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COLLAPSE OR INSTANT SOURCE OF GAS ON STRAIGHT LINE

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Abstract. In the work a partially invariant solution of the rank 2 defect 0 on four-dimensional subalgebra is constructed. The motion of allocated volume of gas is described. The motion of sonic surface is constructed, where the velocity of particles is equal to sound velocity. The motion of the sonic characteristics and sonic conoid is described. The solution specifies gas motion from the whole space towards the straight line for negative time (collapse) and from the line to the whole space for positive time (instant source). The subsonic motion occulies the whole space for infinitely large absolute value of time. The sonic surface moves from infinitely distant points towards the straight line. It is shown, that the sonic characteristics and points of sonic conoid move towards the sonic surface.

Keywords: gas dynamics, partially invariant solution, collapse, sonic conoid.

Inroduction

Gas dynamics equations (GDE)

$$\rho D\vec{u} + \nabla p = 0, D\rho + \rho\nabla \cdot \vec{u} = 0, Dp + \rho c^2\nabla \cdot \vec{u} = 0, \tag{1}$$

where $D = \partial_t + \vec{u} \cdot \nabla$ is the total derivative by the time, $\vec{u}$ is the vector of velocity, p is the pressure, ρ is the density, $c^2 = \frac{\partial p}{\partial \rho}$ is the squared sound velocity, with the equation of state with the factorized density

$$\rho = h(p)S \tag{2}$$

(S is the entropy) admit Lie algebra L_{12} of the operators [1]. A problem of enumerating all the submodules of GDE is formulated in the paper [1]. For this purpose one needs an optimal system of subalgebras. An optimal system of dissimilar subalgebras for GDE with the equation of state (2) is presented in [2]. A system of embedded subalgebras has been composed for the optimal system of subalgebras of L_{12}. It is presented in the form of a graph. A hierarchy of submodules of GDE [3] has been considered on the example of a five-dimensional self-normalized subalgebra. A graph of all subalgebras embedded into it has been composed. All invariant submodels of the graph have been obtained. Examples of regular partially-invariant submodels (RPIS), irregular partially-invariant submodels (IPIS), differentially-invariant submodels (DIS) are given. The submodels are embedded into each other in such a way that the solution of an invariant submodel of the overalgebra is a partial solution of the invariant submodel of the subalgebra. Therefore, the exact solution for the five-dimensional subalgebra is the solution for submodels of a lower dimension [4].

In the present paper the four-dimensional subalgebra of the twelve-dimensional Lie algebra is considered. A partially invariant solution of rank 0, defect 2 is constructed on the subalgebra from the graph Γ_5 of the paper [4]. The motion of particles of a given gas volume (contact characteristics) was also described. The motion of the sonic surface, where the velocity of

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particles coincides with the sound velocity is constructed. Propagation of weak discontinuities [5] on this solution is given by the sonic characteristics and the sonic conoid. The solution defines the gas motion from the whole space in the direction to the straight line when $-\infty < t < 0$ (collapse) and from the straight line into the whole space when $0 < t < +\infty$ (instant source). When $t \to -\infty$ the whole space is occupied by the subsonic motion. The sonic surface moves from the infinitely remote points to some straight line. It is shown, that weak discontinuities are accumulated on the sonic surface.

1. Partially-invariant solution

We construct a partially-invariant solution of range 0, defect 2 on the subalgebra 4.58 from the table 3 of the paper [2]. The basis of the subalgebra in the Cartesian of coordinates is: $\{b\partial_x + \partial_y, t\partial_y + \partial_v, t\partial_z + \partial_w, t\partial_t - u\partial_u - v\partial_v - w\partial_w + 2\rho\partial_\rho\}$. To construct the solution we need to calculate point invariants. From the expression for invariants we find velocities, the pressure and the density are considered to be functions of a general form. The solution has the form:

$$u = t^{-1}u_0, v = t^{-1}(b^{-1}(v_0 - x) + y), w = t^{-1}(w_0 + z), \rho = \rho(t, x, y, z), p = p(t, x, y, z),$$

u_0, v_0, w_0 are constants (invariants). Due to translations admitted by GDE, we make $v_0 = w_0 = 0$. Substituting this presentation of the solution into (1) we obtain the following equations:

$$\begin{gathered} t\rho_t + u_0\rho_x + (y - b^{-1}x)\rho_y + z\rho_z + 2\rho = 0 \Rightarrow \rho = t^{-2}R(x_1, y_1, z_1), \\ tp_t + u_0p_x + (y - b^{-1}x)p_y + zp_z + 2\frac{h}{h'} = 0 \Rightarrow h(p) = t^{-2}H(x_1, y_1, z_1), \\ p_x = u_0t^{-2}\rho, \quad p_y = u_0b^{-1}t^{-2}\rho, \quad p_z = 0, \end{gathered} \tag{3}$$

where $x_1 = x - u_0 \ln|t|, y_1 = t^{-1}(y - b^{-1}(x + u_0)), z_1 = t^{-1}z$. It follows from (3) that $H_{x_1} = H_{y_1} = H_{z_1} = = u_0 = 0$, i.e. $H = 1$. We have obtained the solution

$$u = 0, v = t^{-1}(y - b^{-1}x), w = t^{-1}z, \rho = t^{-2}R(x, v, w), h(p) = t^{-2} \tag{4}$$

on the subalgebra 4.58. The same solution of range 0, defect 2 has been obtained [2] on the subalgebra 4.51:$\{b\partial_x+\partial_y, t\partial_y+\partial_v, t\partial_z+\partial_w, a(t\partial_t+x\partial_x+y\partial_y+z\partial_z)+t\partial_t-u\partial_u-v\partial_v-w\partial_w+2\rho\partial_\rho\}$.

