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GENERALIZED WEBSTER EQUATION: EXACT AND APPROXIMATE RENORMGROUP SYMMETRIES, INVARIANT SOLUTIONS AND CONSERVATION LAWS.

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Abstract. The exact point symmetry group for the generalized Webster type equation, which describes nonlinear acoustic waves in lossy channels with variable cross-sections, is found. It is shown that, for certain types of cross-section profiles S, the admitted three-dimensional point symmetry group is extended and group classification problem for different types of S is solved. Optimal systems of one-dimensional subalgebras of the admitted Lie algebra are revealed and the invariant solutions corresponding to these subalgebras are obtained. Approximate renormgroup symmetries and the corresponding approximate analytic solutions, as well as conservation laws to the generalized Webster equation are derived for channels with constant and smoothly varying or constant cross-sections and arbitrary initial conditions.

Keywords: Webster equation, exact and approximate renormgroup symmetries, invariant solutions, conservation laws.

Introduction

The generalized Webster equation

$$\frac{\partial^2 p}{\partial t^2} - c^2 \frac{\partial^2 p}{\partial x^2} = c^2 \frac{\mathrm{d}\ln S(x)}{\mathrm{d}x}\frac{\partial p}{\partial x} + \frac{\varepsilon}{c^2\rho}\frac{\partial^2 p^2}{\partial t^2} + \frac{b}{\rho}\frac{\partial^3 p}{\partial x \partial t^2} \tag{1}$$

occurs in problems of propagation of the intensive sound [1, 2] in pipes, horns, concentrators and other wavequiding systems with varying cross-section $S(x)$. Here t is the time, x is the space coordinate calculated along the axis of the system, p is the sound pressure, c is the sound velocity, ρ is the medium density. The equation (1) is applicable for pipes, which characteristic width is small as compared with the wave length. Besides, the cross-section is supposed to be smoothly varying along x: the area $S(x)$ should vary a little with the increase of x upon the value of the order of the width of the pipe [3]. The generalized Webster equation (1) differs from the linear Webster equation [4, 5] by the presence of two supplementary terms describing non-linear and dissipative effects: ε, b are parameters of non-linearity and dissipation (the notations are similar to the ones in the book [6]).

Assuming every term in the right-hand side of the equation to be small as compared to its left-hand side, let us consider a running wave. Then, applying the method of smoothly varying profile [6] and following the standard procedure [7] we obtain the evolutionary equation

$$\frac{\partial p}{\partial x} - \frac{\varepsilon}{c^3\rho}p\frac{\partial p}{\partial \tau} - \frac{b}{2c^3\rho}\frac{\partial^2 p}{\partial \tau^2} + \frac{p}{2}\frac{\mathrm{d}\ln S(x)}{\mathrm{d}x} = 0\,, \tag{2}$$

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where $\tau = t - x/c$ is the "slow" time in the coordinate system moving with the sound. Proceeding in this equation from physical variables to the normalized ones,

$$x \to \frac{c}{\omega}\, x\,, \quad \tau \to \frac{\tau}{\omega}\,, \quad p \to p_0 p\,,$$

we rewrite the generalized Webster equation (2) in the form

$$\frac{\partial p}{\partial x} - ap\frac{\partial p}{\partial \tau} - \nu\frac{\partial^2 p}{\partial \tau^2} + \frac{p}{2}\frac{\mathrm{d}\ln S(x)}{\mathrm{d}x} = 0\,, \quad p(0,\tau) = P(\tau)\,. \tag{3}$$

Normalized constants ω, p_0 here have the meaning of characteristic values of frequency and amplitude of the signal, respectively. Two parameters in the equation (3) are given by the following dimensionless combinations of constants:

$$a = \frac{\varepsilon p_0}{c^2\rho}\,, \quad \nu = \frac{b\omega}{2c^2\rho}\,.$$

Their ratio a/ν is called the acoustic Reynolds number [6]. It characterises relative contribution of non-linear and dissipative effects into the distortion of the wave profile. With high values of a/ν nonlinearity predominates, while low values of dissipation dominates here. Without loosing generality in the equation (3) it is possible to assume that $S(0) = 1$.

We can remove the last term in (3) if we replace the variable x by the absorption depending on the coordinate along the channel and given by the function μ:

$$z = \int \frac{1}{\sqrt{S(x)}}\,\mathrm{d}x\,, \quad p\sqrt{S} = u\,, \quad \mu = \nu\sqrt{S(x(z))}\,. \tag{4}$$

Then the equation (3) in the new variables takes the form

$$\frac{\partial u}{\partial z} - au\frac{\partial u}{\partial \tau} - \mu\frac{\partial^2 u}{\partial \tau^2} = 0\,, \quad u(0,\tau) = P(\tau)\,. \tag{5}$$

