 \ , \tag{1}$$

$$L_2 \equiv \Delta^2 v + \frac{1}{2} E \cdot L(w,w) = 0 \ , \tag{2}$$

where Δ is the Laplace operator, nonlinear operators $L(w,v)$ and $L(w,w)$ are determined by the equality

$$L(w,v) = \frac{\partial^2 w}{\partial x^2}\,\frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 w}{\partial y^2}\,\frac{\partial^2 v}{\partial x^2} - 2\frac{\partial^2 w}{\partial x \partial y}\,\frac{\partial^2 v}{\partial x \partial y} \ , \tag{3}$$

d, h, E are known positive constants (d is stiffness for the bend, h is thickness of the plate, E is module of elasticity).

Boundary conditions relating to deformation in the middle surface have the following form:

$$\begin{array}{lll} \text{when} \quad x = 0, a & \sigma_x = \dfrac{\partial^2 v}{\partial y^2} = 0, & \tau_{xy} = \dfrac{\partial^2 v}{\partial x \partial y} = 0, \\ & w = 0, & w_{xx} = 0; \\ \text{when} \quad y = 0, b & \sigma_y = \dfrac{\partial^2 v}{\partial x^2} = -N_y, & \tau_{xy} = -\dfrac{\partial^2 v}{\partial x \partial y} = 0, \\ & w = 0, & w_{yy} = 0. \end{array} \tag{4}$$

where N_y is a longitudinal compressive strain, applied to the edges of the plate along the axis OY, the minus sign specifies that the strain N_y is compressing.

In the paper we consider a problem of bifurcation behaviour of the plate under variation of the parameter N_y. We suggest a new scheme which allows one to determine critical values of this parameter and obtain approximate formulae for the deflection function. The scheme suggested is based on methods of general theory of bifurcations described in [2].

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2. Definition of critical strains

The problem (1)-(4) is more convenient to be transformed into a different form. Assume that $c(x,y) = \frac{x^2}{2}$. It is obvious that the function $c(x,y)$ is the solution of the boundary-value problem:

$$\Delta^2 c = 0\ ,\quad 0 < x < a\ ,\quad 0 < y < b,$$

$$\begin{cases} \dfrac{\partial^2 c}{\partial y^2} = 0\ ,\ \dfrac{\partial^2 c}{\partial x \partial y} = 0 \quad \text{when} \quad x = 0, a; \\ \\ \dfrac{\partial^2 c}{\partial x^2} = 1\ ,\ \dfrac{\partial^2 c}{\partial x \partial y} = 0 \quad \text{when} \quad y = 0, b. \end{cases}$$

Let us determine the function $F(x,y) = v(x,y) + N_y c(x,y)$. Then

$$\Delta^2 v = \Delta^2 F\ ,$$

$$L(w,v) = L(w, F - N_y c) = L(w,F) - N_y L(w,c).$$

Consequently, the functions F and w are the solution of the following boundary-value problem with homogeneous boundary conditions:

$$\widetilde{L_1} \equiv d \cdot \Delta^2 w - h \cdot L(w,F) + hN_y L(w,c) = 0, \tag{5}$$

$$\widetilde{L_2} \equiv \Delta^2 F + \frac{1}{2} E \cdot L(w,w) = 0; \tag{6}$$

$$\begin{array}{lll} \text{when} \ \ x = 0, a & \sigma_x = \dfrac{\partial^2 F}{\partial y^2} = 0\ ,\ \ \tau_{xy} = \dfrac{\partial^2 F}{\partial x \partial y} = 0\ ,\ \ w = 0\ ,\ \ w_{xx} = 0; \\ \\ \text{when} \ \ y = 0, b & \sigma_y = \dfrac{\partial^2 F}{\partial x^2} = 0\ ,\ \ \tau_{yx} = -\dfrac{\partial^2 F}{\partial x \partial y} = 0\ ,\ \ w = 0\ ,\ \ w_{yy} = 0. \end{array} \tag{7}$$

The strain N_y is considered here as a real parameter.