2. Motion of particles and volumes

A gas particle moves according to the equation

$$\frac{d\vec{x}}{dt} = \vec{u}(\vec{x}, t), \vec{x} = (x, y, z). \tag{5}$$

The family of integral curves of the equation (5) are world lines of particles in $\mathbb{R}^4$. The projection of world lines on $\mathbb{R}^3(\vec{x})$ are trajectories of particles. With formulae (4) the equations (5) have the integrals

$$\left\{ \begin{array}{l} x = x_0, \\ y = b^{-1}x_0 + v_0t, \\ z = w_0t, \end{array} \right. \tag{6}$$

where x_0, v_0, w_0 are constants that are global Lagrangian coordinates. The Jacobi matrix $J = \dfrac{\partial(x, y, z)}{\partial(x_0, v_0, w_0)} = \left\| \begin{matrix} 1 & 0 & 0 \\ b^{-1} & t & 0 \\ 0 & 0 & t \end{matrix} \right\|$ has the determinant equal to t^2. When $t = 0$ the determinant of the matrix J is equivalent to 0, the rank of the matrix is equal to 1. Consequently, when $t = 0$ all the particles fall on the straight line $l : by = x, z = 0$, which is the manifold of collapse when $-\infty < t < 0$ or the manifold of an instant source when $0 < t < +\infty$. If α is an angle between the axis Oy and the straight line l, then $\tan \alpha = b$.

Remark. The transformation of $t \to -t, v \to -v, w \to -w$ leaves the solution (4) invariant. Hence when $t > 0$ there exist an instant source (explosion) with motion of particles along the same world lines as during the collapse $t < 0$, but in the opposite direction. Consequently, it is sufficient to consider the motion of particles either when $-\infty < t < 0$ or when $0 < t < +\infty$.

The formulae (6) define the straight line in $\mathbb{R}^4$ and a parametric equation of the straight line (trajectory) in $\mathbb{R}^3(\vec{x})$, $(0, v_0, w_0)$ is the directing vector of the straight line in $\mathbb{R}^3$. Particles move along straight lines in the planes parallel to the plane yOz.

Let us consider motion of a given gas volume when $t < 0$. For this purpose it is sufficient to consider the motion of the section of this volume by a plane, which is parallel to the plane yOz, since the motion is similarly in other parallel planes. Let the section by the plane $x = x_0$ when $t = t_0 < 0$ is a circle of radius R_0 with the centre (y_{01}, z_{01}) :

$$(y_0 - y_{01})^2 + (z_0 - z_{01})^2 = {R_0}^2. \tag{7}$$

The following cases are possible.

a) The circle lies in the plane parallel to yOz, so that the point M of the crossing l with the plane is outside the circle. The coordinates of the centre of the circle at the moment t_0 are given by the equations

$$\begin{cases} y_{01} = b^{-1}x_0 + v_{01}t_0, \\ z_{01} = w_{01}t_0, \end{cases} \tag{8}$$

where v_{01}, w_{01} are velocities of the centre of the circle. The points on the boundary of the domain at the moment t_0 are given by the equations

$$\begin{cases} y_0 = b^{-1}x_0 + v_0t_0, \\ z_0 = w_0t_0. \end{cases} \tag{9}$$

Substitution of (8), (9) into (7) provides connection between global Lagrangian coordinates of the points on the boundary of the domain and the centre

$$(v_0 - v_{01})^2 + (w_0 - w_{01})^2 = \frac{{R_0}^2}{{t_0}^2}. \tag{10}$$

Substitution of (6) into (10) provides the equation of motion of the domain boundary

$$(y - b^{-1}x_0 - v_{01}t)^2 + (z - w_{01}t)^2 = \frac{{R_0}^2t^2}{{t_0}^2}. \tag{11}$$

When $t \to 0$ the radius $\dfrac{{R_0}^2t^2}{{t_0}^2}$ decreases and at the moment $t = 0$ is reduced to a point. When $t > 0$ particles move in the same plane symmetrically with respect to the point $M(x_0, b^{-1}x_0, 0)$ (see the remark, Figure 1).

b) The circle lies in the plane, parallel to yOz, so that the point M of the crossing l with the plane is on the circle. The coordinates of the centre of the circle (8) at the moment $t_0 < 0$ satisfy the condition

$$\left(y_{01} - b^{-1}x_0\right)^2 + {z_{01}}^2 = {R_0}^2 \Rightarrow {v_{01}}^2 + {w_{01}}^2 = \frac{{R_0}^2}{{t_0}^2}.$$

Then the equation of the motion of the plane boundary takes the form

$$\left(y - b^{-1}x_0\right)\left(y - b^{-1}x_0 - 2v_{01}t\right) + z\left(z - 2w_{01}t\right) = 0.$$