Introducing the new variable q connected with u by $u = 2(\partial q/\partial \tau)$, we write the modified generalized Webster equation

$$F \equiv \frac{\partial q}{\partial z} - a\left(\frac{\partial q}{\partial \tau}\right)^2 - \mu\frac{\partial^2 q}{\partial \tau^2} = 0\,, \quad q(0,\tau) = Q(\tau) \tag{6}$$

instead of (3). The generalized Webster equation in the forms (5) and (6) is used not only as a model of wave propagation in the pipes [8], but also for the calculation of acoustic field in inhomogeneous media in the geometric acoustics approximation [1, 9] functioning therewith as transfer equation, written in ray coordinates. The axis of the ray pipe is a geometric ray calculated from the eikonal equation, and the function $S(x)$ is a cross-section of the ray pipe. Application of the equations (5) and (6) containing, unlike linear Webster equations, supplementary contributions responsible for non-linear effects and absorption, opens up possibilities for investigating problems of propagation of sound waves of a finite amplitude in absorbing media, in particular, for acoustic sounding of media, through which sound waves propagate [10].

One of the most effective instruments of constructing solutions of the generalized Webster equation is application of methods of modern group analysis. In the present paper we apply this approach to the symmetries of the Webster equations. Such a problem has already been analysed for the generalized Burgers' equation (5), known also as the Burgers' equation with variable viscosity. E.g. in [11] exact symmetries were found, the cases of extension of the admitted group with different values of $S(x)$ were pointed out and finite transformations of the group were constructed. A form of representation of an invariant solution for the operators, extending the admitted group of symmetries, the type of reduction of the generalized Burgers' equation to an ordinary differential equation were found there. Finite transformations between the generalized Burgers' equation (5) with different types of $S(x)$ were discussed in the papers [12, 13]. Discussion of more complex variants (5), extending this equation to the case of S depending on a greater number of variables or to the case of two-dimensional geometry, can be found, for example, in [14, 15]. In the present paper we concentrate on the Webster equation (6) which is not obtained from (5) by a point transformation.

The paper is divided into five sections. In the second section we find a group of point transformations for a modified generalized Webster equation and show, that for the profiles of the cross-section of a special type this group of symmetries extends. We solve the problem of group classification of the equation (6) with respect to the function $S(x)$ and construct optimal systems of one-dimensional

subalgebras. We also find invariant solutions for certain subalgebras. The third section of the paper is dedicated to construction of approximate symmetries and approximate analytical solutions of the generalized Webster equation for arbitrarily varying cross-sections and arbitrary initial condition. The small parameter in these constructions is slowness of varying of the profile of the wave conducting cross-section $S(x)$. The fourth section of the paper is dedicated to construction of conservation laws for the Webster equation. In the fifth section we shortly formulate general results of the paper.

1. Symmetry group and invariant solutions of the generalized Webster equation

1.1. Lie group transformations of the generalized Webster equation. *The infinitesimal operator* of the group of Lie point transformations of the generalized Webster equation (6) has the form

$$X = \xi^1(z,\tau,q)\frac{\partial}{\partial z} + \xi^2(z,\tau,q)\frac{\partial}{\partial \tau} + \eta(z,\tau,q)\frac{\partial}{\partial q}\,. \tag{7}$$

To find coefficients of the infinitesimal operator (7) we apply the standard method [16, 17]. The operator action (7) extends on the derivatives by the formula

$$\tilde{X} = X + \zeta_1\frac{\partial}{\partial q_z} + \zeta_2\frac{\partial}{\partial q_\tau} + \zeta_{22}\frac{\partial}{\partial q_{\tau\tau}} \tag{8}$$

and we write the invariance condition of the equation (6), considered as a differential manifold, under the action of the operator (8):

$$\{\tilde{X}F\}\big|_{F=0} \equiv \left\{\zeta_1 - 2a\frac{\partial q}{\partial \tau}\,\zeta_2 - \mu\zeta_{22} - \frac{\mathrm{d}\mu(z)}{\mathrm{d}z}\frac{\partial^2 q}{\partial \tau^2}\,\xi^1\right\}\Big|_{F=0} = 0\,. \tag{9}$$

The coefficients ζ_1, ζ_2 and ζ_{22} in the operator (8) and in the *determining equation* (9) are obtained by the prolongation formulae

$$\begin{aligned}
\zeta_1 &= D_z(\eta) - q_z D_z(\xi^1) - q_\tau D_z(\xi^2)\,,\\
\zeta_2 &= D_\tau(\eta) - q_z D_\tau(\xi^1) - q_\tau D_\tau(\xi^2)\,,\\
\zeta_{22} &= D_\tau(\zeta_2) - q_{z\tau} D_\tau(\xi^1) - q_{\tau\tau} D_\tau(\xi^2)
\end{aligned} \tag{10}$$

by using the operators of total differentiation

$$\begin{aligned}
D_z &= \frac{\partial}{\partial z} + q_z\frac{\partial}{\partial q} + q_{zz}\frac{\partial}{\partial q_z} + q_{z\tau}\frac{\partial}{\partial q_\tau} + \dots\,,\\
D_\tau &= \frac{\partial}{\partial \tau} + q_\tau\frac{\partial}{\partial q} + q_{z\tau}\frac{\partial}{\partial q_z} + q_{\tau\tau}\frac{\partial}{\partial q_\tau} + \dots\,.
\end{aligned} \tag{11}$$