The problem (5)-(7) was studied in many papers, in particular, its solvability was proved in [3] (p. 352–354) and the structure of a set of solutions was investigated in (p. 361–364).

In case of any value of the parameter N_y the problem (5)–(7) has a trivial solution $w(x,y) \equiv 0$, $F(x,y) \equiv 0$, but the zero solution is not always unique. This corresponds to a well known experimental fact: a plate can have several different forms of balance in case of one and the same strain. As a rule, only one form of balance is required. Transformation into other forms can cause a failure of the construction. In this connection it becomes necessary to foresee such a transformation, which is reduced to finding critical values of the strains N_y or bifurcation points of the problem (5)–(7).

From the point of view of the general theory of bifurcations, presence of critical values of the strains N_y does not denote a qualitative varying of the form of balance of the plate in case of strain transformation through such critical values. In other words, in problems of bifurcation points there are usually necessary and sufficient conditions of bifurcation. The necessary condition is connected with the fact that the corresponding linearised equations have nontrivial solutions, and the sufficient condition is connected with the transversal behaviour of the corresponding eigenvalues of the linear problem. Though the necessary condition is also the sufficient one in problems of deflations of plates.

Alongside with (5)-(7) we also consider the linear boundary-value problem

$$d \cdot \Delta^2 w = -N_y h \cdot L(w,c), \tag{8}$$

$$\begin{array}{lll} \text{for} \ \ x = 0, a & w = w_{xx} = 0; \\ \text{for} \ \ y = 0, b & w = w_{yy} = 0. \end{array} \tag{9}$$

In compliance with [3] we say that λ_0 is a bifurcation points of the problem (5)–(7), if the linear problem (8)–(9) has a nontrivial solution when $N_y = \lambda_0$.

The solution of the problem (8)-(9) can be presented in the form

$$w(x,y)=\sum_{k=1}^{\infty}\sum_{m=1}^{\infty}B_{km}\sin\frac{\pi kx}{a}\sin\frac{\pi my}{b}.$$

Then the critical strain N_y is determined from the equations $\frac{hm^2N_y}{\pi^2b^2d}=\left[\frac{k^2}{a^2}+\frac{m^2}{b^2}\right]^2$; therefore every pair k and m is put into correspondence by some bifurcation point. The lowest value of N_y is achieved when $k=1, m=1$, and

$$N_y^*=\frac{\pi^2d}{h}\cdot\frac{(a^2+b^2)^2}{a^4b^2}.$$

The number N_y^* is the lowest bifurcation point of the problem (5)-(7). A similar result was obtained, for example, in [1].

3. Auxiliary data from theory of bifurcations operator equations

To study problems of approximate construction of deflection functions in case of transition of the parameter N_y through the bifurcation points, we give some results from the paper [2] in a convenient form. Let us consider the operator equation

$$x=A(\lambda)x+a(x,\lambda), \tag{10}$$

where $A(\lambda)$ is a linear completely continuous operator $\|a(x,\lambda)\|=o(\|x\|)$, $\|x\|\to 0$ acting in the Hilbert space H.

The number λ_0 is called a bifurcation point of small solutions of the equation (10), if there is a sequence $\lambda_n\to\lambda_0$ such that $\lambda=\lambda_n$ the equation (10) has nontrivial solutions x_n such that $x_n\to 0$ when $n\to\infty$. The bifurcation points of the equation (10) should be searched only among those λ_0, for which the operator $A(\lambda_0)$ has the eigenvalue 1.

Let the following condition hold:

U1. The number 1 is a simple eigenvalue of the operator $A(\lambda_0)$.

Then there is a nontrivial vector e_0 such that $A(\lambda_0)e_0=e_0$. The adjoint operator $A^*(\lambda_0)$ also has a similar eigenvalue 1, correspondingly the eigenvector g_0. The vectors e_0 and g_0 can be chosen from the condition: $\|e_0\|=1, (e_0,g_0)=1$. Let the following condition hold alongside with U1

U2. $(A'(\lambda_0)e_0,g_0)\neq 0$, where $A'(\lambda)$ is a derivative of the operator $A(\lambda)$ with respect to the parameter λ.