When $t \to 0$ particles are retracted from the circle into the point on the straight line. The point of contact M of the circle and the straight line l remain immobile. When $t > 0$ particles move symmetrically with respect to the point M (Figure 2).

c) The point M of crossing of the straight line l and the plane parallel to the plane yOz is inside the circle. When $t \to 0$ particles are retracted from the circle to the point on the straight

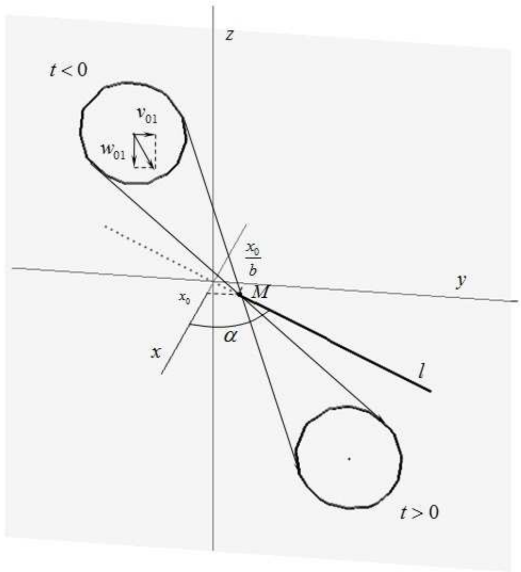


FIGURE 1. Motion of the circular cross-section. The point M is on the circle

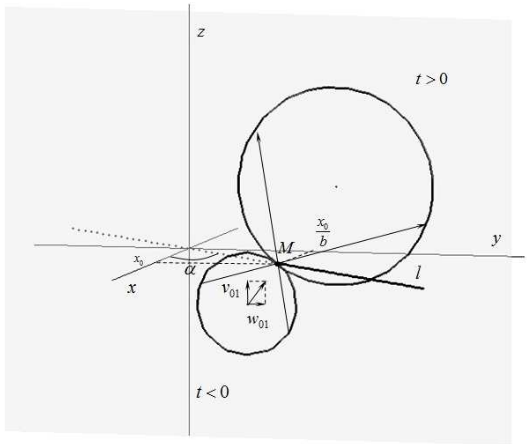


FIGURE 2. Motion of the circular cross-section. The point M is outside the circle

line. When $t > 0$ particles situated at one point of the straight line l ($\rho = \infty$) scatter on all the plane yOz symmetrically with respect to the point M(Figure 3).

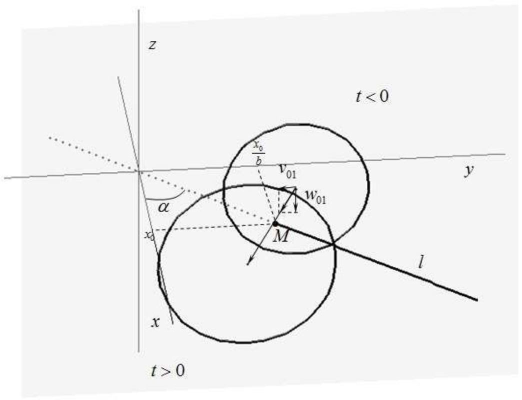


FIGURE 3. Motion of the circular cross-section. The point M is inside the circle

3. MOTION OF THE SONIC SURFACE

By the sonic surface we understand the surface, where the particles velocity is equal to the sound velocity

$$u^2 + v^2 + w^2 = c^2. \tag{12}$$

For the equation of state with the factorized density (2) the sound velocity is determined by the expression $c^2 = \dfrac{h(p)}{\rho h'(p)}$. On the solution (4) the equation of the sonic surface (12) takes the form

$$\left(y - b^{-1}x\right)^2 + z^2 = \frac{t^2 K(t)}{R\left(x, t^{-1}(y - b^{-1}x), t^{-1}z\right)}, \tag{13}$$

where $K^{-1} = h'\left(g(t^{-2})\right)$, g is the inverse function to h. Let us consider the polytropic gas with the equation of the state

$$p = B(S)\rho^{\gamma}, 1 < \gamma < 2. \tag{14}$$

It follows from (2) and (14), that $h(p) = p^{1/\gamma}$. Then in (13) $t^2K(t) = \gamma t^{4-2\gamma}$. The equation (13) takes the form

$$\left(y - b^{-1}x\right)^2 + z^2 = \frac{\gamma t^{4-2\gamma}}{R\left(x, t^{-1}(y - b^{-1}x), t^{-1}z\right)}. \tag{15}$$

Therefore, the form of the sonic surface depends on the form of the function $R(x, v, w)$ and the choice of this function determines the motion of the sonic surface. Let us consider different types of sonic surfaces.

1) Let $R = \left(t^{-2}(y - b^{-1}x)^2 + t^{-2}z^2\right)^{-1} x^{-1}$. Then the corresponding sonic surface is the plane $x = \dfrac{1}{\gamma}t^{4-2\gamma}$. When $-\infty < t < 0$ the sonic surface moves in the direction to the plane $x = 0$ from the positive values x. The domain with supersonic velocities is beyond the front of motion of the sonic surface, and the domain of subsonic velocities is in front of it. When $0 < t < +\infty$ the sonic surface moves from the plane $x = 0$ backwards. The domain with supersonic velocities is in front of the front. It is beyond the front of motion of a sonic surface in case of subsonic velocities.