Substitution of (10) into the determining equation (9) provides the overdetermined system of linear equations for the coefficients ξ^1, ξ^2 and η, which solution has the form:

$$\begin{aligned}
&\xi^1 = b(z)\,,\quad b(z) = \beta_0 + \beta_1 z + \beta_2 z^2\,,\quad \xi^2 = c_0 + c_1 z + \frac{\tau}{2}\left(M + \frac{\partial b}{\partial z}\right)\,,\\
&\eta = k(z,\tau)\exp\left(-\frac{aq}{\mu}\right)\delta_{M,0} + c_2 - \frac{\tau}{2a}c_1 + Mq - \frac{1}{4a}\frac{\partial^2 b}{\partial z^2}\left(\frac{\tau^2}{2} + \int \mu\,\mathrm{d}z\right)\,.
\end{aligned} \tag{12}$$

In the equations (12) the functions $b(z)$ and $\mu(z)$ and the constant $M = \mathrm{const}$ are connected by the relationship

$$M\left(\frac{\mathrm{d}}{\mathrm{d}z}\ln(\mu(z))\right)^{-1} = b(z)\,,\quad M = \mathrm{const} \neq 0\,. \tag{13}$$

The function $k(z,\tau)$ differs from zero only in case of the channel of constant cross-section with $\mathrm{d}\mu/\mathrm{d}z = 0$, when in (12) it is possible to assume that $M = 0$. Then the linear parabolic equation follows:

$$\frac{\partial k}{\partial z} - \mu\frac{\partial^2 k}{\partial \tau^2} = 0\,. \tag{14}$$

Therefore, the group of point transformations admitted by the equation (6) with the arbitrary profile of inhomogeneity $\mu(z)$ is generated by three infinitesimal operators:

$$X_1 = \frac{\partial}{\partial \tau}\,,\quad X_2 = \frac{1}{a}\frac{\partial}{\partial q}\,,\quad X_3 = z\frac{\partial}{\partial \tau} - \frac{\tau}{2a}\frac{\partial}{\partial q}\,. \tag{15}$$

The first two operators in this list are evident from the physical point of view: they correspond to translations: in the variables τ and q, and the last operator corresponds to the group of the Galilean transformations. An extension of the group generators (15) occurs for the profiles of a cross-section of a special form following the classical relationship (13), and the supplementary symmetry operator X_4 is

$$X_4 = b\frac{\partial}{\partial z} + \frac{\tau}{2}\left(M + \frac{\mathrm{d}b}{\mathrm{d}z}\right)\frac{\partial}{\partial \tau} + \left(Mq - \frac{(\tau^2 + 2\int \mu\,\mathrm{d}z)}{8a}\frac{\mathrm{d}^2 b}{\mathrm{d}z^2}\right)\frac{\partial}{\partial q}\,. \tag{16}$$

The operators (15), (16) span the four-dimensional Lie algebra $L_4 = \{X_1, X_2, X_3, X_4\}$ with the table of commutators given in Table 1.

TABLE 1. Commutators of algebra L_4

	X_1	X_2	X_3	X_4
X_1	0	0	$-X_2/2$	$(M+\beta_1)X_1/2 + \beta_2 X_3$
X_2	0	0	0	MX_2
X_3	$X_2/2$	0	0	$(M-\beta_1)X_3/2 - \beta_0 X_1$
X_4	$-(M+\beta_1)X_1/2 - \beta_2 X_3$	$-MX_2$	$(\beta_1 - M)X_3/2 + \beta_0 X_1$	0

The classifying relationship (13) is a first-order differential equation for the function $\mu(z)$, which is integrated in the explicit form and gives a three-parameter (defined by the parameters β_0, β_1 and β_2) family of curves in the space $\{z, \mu\}$:

$$\ln(\mu/\nu) = d(z) \equiv M\int\limits_0^z \frac{\mathrm{d}y}{b(y)}\,, \tag{17}$$

where the form of the function $d(z)$ depends on the relationship between the parameters β_i $(i = 0, 1, 2)$,

$$d(z) = \begin{cases} \frac{2M}{\sqrt{4\beta_0\beta_2 - \beta_1^2}}\left[\arctan\frac{\beta_1 + 2\beta_2 z}{\sqrt{4\beta_0\beta_2 - \beta_1^2}} - \arctan\frac{\beta_1}{\sqrt{4\beta_0\beta_2 - \beta_1^2}}\right], & \beta_1^2 < 4\beta_0\beta_2\,, \\ \frac{M}{\sqrt{\beta_0\beta_2}}\frac{z}{z + \sqrt{\beta_0/\beta_2}}\,, & \beta_1^2 = 4\beta_0\beta_2\,, \\ \frac{M}{\sqrt{\beta_1^2 - 4\beta_0\beta_2}}\ln\frac{(\sqrt{\beta_1^2 - 4\beta_0\beta_2} - \beta_1 - 2\beta_2 z)(\sqrt{\beta_1^2 - 4\beta_0\beta_2} + \beta_1)}{(\sqrt{\beta_1^2 - 4\beta_0\beta_2} + \beta_1 + 2\beta_2 z)(\sqrt{\beta_1^2 - 4\beta_0\beta_2} - \beta_1)}\,, & \beta_1^2 > 4\beta_0\beta_2\,. \end{cases} \tag{18}$$