Theorem 1. *Let conditions U1 and U2 hold. Then λ_0 is the bifurcation point of the equation (10).*

Bifurcating solutions of the equation (10) usually make up continuous branches $x=x(\lambda)$, where $x(\lambda)$ is a continuous function and $x(\lambda)\neq 0$ when $\lambda\neq\lambda_0$ and $x(\lambda)\to 0$ when $\lambda\to\lambda_0$. It is often convenient to search for the function $x=x(\lambda)$ in a parameter form $x=x(\varepsilon)$ and $\lambda=\lambda(\varepsilon)$, where ε is an auxiliary small parameter. We call the functions $x(\varepsilon)$ and $\lambda(\varepsilon)$ asymptotic formulae for bifurcating solutions of the equations (10).

Theorem 2. *Let the nonlinearity $a(x,\lambda)$ have the form $a(x,\lambda)=a_3(x,\lambda)+\varphi(x,\lambda)$, where $a_3(kx,\lambda)=k^3a(x,\lambda)$, $\varphi(x,\lambda)=o(\|x\|^3)$. Then the asymptotic formulae take the form:*

$$\begin{cases} x(\varepsilon)=\varepsilon e_0+\varepsilon^3e_1+o(\varepsilon^3),\\ \lambda(\varepsilon)=\lambda_0+\varepsilon^2\lambda_1+o(\varepsilon^2),\end{cases}$$

$$e_1=\Gamma_0a_3(e_0,\lambda_0),\quad \lambda_1=-\frac{\big(a_3(e_0,\lambda_0),g_0\big)}{(A'e_0,g_0)}\ .$$

Here $A' = A'(\lambda_0)$ is a linear operator Γ_0 and it is inverse to the operator $Bh = h - \lambda_0(h, g_0)A'(\lambda_0)e_0 - A(\lambda_0)h$.

4. Basic statements

Let $W_2^2(\Omega)$ be the Sobolev space, $\overset{\circ}{W}{}_2^2(\Omega)$ be a subspace of the space $W_2^2(\Omega)$, obtained by means of closure of the set of all infinitely differentiated functions with carriers in Ω (i.e. the subspace of the Sobolev space $W_2^2(\Omega)$ with elements satisfying the corresponding homogeneous conditions).

To study the problem of asymptotic functions, describing nonlinear deflections of the problem (5)–(7) during transition trough the critical strains, we suggest the following scheme.

Let us denote the operator corresponding to the solution of the boundary-value problem to the functions w via $A_0 : \overset{\circ}{W}{}_2^2(\Omega) \to \overset{\circ}{W}{}_2^2(\Omega)$

$$\Delta^2 F = -\frac{1}{2} E \cdot L(w, w), \tag{11}$$

$$\begin{aligned} &\text{when} \quad x = 0, a \qquad \sigma_x = \frac{\partial^2 F}{\partial y^2} = 0 \ , \quad \tau_{xy} = \frac{\partial^2 F}{\partial x \partial y} = 0 \ ; \\ &\text{when} \quad y = 0, b \qquad \sigma_y = \frac{\partial^2 F}{\partial x^2} = 0 \ , \quad \tau_{yx} = -\frac{\partial^2 F}{\partial x \partial y} = 0 \ . \end{aligned} \tag{12}$$

Let us consider the functional determined on $\overset{\circ}{W}{}_2^2(\Omega)$

$$f(w) = \int_0^a \int_0^b \left(d \cdot (\Delta w)^2 - \frac{h}{2} L(w, w) A_0(w) \right) dxdy \ .$$

This functional is continuously differentiable by Frechet on $\overset{\circ}{W}{}_2^2(\Omega)$, and its gradient $\nabla f : \overset{\circ}{W}{}_2^2 \to \overset{\circ}{W}{}_2^2$ is determined (see [3]) by the equality

$$\Big(\nabla f(w), \varphi \Big) = 2 \int_0^a \int_0^b \Big(d \cdot \Delta w \Delta \varphi - h L(w, A_0(w)) \varphi \Big) dxdy \ ,$$

where $\varphi = \varphi(x, y)$ is an arbitrary function from $\overset{\circ}{W}{}_2^2$.