2) Let $R = x^{-1}$. Then the equation of the sonic surface takes the form $(y - b^{-1}x)^2 + z^2 = \gamma x t^{4-2\gamma}$. This equation describes an elliptical paraboloid. When $t \to -\infty$ the sonic surface asymptotically converges to the plane $x = 0$. When $t \to 0$ the elliptical paraboloid degenerates into the straight line $by = x, z = 0$. Inside the sonic paraboloid $(y - b^{-1}x)^2 + z^2 < \gamma x t^{4-2\gamma}$ the particles velocity $\sqrt{(y - b^{-1}x)^2 + z^2}|t|^{-1} < \sqrt{\gamma x}|t|^{1-\gamma}$ is less

than the sound velocity $c = \sqrt{\gamma x}|t|^{1-\gamma}$. Outside the sonic paraboloid the particles velocity is supersonic. When $t \to +\infty$ the sonic surface asymptotically converges to the plane $x = 0$.

3) Let us consider in details the case, when the function R is constant ($R = \rho_0$). The sonic surface is given by the equation

$$\left(y - b^{-1}x\right)^2 + z^2 = {c_0}^2 t^{4-2\gamma}, \tag{16}$$

where ${c_0}^2 = \gamma {\rho_0}^{-1}$. The sound velocity in all points of the space is equal to $c = c_0|t|^{1-\gamma}$. The sonic surface is an elliptical cylinder with a straight line parallel to the axis of the cylinder ($by = x, z = 0$) and with the directing circle $y^2 + z^2 = {c_0}^2 t^{4-2\gamma}$ in the plane $x = 0$. Let us rotate the axes x and y by the angle α around the axis z so that the axis x transforms into the straights line $by = x, z = 0$. Then in the new variables (x', y', z') the directing curve is the ellipse ${y'}^2(1 + b^{-2}) + {z'}^2 = {c_0}^2 t^{4-2\gamma}$. When $t \to 0$ the cylinder collapses on the axis x'. Outside the sound cylinder $\sqrt{(y - b^{-1}x)^2 + z^2} > c_0|t|^{2-\gamma}$ the particles velocity $\sqrt{(y - b^{-1}x)^2 + z^2}|t|^{-1} > c_0|t|^{1-\gamma} = c$ is higher than the sound velocity. Inside the sound cylinder the particles velocity is subsonic. Since when $t \to 0$ the sound velocity increases in all the space, whereas the velocity of every particle remains unchanged, then particles outside the sonic surface after some time appear inside the sonic surface. Velocities of the particles, which used to be supersonic, become subsonic after they pass the sonic surface.

Proposition. Any particle belonging to the sound cylinder at some moment of time t_0 can appear inside the sound cylinder at the sequent moment of time $t_0 < t < 0$.

Proof. Let us consider the section of the sonic surface with the plane $x = 0$. In the plane yOz we obtain the section of the sonic surface which is a circle $y^2 + z^2 = {c_0}^2 t^{4-2\gamma}$. At the moment $t_0 < 0$ we take the point $A(0, 0, z_0)$ on the sound cylinder $z_0 = c_0|t_0|^{2-\gamma}$, then the particle belonging to the point A, according to (6), has the Lagrangian coordinates $x_0 = 0, v_0 = 0, z_0 = w_0 t_0$. This particle moves according to the law $x = 0, y = 0, z = z_0 t/t_0 = c_0|t|^{1-\gamma}|t|$. The radius of the section of the sound cylinder varies according to the law $r = c_0|t|^{2-\gamma}$. Let us compare the laws of motion of particles and the point A of the sonic surface when $t_0 < t < 0, |t| < |t_0|$: $\dfrac{z}{r} = \left(\dfrac{|t_0|}{|t|}\right)^{1-\gamma} < 1$. Therefore, the particle appears inside the sound cylinder.

4. Motion of sound characteristics

Propagation of weak discontinuities (further perturbations) is given by sound characteristics and a sonic conoid. To find the equations of characteristics given by the equation $F(t, \vec{x}) = \mathrm{const}$ we solve the equation [5]

$$F_t + uF_x + vF_y + wF_z = \pm c\sqrt{{F_x}^2 + {F_y}^2 + {F_z}^2}. \tag{17}$$

We consider the Cauchy problem for (17): $F(t_0, \vec{x}) = F_0(\vec{x})$. Then the solution of the Cauchy problem determines the characteristics passing through the surface $F_0(\vec{x}) = \mathrm{const}$. The solution of the Cauchy problem is constructed by the method of characteristics. Characteristics of the equation (17) are called bicharacteristics of the initial GDE [5]. They satisfy the equations

$$\frac{d\vec{x}}{dt} = \vec{u} \pm c\frac{\nabla F}{|\nabla F|}, \frac{dF_j}{dt} = -\vec{u_j} \cdot \nabla F \mp c_j|\nabla F|, j = x, y, z.$$