The choice $M = 0$ corresponds to the channel with the constant cross-section with $\mathrm{d}\mu(z)/\mathrm{d}z = 0$, for which the classifying relationship (13) is satisfied with any β_i. This gives three operators

$$X_{41} = \frac{\partial}{\partial z}\,, \quad X_{42} = z\frac{\partial}{\partial z} + \frac{\tau}{2}\frac{\partial}{\partial \tau}\,, \quad X_{43} = z^2\frac{\partial}{\partial z} + \tau z\frac{\partial}{\partial \tau} - \frac{(\tau^2 + 2z\nu)}{4a}\frac{\partial}{\partial q}\,, \tag{19}$$

instead of the operator X_4. The first operator X_{41}, gives the translation along the axis z, the second, X_{42}, provides expansions, and the last operator, X_{43}, corresponds to the group of projective transformations. Apart from the operators (19), the generalized Webster equation also admits an operator of the infinite subgroup for the channel of the constant cross-section with $\mu \equiv \nu$

$$X_\infty = k(z,\tau)\exp\left(-\frac{aq}{\mu}\right)\frac{\partial}{\partial q}\,, \qquad \frac{\partial k}{\partial z} - \mu\frac{\partial^2 k}{\partial \tau^2} = 0\,. \tag{20}$$

Therefore the linear parabolic equation, satisfied by the function of two variables $k(z,\tau)$ can be written in the variables $\{x, \tau\}$, i.e.

$$\frac{\partial k}{\partial x} - \nu\frac{\partial^2 k}{\partial \tau^2} = 0\,.$$

We are intended to apply the last fact to construction of an approximate point symmetry for the equation (6). Let us note, that the group of symmetries (15), (19) and (20) is well known in the theory of the modified Burgers' equation [17], to which in this case the generalized Webster equation is reduced.

1.2. The equivalence group for the generalized Webster equation. To exclude the arbitrary way in the choice of the coefficients β_i and M in the function $\mu(z)$, we apply the equivalence transformations defining the form of the group generator X_4, i.e. the transformations conserving the form of the system of the equations

$$\frac{\partial q}{\partial z} - a\left(\frac{\partial q}{\partial \tau}\right)^2 - \mu\frac{\partial^2 q}{\partial \tau^2} = 0\,, \quad \frac{\partial \mu}{\partial \tau} = 0, \quad \frac{\partial \mu}{\partial q} = 0\,. \tag{21}$$

The infinitesimal operator of the equivalence group for the system of equations (21) has the form

$$E = \xi^1(z,\tau,q)\frac{\partial}{\partial z} + \xi^2(z,\tau,q)\frac{\partial}{\partial \tau} + \eta(z,\tau,q)\frac{\partial}{\partial q} + \vartheta(z,\tau,q,\mu)\frac{\partial}{\partial \mu}\,. \tag{22}$$

The coefficients of the operator (22) are calculated from the invariance conditions of the equations (21) under influence of the prolonged operator

$$\tilde{E} = E + \zeta_1\frac{\partial}{\partial q_z} + \zeta_2\frac{\partial}{\partial q_\tau} + \zeta_{22}\frac{\partial}{\partial q_{\tau\tau}} + \omega_0\frac{\partial}{\partial \mu_q} + \omega_1\frac{\partial}{\partial \mu_z} + \omega_2\frac{\partial}{\partial \mu_\tau}\,. \tag{23}$$

The coordinates ζ_1, ζ_2 and ζ_{22} are given by the formulae (10), and the coordinates ω_i are given by similar formulae with a slight difference:

$$\begin{aligned}
&\omega_0 = \tilde{D}_q(\vartheta) - \mu_z\tilde{D}_q(\xi^1) - \mu_\tau\tilde{D}_q(\xi^2) - \mu_q\tilde{D}_q(\eta)\,,\\
&\omega_1 = \tilde{D}_z(\vartheta) - \mu_z\tilde{D}_z(\xi^1) - \mu_\tau\tilde{D}_z(\xi^2) - \mu_q\tilde{D}_z(\eta)\,,\\
&\omega_2 = \tilde{D}_\tau(\vartheta) - \mu_z\tilde{D}_\tau(\xi^1) - \mu_\tau\tilde{D}_\tau(\xi^2) - \mu_q\tilde{D}_\tau(\eta)\,,\\
&\tilde{D}_z = \frac{\partial}{\partial z} + \mu_z\frac{\partial}{\partial \mu}\,,\ \tilde{D}_\tau = \frac{\partial}{\partial \tau} + \mu_\tau\frac{\partial}{\partial \mu}\,,\ \tilde{D}_q = \frac{\partial}{\partial q} + \mu_q\frac{\partial}{\partial \mu}\,.
\end{aligned} \tag{24}$$