The problem of finding solutions of the system (5)-(7) is equivalent (see [3]) to the problem of finding solutions of the equation

$$\nabla f(w) = -N_y \nabla g(w) \ , \tag{13}$$

where the functions $g(w)$ is determined by the equality

$$g(w) = h \int_0^a \int_0^b L(w, c) w dxdy.$$

Let us denote operators determined by the equalities

$$(Bw, \varphi) = 2 \int_0^a \int_0^b d \cdot \Delta w \Delta \varphi dxdy \ , \quad (Dw, \varphi) = -2h \int_0^a \int_0^b L(w, c) \varphi dxdy$$

via $B : \overset{\circ}{W}{}_2^2 \to \overset{\circ}{W}{}_2^2$ and $D : \overset{\circ}{W}{}_2^2 \to \overset{\circ}{W}{}_2^2$.

Theorem 3. *The equation $(B - \lambda D)w = 0$ when $\lambda = \lambda_0 = N_y^*$ has a nontrivial solution $w = C \sin \dfrac{\pi x}{a} \sin \dfrac{\pi y}{b}$, where C is an arbitrary constant.*

The Theorem holds because the function $w = C\sin\frac{\pi x}{a}\sin\frac{\pi y}{b}$ is the solution of the problem (8)-(9).

We rewrite the equation (13) in the following form

$$w = A(\lambda)w + a_3(w)\ , \quad w \in \overset{\circ}{W}{}_2^2\ , \quad \lambda \in \mathbb{R}\ , \tag{14}$$

where $A(\lambda) = I + B - \lambda D$, $a_3(w) = -b(w)$; here $b(w)$ is a nonlinear operator determined by the equality

$$\big(b(w), \varphi\big) = 2h\int\limits_0^a\int\limits_0^b L\big(w, A_0(w)\big)\varphi\, dxdy\ .$$

Theorem 4. *The value* $\lambda_0 = N_y^* = \frac{\pi^2 d}{h}\cdot\frac{(a^2+b^2)^2}{a^4b^2}$ *is a bifurcation point of the equation (14).*

Proof. The critical strains of the problem (5)-(7) coincide with bifurcation points of the operator equation (14). In its turn, the equation (14) is similar to the equation (10). Therefore to prove Theorem 4 it is sufficient to state that all conditions of Theorem 1 hold for the operator equation (14).

Let us verify fulfilment of condition U1.

Since the problem (8)-(9) has a unique (with precision to a factor) nontrivial solution $w = C\sin\frac{\pi x}{a}\sin\frac{\pi y}{b}$ when $\lambda = \lambda_0 = N_y^*$, then the operator $A(\lambda)w$ has a simple eigenvalue 1 corresponded by the function w when $\lambda = \lambda_0 = N_y^*$. Assume that $w = e_0$; the constant C can be choosing so from the condition $(e_0, e_0) = 1$, hence $C = \frac{2\sqrt{a^3b^3}}{\pi^2(a^2+b^2)}$. Therefore, condition U1 holds.

Let us verify condition U2. The operator $A(\lambda)w$ is selfadjount in the space $\overset{\circ}{W}{}_2^2$, therefore we can choose the function $g_0(x, y) = e_0(x, y)$ as an eigenvector of the adjoint operator. Therefore we have

$$A'e_0 = -De_0 = -h\frac{\pi^2}{b^2}e_0,$$

$$(A'(\lambda_0)e_0, g_0) = (-De_0, e_0) = -h\frac{\pi^2}{b^2} \neq 0\ .$$

Therefore, condition U2 also holds. Consequently, $\lambda_0 = N_y^*$ is a critical strain of the problem (5)-(7) or the equation (14). The Theorem has been proved.