For the solution (4) in the case $R=\rho_0$ the equations of bicharacteristics for the polytropic gas have the form

$$\begin{gathered}\frac{dt}{-1}=\frac{dx}{\mp c_0|t|^{1-\gamma}\dfrac{F_x}{|\nabla F|}}=\frac{dy}{-\left(\dfrac{y}{t}-\dfrac{x}{bt}\pm c_0|t|^{1-\gamma}\dfrac{F_y}{|\nabla F|}\right)}=\\ =\frac{dz}{-\left(\dfrac{z}{t}\pm c_0|t|^{1-\gamma}\dfrac{F_z}{|\nabla F|}\right)}=\frac{-btdF_x}{F_y}=\frac{tdF_y}{F_y}=\frac{tdF_z}{F_z}.\end{gathered}\tag{18}$$

The integrals of the system (18) are as follows:

$$\begin{gathered}F_x=F_1-\frac{F_2}{bt},\quad F_y=\frac{F_2}{t},\quad F_z=\frac{F_3}{t},\\ |\nabla F|=|t|^{-1}\sqrt{(F_1t-b^{-1}F_2)^2+{F_2}^2+{F_3}^2},\end{gathered}\tag{19}$$

$$\begin{aligned}x&=\mp\int_{t_0}^{t}\frac{c_0|t|^{1-\gamma}\left(F_1t-b^{-1}F_2\right)dt}{\sqrt{(F_1t-b^{-1}F_2)^2+{F_2}^2+{F_3}^2}}+x_0,\\ y&=\mp t\int_{t_0}^{t}\frac{c_0|t|^{1-\gamma}\left((1+b^{-2})F_2-b^{-1}tF_1\right)dt}{t\sqrt{(F_1t-b^{-1}F_2)^2+{F_2}^2+{F_3}^2}}\mp\\ &\mp b^{-1}\int_{t_0}^{t}\frac{c_0|t|^{1-\gamma}\left(F_1t-b^{-1}F_2\right)dt}{\sqrt{(F_1t-b^{-1}F_2)^2+{F_2}^2+{F_3}^2}}+b^{-1}x_0+\left(y_0-b^{-1}x_0\right)\frac{t}{t_0},\\ z&=\mp t\int_{t_0}^{t}\frac{c_0|t|^{1-\gamma}F_3dt}{t\sqrt{(F_1t-b^{-1}F_2)^2+{F_2}^2+{F_3}^2}}+z_0\frac{t}{t_0},\end{aligned}\tag{20}$$

where $x_0,y_0,z_0,F_i,i=1,2,3$ are constants. When $t=t_0$ the values x_0,y_0,z_0 of the integrals (20) satisfy the equation $F_0(x_0,y_0,z_0)=\text{const},\nabla F|_{t=t_0}=\nabla_0F_0$. Whence we find values of constant integrals (19) $F_1=F_{0x_0}+b^{-1}F_{0y_0},F_2=t_0F_{0y_0},F_3=t_0F_{0z_0}$. There are two free parameters remaining in the equations (20). Therefore, the equations (20) give moving two-dimensional surface in $\mathbb{R}^3$.

Let the initial velocity when the perturbation occurs at the moment of time $-\infty<t_0<0$ be an elliptical cylinder, coaxial with the sonic cylinder

$$F_0:(y_0-b^{-1}x_0)^2+(1+b^{-2}){z_0}^2={R_0}^2.$$

Then the equations of sonic characteristics (20) passing through F_0:

$$F_-:(y-b^{-1}x)^2+(1+b^{-2})z^2=t^2\left(\frac{R_0}{|t_0|}-c_0\frac{\sqrt{1+b^{-2}}}{(\gamma-1)}\left(|t|^{1-\gamma}-|t_0|^{1-\gamma}\right)\right)^2={R_-}^2,$$

$$F_+:(y-b^{-1}x)^2+(1+b^{-2})z^2=t^2\left(\frac{R_0}{|t_0|}+c_0\frac{\sqrt{1+b^{-2}}}{(\gamma-1)}\left(|t|^{1-\gamma}-|t_0|^{1-\gamma}\right)\right)^2={R_+}^2,$$

define elliptical cylinders which semiaxes vary in time. There are two possible cases.

1) If the perturbation occurs on the surface covering the sonic cylinder $\left(\dfrac{R_0}{\sqrt{1+b^{-2}}}>c_0|t_0|^{2-\gamma}\right)$, then it moves inside this surface along the characteristics F_- and F_+ in the supersonic motion.