The variables z, τ and q are considered to be independent, and μ is a dependent (differential) variable. The action of (23) on (21) provides the system of defining equations

$$\left\{\zeta_1 - 2a\frac{\partial q}{\partial \tau}\zeta_2 - \mu\zeta_{22} - \vartheta\frac{\partial^2 q}{\partial \tau^2}\right\}\Big|_{(21)} = 0\,, \quad \{\omega_0\}\big|_{(21)} = 0\,, \quad \{\omega_2\}\big|_{(21)} = 0\,. \tag{25}$$

Solution of the system (25) gives generators of the equivalence group:

$$\begin{aligned}
&E_1 = \frac{\partial}{\partial \tau}\,, \quad E_2 = \frac{1}{a}\frac{\partial}{\partial q}\,, \quad E_3 = z\frac{\partial}{\partial \tau} - \frac{\tau}{2a}\frac{\partial}{\partial q}\,,\\
&E_4 = \frac{\partial}{\partial z}\,, \quad E_5 = 2z\frac{\partial}{\partial z} + \tau\frac{\partial}{\partial \tau}\,, \quad E_6 = z\frac{\partial}{\partial z} - q\frac{\partial}{\partial q} - \mu\frac{\partial}{\partial \mu}\,.
\end{aligned} \tag{26}$$

Finite transformations of the equivalence group generated by the six-dimensional Lie algebra spanned by the operators (26), have the form

$$\begin{aligned}
&\bar{z} = (z+b_4)\mathrm{e}^{2b_5+b_6}\,, \quad \bar{\tau} = (\tau+b_1)\mathrm{e}^{b_5} + b_3(z+b_4)\mathrm{e}^{2b_5+b_6}\,,\\
&\bar{q} = (q+ab_2)\mathrm{e}^{-b_6} - \frac{b_3^2}{4a}(z+b_4)\mathrm{e}^{2b_5+b_6} - \frac{b_3}{2a}(\tau+b_1)\mathrm{e}^{b_5}\,, \quad \bar{\mu} = \mu\mathrm{e}^{-b_6}\,.
\end{aligned} \tag{27}$$

Application of the transformations (27) to the classifying relationship (13) allows to single out the classes of functions μ presented in Table 2. We reduce to them all inhomogeneous profiles for which extension of the basic algebra (15) occurs.

It is worth note the relationship of the basic types of profiles of acoustic wave conductors μ/ν (degree, linear fraction and exponential dependences of three types) resulting from the classifying relationship (13) under the group analysis of the equation (6), with those obtained for the generalized Burgers equation (5) in [11].

1.3. Optimal systems of one-dimensional subalgebras for the generalized Webster equation. To construct the optimal system of one-dimensional subalgebras $\Theta(L)$ of the algebra L spanned by the operators (15), (16), it is necessary to calculate the group of inner automorphisms. The inner automorphism of the algebra L given by any of the operators X_i of this algebra is constructed by solving the Cauchy problem

$$\frac{\mathrm{d}\bar{X}}{\mathrm{d}a_i} = \left[X_i, \bar{X}\right]\,, \quad \bar{X}\big|_{a_i=0} = X\,, \tag{28}$$

TABLE 2. Admitted values of the function μ/ν in accordance with (13)

$\beta_2 = 0$	$\beta_1 = 0,\ \beta_0 \neq 0$	$\exp(\pm z)$
	$\beta_1 \neq 0,\ \beta_0 = \text{arbitrary}$	$(1+z)^{M/\beta_1}$
$\beta_2 \neq 0$	$\beta_1^2 < 4\beta_0\beta_2$	$\exp\left[\frac{4M\beta_2}{4\beta_0\beta_2-\beta_1^2}\arctan z\right]$
	$\beta_1^2 > 4\beta_0\beta_2$	$\left(\frac{1-z}{1+z}\right)^{2M\beta_2/(4\beta_0\beta_2-\beta_1^2)}$
	$\beta_1^2 = 4\beta_0\beta_2$	$\exp\left(\pm\frac{z}{1+z}\right)$

where the operators $\bar{X}$ are represented by expansion in the basis operators X_i:

$$\bar{X} = \sum_{i=1}^{4} \bar{k}_i X_i\,, \quad \bar{k}_i = \bar{k}_i(a_i, k_1, k_2, k_3, k_4)\,. \tag{29}$$

Applying the equation (29) and Table 1 of commutators, we write the inner automorphisms as the transformations of the coordinates of the operator $\bar{X}$ in the form of Table 3 where $\Omega = \sqrt{\beta_1^2 - 4\beta_0\beta_2}$.