Theorem 5. *The bifurcation solutions* w_ε *of the equation* (14) *and corresponding values of the parameter* $\lambda_\varepsilon = \lambda(w_\varepsilon)$ *can be presented in the form*

$$w_\varepsilon = \varepsilon e_0 + \varepsilon^3 e_1 + o(\varepsilon^3)\ , \quad \lambda_\varepsilon = \lambda_0 + \varepsilon^2\lambda_1 + o(\varepsilon^2)\ ,$$

where $\varepsilon > 0$ *is a small parameter,*

$$e_1 = \Gamma_0 a_3(e_0)\ , \quad \lambda_1 = \frac{\big(a_3(e_0), e_0\big)b^2}{h\pi^2}\ .$$

The operator is calculated (see [2]*) by the formula* $\Gamma_0 y = h_0 + h^0$ $\Gamma_0 : \overset{\circ}{W}{}_2^2 \to \overset{\circ}{W}{}_2^2$ *with any* $y \in: \overset{\circ}{W}{}_2^2$*, where*

$$h_0 = \frac{(y, e_0)e_0}{\lambda_0(De_0, e_0)}\ , \quad h^0 = (I - A(\lambda_0))^{-1}\left[y - \frac{(y, e_0)De_0}{(De_0, e_0)}\right]\ .$$

Validity of this statement results from Theorem 2.

Direct calculations show that the value λ_1 is presented in the form of a fraction

$$\lambda_1 = \frac{\lambda_{11}}{\lambda_{12}} ,$$

in which

$$\lambda_{11} = 8192 \cdot Eab^3\pi^2(\widetilde{\lambda}^4 - 32\pi^4)\times$$
$$\times\left(2(1+\sin\widetilde{\lambda}\sinh\widetilde{\lambda})\left(8\pi^4(a^2-b^2)^2+\widetilde{\lambda}^4(a^4+b^4)\right) - (16\pi^4+\widetilde{\lambda}^4)(3b^4+3a^4-2a^2b^2)\right) ,$$
$$\lambda_{12} = \widetilde{\lambda}^4(a^2+b^2)^4(16\pi^4-\widetilde{\lambda}^4)^3\left[\left(\frac{a^2}{b^2}+\frac{b^2}{a^2}\right)\left(\alpha-\frac{2}{\widetilde{\widetilde{\lambda}}}\right)^2\alpha^2+2\left(1-\frac{\alpha}{\widetilde{\widetilde{\lambda}}}\right)^2\right] .$$

The constants α and $\widetilde{\lambda}$ are determined from the relationships

$$\cos\widetilde{\lambda}\cdot \mathrm{ch}\widetilde{\lambda} = 1 , \quad \alpha = \frac{\sin\widetilde{\lambda}-\mathrm{sh}\widetilde{\lambda}}{\cos\widetilde{\lambda}-\mathrm{ch}\widetilde{\lambda}} .$$

The choice of values α and $\widetilde{\lambda}$ is determined by the necessity to satisfy conditions of the problem (11)-(12).

The value e_1 can be also calculated, but this demands significantly longer calculations. Let us present only a scheme of calculations for e_1.

Since $e_1 = \Gamma_0 a_3(e_0, \lambda_0)$, where Γ_0 is inverse to the operator $Bh = h - \lambda_0(h, g_0)A'(\lambda_0)e_0 - A(\lambda_0)h$, then $Be_1 = a_3(e_0, \lambda_0)$. According to Theorem 5 we have $e_1 = h = h_0 + h^0$. As a result we arrive to the equation

$$a_3(e_0, \lambda_0) = h_0 + h^0 - \lambda_0(h_0 + h^0, e_0)A'(\lambda_0)e_0 - A(\lambda_0)(h_0 + h^0),$$

which, in its turn, is simplified if we take into account that

$$(h^0, e_0) = 0, \quad Ah_0 = h_0, \quad A'(\lambda_0)e_0 = -De_0, \quad \lambda_0 = N_y^*, \quad h_0 = \frac{(a_3, e_0)b^2 e_0}{N_y^* h\pi^2}.$$

It remains only to calculate the function h^0, solving the equation

$$a_3(e_0, \lambda_0) = h^0 + (a_3, e_0)e_0 - Ah^0$$

by means of the method of indefinite coefficients.

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