The perturbation which moves along the F_- characteristics passes the sonic cylinder when

$$\frac{c_0|t_0|\left(\sqrt{1+b^{-2}}+(\gamma-1)\right)}{R_0(\gamma-1)+\sqrt{1+b^{-2}}c_0|t_0|^{2-\gamma}} \leqslant |t|^{\gamma-1} \leqslant \frac{c_0|t_0|\gamma\sqrt{1+b^{-2}}}{R_0(\gamma-1)+\sqrt{1+b^{-2}}c_0|t_0|^{2-\gamma}}$$

and collapses into the straight line $by = x, z = 0$ when $|t|^{\gamma-1} = \dfrac{c_0|t_0|\sqrt{1+b^{-2}}}{R_0(\gamma-1)+\sqrt{1+b^{-2}}c_0|t_0|^{2-\gamma}}$. Further the perturbation reflects from the straight line and moves through the sonic surface once again when

$$\frac{c_0|t_0|\sqrt{1+b^{-2}}(2-\gamma)}{R_0(\gamma-1)+\sqrt{1+b^{-2}}c_0|t_0|^{2-\gamma}} \leqslant |t|^{\gamma-1} \leqslant \frac{c_0|t_0|\left(\sqrt{1+b^{-2}}-(\gamma-1)\right)}{R_0(\gamma-1)+\sqrt{1+b^{-2}}c_0|t_0|^{2-\gamma}}.$$

The greater of the semiaxes F_- reaches the layer of the radius r^* (the stop point, see Figure 4) when

$$|t|^{\gamma-1} = \frac{c_0|t_0|\sqrt{1+b^{-2}}(2-\gamma)}{R_0(\gamma-1)+\sqrt{1+b^{-2}}c_0|t_0|^{2-\gamma}}.$$

When $t = 0$ the perturbation collapses on the straight line. The perturbation propagating along the F_+ characteristics does not move through the sonic surface and collapses upon the straight line $by = x, z = 0$ when $t = 0$ (Figure 4).

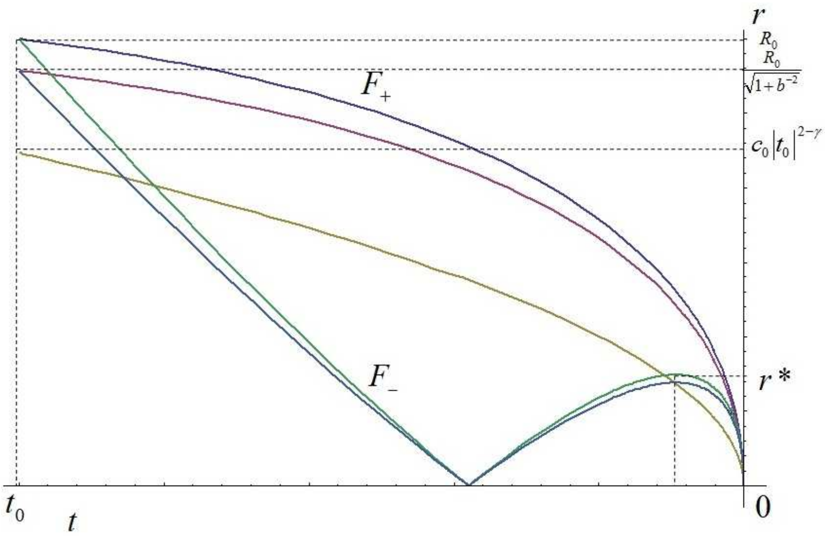


FIGURE 4. Characteristics with $t < 0$

2) If the sonic cylinder covers the surface where the perturbation occurred ($R_0 < c_0|t_0|^{2-\gamma}$), then the part of the perturbation F_+ moves outside $F_0 = \text{const}$, where the perturbation occurred and the other part F_- moves inside $F_0 = \text{const}$. The perturbation which moved along the F_+ characteristics crosses the sound cylinder when

$$\frac{c_0|t_0|\sqrt{1+b^{-2}}(2-\gamma)}{\sqrt{1+b^{-2}}c_0|t_0|^{2-\gamma}-R_0(\gamma-1)} \leqslant |t|^{\gamma-1} \leqslant \frac{c_0|t_0|\left(\sqrt{1+b^{-2}}-(\gamma-1)\right)}{\sqrt{1+b^{-2}}c_0|t_0|^{2-\gamma}-R_0(\gamma-1)},$$

and the greater of the semiaxes F_+ reaches the layer of the radius r_+^* when

$$|t|^{\gamma-1} = \frac{c_0|t_0|\sqrt{1+b^{-2}}(2-\gamma)}{\sqrt{1+b^{-2}}c_0|t_0|^{2-\gamma}-R_0(\gamma-1)}.$$

Further the perturbation stops, moves backwards and collapses on the straight line $t = 0$ (Figure 5). The perturbation moving by F_- characteristics, reaches the straight line $by =$

$x, z = 0$ when $|t|^{\gamma-1} = \dfrac{c_0|t_0|\sqrt{1+b^{-2}}}{R_0(\gamma-1)+\sqrt{1+b^{-2}}c_0|t_0|^{2-\gamma}}$, reflects from it, crosses the sonic cylinder when

$$\frac{c_0|t_0|\sqrt{1+b^{-2}}(2-\gamma)}{R_0(\gamma-1)+\sqrt{1+b^{-2}}c_0|t_0|^{2-\gamma}} \leqslant |t|^{\gamma-1} \leqslant \frac{c_0|t_0|\left(\sqrt{1+b^{-2}}-(\gamma-1)\right)}{R_0(\gamma-1)+\sqrt{1+b^{-2}}c_0|t_0|^{2-\gamma}}.$$

When $|t|^{\gamma-1} = \dfrac{c_0|t_0|\sqrt{1+b^{-2}}(2-\gamma)}{R_0(\gamma-1)+\sqrt{1+b^{-2}}c_0|t_0|^{2-\gamma}}$ the greater semiaxis F_- reaches the value r_-^* (the stop point), and when $t = 0$ the perturbation collapses again on the straight line (Figure 5).