TABLE 3. Inner automorphisms of L_4

	$\bar{k}_1$	$\bar{k}_2$	$\bar{k}_3$	$\bar{k}_4$
A_1	$k_1 + (M+\beta_1)k_4a_1$	$k_2 - k_3a_1 - \beta_2k_4a_1^2$	$k_3 + 2\beta_2k_4a_1$	k_4
A_2	k_1	$k_2 + Mk_4a_2$	k_3	k_4
A_3	$k_1 - 2\beta_0k_4a_3$	$k_2 + k_1a_3 - \beta_0k_4a_3^2$	$k_3 + k_4(M-\beta_1)a_3$	k_4
A_4	$(1/\Omega)\mathrm{e}^{-Ma_4}\,[k_1\Omega\cosh(\Omega a_4)$ $+ (2k_3\beta_0 - k_1\beta_1)\sinh(\Omega a_4)]$	$k_2\mathrm{e}^{-2Ma_4}$	$(1/\Omega)\mathrm{e}^{-Ma_4}\,[k_3\Omega\cosh(\Omega a_4)$ $+ (k_3\beta_1 - 2k_1\beta_2)\sinh(\Omega a_4)]$	k_4

Knowledge of the inner automorphisms allows to construct optimal systems of one-dimensional subalgebras $\Theta_1(L_4)$ presented in Table 4.

1.4. Invariant solutions of the generalized Webster equation. In this section we consider some types of invariant solutions for the generalized modified Webster equation obtained using one-dimensional subalgebras from Table 4. The most interesting and non-trivial invariant solutions are obtained using the operator X_4 and its linear combinations with the operators X_1 and X_3. Applying the standard procedure [16] of representation of an invariant solution to invariants of the operator of the admitted group, we produce the type of invariant solutions for these combinations.

1. The invariant solution for the operator X_4:

$$q = \mu U(\lambda) - \frac{\beta_2}{2aM}\left(z\mu - \int \mu\,\mathrm{d}z\right) - \mu\frac{\beta_2}{4a}\lambda^2 z\,, \quad \lambda = \frac{\tau}{\sqrt{b\mu}}\,. \tag{30}$$

2. The invariant solution for the operator $X_4 + \alpha X_1$:

$$\begin{gathered} q = \mu U(\lambda) - \frac{\beta_2}{2aM}\left(z\mu - \int \mu\,\mathrm{d}z\right) - \mu\frac{\beta_2}{4a}\left(\lambda^2 z - 2\lambda\int \lambda_1\mathrm{d}z + \int \lambda_1^2\mathrm{d}z\right)\,, \\ \lambda = \frac{\tau}{\sqrt{b\mu}} + \lambda_1(z)\,, \quad \lambda_1(z) = -\alpha\int \frac{1}{b\sqrt{b\mu}}\,\mathrm{d}z\,. \end{gathered} \tag{31}$$

TABLE 4. Optimal system $\Theta_1(L_4)$ of one-dimensional subalgebras

$\beta_2 = 0$	$\beta_1 = 0$	$\beta_0 \neq 0$	$\{X_1\}, \{X_2\}, \{X_3\}, \{X_4 + \alpha X_3\}, \alpha =$ arbitrary
	$\beta_1/M = -1$	$\beta_0 \neq 0$	$\{X_1\}, \{X_2\}, \{X_3\}, \{X_3 \pm X_1\}, \{X_4 + \alpha X_3\}, \alpha =$ arbitrary
		$\beta_0 = 0$	$\{X_1\}, \{X_2\}, \{X_3\}, \{X_3 \pm X_1\}, \{X_4\}, \{X_4 \pm X_1\}$
	$\beta_1/M = 1$	$\beta_0 \neq 0$	$\{X_1\}, \{X_2\}, \{X_3\}, \{X_3 \pm X_1\}, \{X_4 + \alpha X_3\}, \alpha =$ arbitrary
		$\beta_0 = 0$	$\{X_1\}, \{X_2\}, \{X_3\}, \{X_3 \pm X_1\}, \{X_4\}, \{X_4 \pm X_3\}$
	$\beta_1/M \neq \pm 1$	$\beta_0 =$ arbitrary	$\{X_1\}, \{X_2\}, \{X_3\}, \{X_3 \pm X_1\}, \{X_4\}$
$\beta_2 \neq 0$	$\beta_1 = 0$	$\beta_0 = 0$	$\{X_1\}, \{X_2\}, \{X_3\}, \{X_3 \pm X_1\}, \{X_4\}$
	$\beta_1 = 0$	$\beta_0 \neq 0$	$\{X_1\}, \{X_2\}, \{X_3\}, \{X_3 \pm X_1\}, \{X_4 + \alpha X_3\}, \alpha =$ arbitrary
	$\beta_1/M = 1$	$\beta_0 = 0$	$\{X_1\}, \{X_2\}, \{X_3\}, \{X_3 \pm X_1\}, \{X_4 + \alpha X_1\}, \alpha =$ arbitrary
	$\beta_1/M = -1$	$\beta_0 = 0$	$\{X_1\}, \{X_2\}, \{X_3\}, \{X_3 \pm X_1\}, \{X_4\}, \{X_4 \pm X_1\}$
	$\beta_1/M \neq \pm 1$	$\beta_0 = 0$	$\{X_1\}, \{X_2\}, \{X_3\}, \{X_3 \pm X_1\}, \{X_4\}$
	$\beta_1\beta_0 \neq 0$	$\beta_1/M = \pm 1$	$\{X_1\}, \{X_2\}, \{X_3\}, \{X_3 \pm X_1\}, \{X_4 + \alpha X_3\}, \alpha =$ arbitrary
	$\beta_1\beta_0 \neq 0$	$\beta_1/M \neq \pm 1$	$\{X_1\}, \{X_2\}, \{X_3\}, \{X_3 \pm X_1\}, \{X_4\}$

3. The invariant solution for the operator $X_4 + \alpha X_3$:

$$q = \mu U(\lambda) - \alpha \frac{\mu}{2a}\left(\lambda \int \frac{1}{\sqrt{b\mu}}\,\mathrm{d}z - \int \frac{\lambda_3}{\sqrt{b\mu}}\,\mathrm{d}z\right) - \mu\frac{\beta_2}{4a}\left(\lambda^2 z - 2\lambda \int \lambda_3\,\mathrm{d}z + \int \lambda_3^2\,\mathrm{d}z\right) - \\ - \frac{\beta_2}{2aM}\left(z\mu - \int \mu\,\mathrm{d}z\right), \quad \lambda = \frac{\tau}{\sqrt{b\mu}} + \lambda_3(z), \quad \lambda_3(z) = -\alpha \int \frac{z}{b\sqrt{b\mu}}\,\mathrm{d}z\,. \tag{32}$$

The substitution of the representations (30), (31) and (32) in the initial equation (6) provides the unique factor-equation for the function $U(\lambda)$ for all the three cases, the second-order ordinary differential equation:

$$\frac{\mathrm{d}^2U}{\mathrm{d}\lambda^2} + a\left(\frac{\mathrm{d}U}{\mathrm{d}\lambda}\right)^2 + (M + \beta_1)\frac{\lambda}{2}\left(\frac{\mathrm{d}U}{\mathrm{d}\lambda}\right) - MU + \frac{\beta_0\beta_2}{4a}\lambda^2 = 0\,. \tag{33}$$

The equation (33) with arbitrary coefficients does not have any symmetry. If the coefficients of the equation are connected by some definite relationship, then the equation (33) admits the group of point transformations with the operator

$$X = \frac{\partial}{\partial\lambda} - \frac{M + \beta_1}{4a}\frac{\partial}{\partial U}, \quad 4\beta_0\beta_2 = \beta_1^2 - M^2\,. \tag{34}$$

Choosing the invariant of this operator as a new dependent variable $V(\lambda)$ we obtain an autonomous equation for it

$$\frac{\mathrm{d}^2V}{\mathrm{d}\lambda^2} + a\left(\frac{\mathrm{d}V}{\mathrm{d}\lambda}\right)^2 - MV = 0\,, \quad V = U + \frac{M + \beta_1}{8a}\lambda^2 + \frac{M + \beta_1}{4a}\,. \tag{35}$$

Its solution is written in the quadratures,

$$\lambda = \int \left(C_0 \exp(-2aV) + (M/2a^2)(2aV - 1)\right)^{-1/2}\mathrm{d}V + C_1\,, \quad C_0, C_1 = \mathrm{const}\,. \tag{36}$$

2. The group of approximate symmetries and approximate group invariant solutions of the generalized Webster equation

The invariant solutions of the modified generalized Webster equation (6) constructed in the previous section describe propagation of acoustic perturbations in wavequides of the variable cross-section with