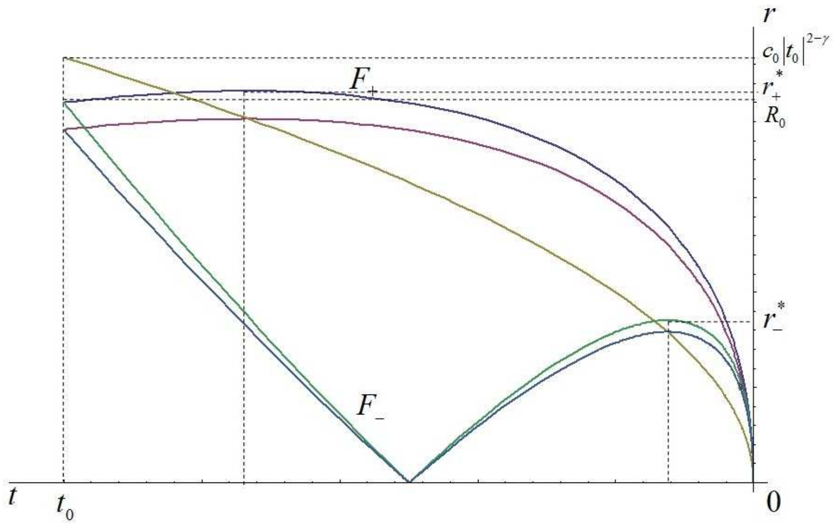


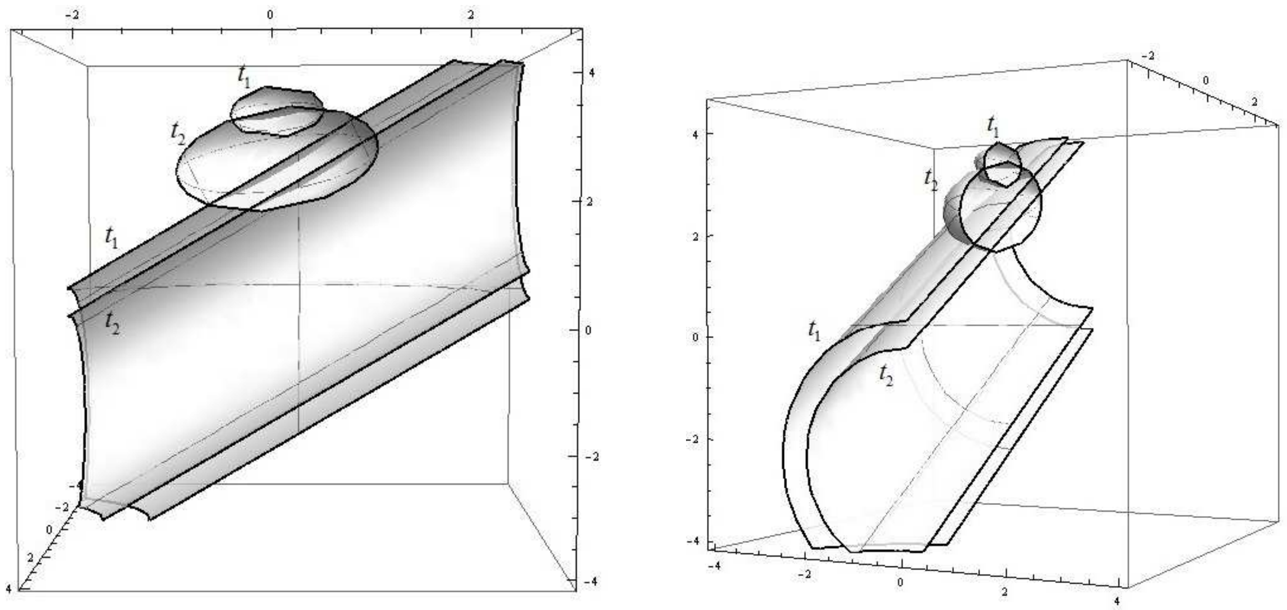
Figure 5. Characteristics with $t < 0$

5. Motion of the characteristic conoid

The characteristic conoid is a geometric place of all bicharacteristics (18), coming out of the point $P(t_0, \vec{x_0})$ [5]. When $t = t_0$ the integrals (20) satisfy the conditions $\vec{x}(t_0) = \vec{x_0}$, and the dilation of the parameters $F_j, j = 1, 2, 3$ does not change the values of the integrals. Therefore one can assume that ${F_1}^2 + {F_2}^2 + {F_3}^2 = 1$, and for the parameters one can introduce spherical coordinates on a unit sphere. Hence (20) defines the surface (conoid) in a parametric form moving in $\mathbb{R}^3$.

The integrals (20) are not calculated analytically therefore we have conducted numerical calculations in Maple 12. Let us consider the motion of the conoid from some fixed point. The following cases are possible.

1) Let this point at the moment of time $t_0 < 0$ is in the domain of the supersonic motion (outside the sonic cylinder). Then, as it results from considerations for sonic characteristics, the conoid moves, in the supersonic domain, in the direction of the sonic cylinder. When $t \to 0$ some points of the conoid cross the sonic cylinder, reach the axis of the cylinder and cross the other side of the sonic cylinder. At some moment these points stop at some distance from the sonic cylinder and start moving backwards. The points of the conoid in the subsonic domain move along the axis of the cylinder. All the points of the conoid collapse upon the straight line l when $t = 0$ (Figure 6, 7).



a) View from the side of the axis z b) View from the side of the straight line l

FIGURE 6. Two positions of the sonic surface and the conoid at the moments $t_1 < t_2 < 0$ when the apex of the conoid is in the supersonic domain

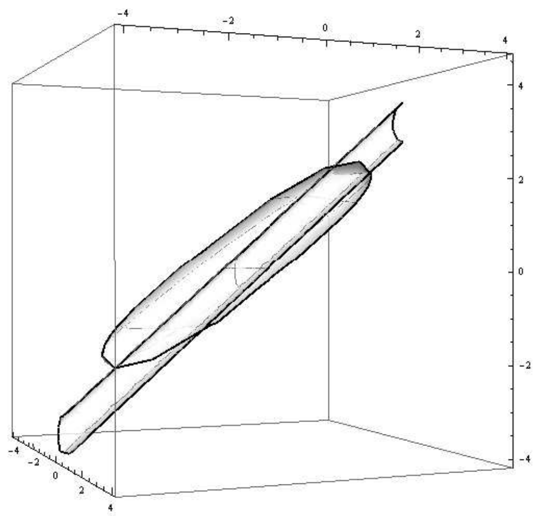
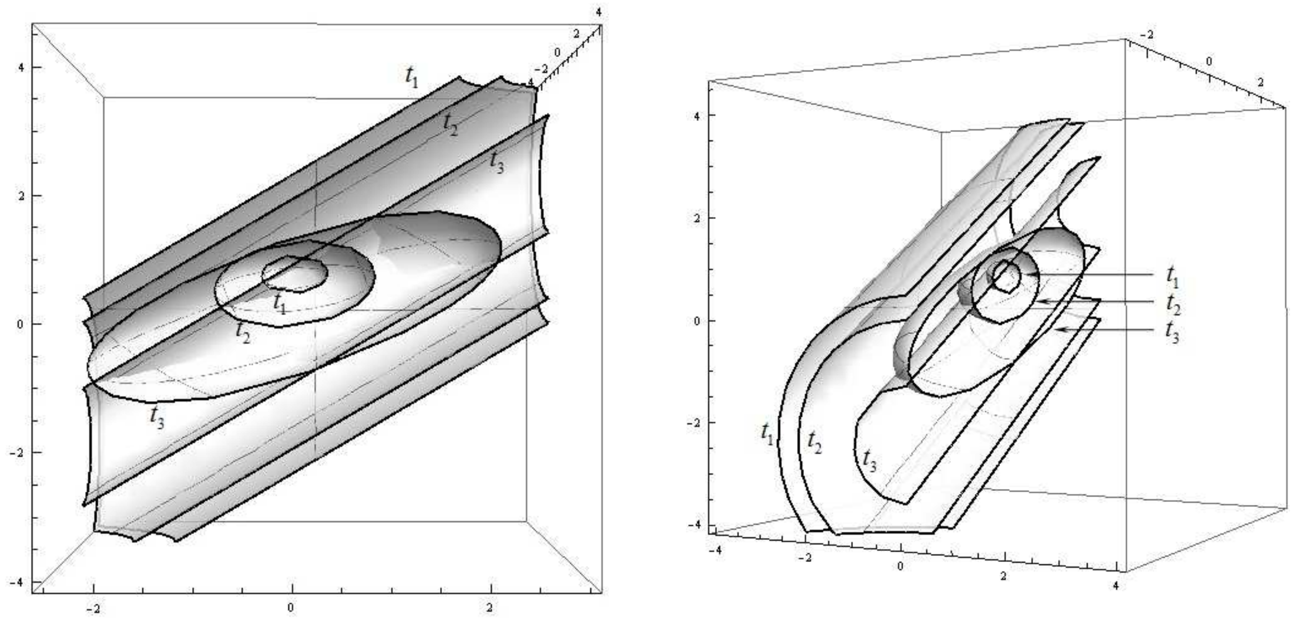


FIGURE 7. Position of the sonic surface and the conoid, when the conoid intersects the sonic cylinder twice

2) Let the beginning of the conoid at the moment of time $t_0 < 0$ appears in the domain of the subsonic motion (inside the sonic cylinder). Then as $t \to 0$ some points of the conoid move into the supersonic zone, moving off the sonic cylinder. At some moment these points stop and start moving to the sonic cylinder. Other points which have remained in the subsonic domain move along the axis of the cylinder (Figure 8). All the points collapse on the straight line l when $t = 0$.



a) View from the side of the axis z b) View from the side of the straight line l

FIGURE 8. Three positions of the sonic surface and the conoid at the moments $t_3 < t_1 < t_2 < 0$ when the apex of the conoid is in the subsonic domain

Thus, we have shown that sonic characteristics and points of the sonic conoid in the course of time ($t < 0$) approach the sonic surface.

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