$\mathrm{d}\mu/\mathrm{d}z \neq 0$. The law of the cross-section varying along the axis of a wavequide, though, is not arbitrary, and satisfies the supplementary constraint resulting from the classifying relationship (13). The choice of the type of the initial perturbation $Q(\tau)$ (boundary of initial condition) in the equation (6) is also not arbitrary, it results from the solution of the factor-equation (33), occurring in the process of searching for the group invariant solution. It is possible to release the constraints for the type of dependence of the cross-section profile of the acoustic channel on the coordinate along its axis and for the choice of initial conditions, applying approximate symmetries, to which construction we dedicated the present section. The challenging motive for the use of the approximate symmetries in the present problem is the fact, that in the channel of the constant cross-section $\mathrm{d}\mu/\mathrm{d}z = 0$, boundary-value problem (6) has an exact analytical solution in case of *arbitrary* smooth boundary conditions. This solution can be constructed with the application of symmetries of the *renormgroup* type, for which the orbit of the group coincides with the solution of the boundary-value problem (6), i.e. under the influence of the group the solution of the boundary-value problem transforms again into the solution of this problem with the same boundary parameters. The general description of the algorithm of renormgroup symmetries construction and its substantiation can be found in [18, 19] (see also [16, §29]), and details of calculations by the example of solution of the boundary-value problem for the modified Burgers' equation are given in [20]. The opportunity to construct renormgroup symmetries for arbitrary boundary conditions in the channel of a constant cross-section is stipulated by existence of the infinite dimensional Lie subalgebra with the operator (20). In the case of the acoustic channel of the variable cross-section $\mathrm{d}\mu/\mathrm{d}z \neq 0$ considered here this symmetry does not exist. The aim of the present section is to construct approximate symmetries, which exist in the system with a small parameter, connected with comparative smoothness of varying of the area of the cross-section along the axis of the waveguiding system, i.e. infinitesimality of the derivative $\mathrm{d}(\ln\mu(z))/\mathrm{d}z \equiv \mu_z/\mu \ll 1$ (the lower index here denotes a derivative by the respective argument).

Calculation of the approximate group for the problem (6) is slightly different from the standard approach [21, 22] in the meaning, that the initial equation itself does not contain a small parameter, this parameter occurs in the process of construction and solution of the determining equation of the group. Following the scheme applied for the modified Burgers' equation we are intended to search for the infinitesimal operator of an approximate group in the expanded space of the variables $\{z, \tau, a, q\}$,

$$X^a = \xi^1(z,\tau,a,q)\frac{\partial}{\partial z} + \xi^2(z,\tau,a,q)\frac{\partial}{\partial \tau} + \xi^3(z,\tau,a,q)\frac{\partial}{\partial a} + \eta(z,\tau,a,q)\frac{\partial}{\partial q}\,. \tag{37}$$

Extension of action of the infinitesimal operator (37) upon variables is written by analogy with (8)

$$\begin{aligned}
\tilde{X}^a &= X^a + \zeta_1\frac{\partial}{\partial q_z} + \zeta_2\frac{\partial}{\partial q_\tau} + \zeta_{22}\frac{\partial}{\partial q_{\tau\tau}}\,,\\
\zeta_1 &= D_z(\eta) - q_z D_z(\xi^1) - q_\tau D_z(\xi^2) - q_a D_z(\xi^3)\,,\\
\zeta_2 &= D_\tau(\eta) - q_z D_\tau(\xi^1) - q_\tau D_\tau(\xi^2) - q_a D_\tau(\xi^3)\,,\\
\zeta_{22} &= D_\tau(\zeta_2) - q_{z\tau} D_\tau(\xi^1) - q_{\tau\tau} D_\tau(\xi^2) - q_{a\tau} D_\tau(\xi^3)\,,
\end{aligned} \tag{38}$$

with the application of supplemented operators of complete differentiation

$$\begin{aligned}
D_z &= \frac{\partial}{\partial z} + q_z\frac{\partial}{\partial q} + q_{zz}\frac{\partial}{\partial q_z} + q_{\tau z}\frac{\partial}{\partial q_\tau} + q_{az}\frac{\partial}{\partial q_a} + \dots\,,\\
D_\tau &= \frac{\partial}{\partial \tau} + q_\tau\frac{\partial}{\partial q} + q_{z\tau}\frac{\partial}{\partial q_z} + q_{\tau\tau}\frac{\partial}{\partial q_\tau} + q_{a\tau}\frac{\partial}{\partial q_a} + \dots\,.
\end{aligned} \tag{39}$$

The action of the operator (38) on (6) gives the determining equation

$$\{\tilde{X}^a F\}\big|_{F=0} \equiv \left\{\zeta_1 - 2a\frac{\partial q}{\partial \tau}\,\zeta_2 - \mu\zeta_{22} - \frac{\mathrm{d}\mu(z)}{\mathrm{d}z}\frac{\partial^2 q}{\partial \tau^2}\,\xi^1 - \left(\frac{\partial q}{\partial \tau}\right)^2 \xi^3\right\}\Big|_{F=0} = 0\,, \tag{40}$$

which after simplification is reduced to the following system of equations for coordinates of the operator (37)

$$\eta_z - \mu\eta_{\tau\tau} = 0\,, \tag{41}$$

$$\xi^1_z - 2\xi^2_\tau + (\mu_z/\mu)\xi^1 = 0\,, \tag{42}$$

$$\xi^2_z - \mu\xi^2_{\tau\tau} + 2(a\eta_\tau + \mu\eta_{q\tau}) = 0\,, \tag{43}$$

$$a(\xi^1_z - 2\xi^2_\tau) + \xi^3 + a\eta_q + \mu\eta_{qq} = 0\,, \tag{44}$$

$$\xi^1 = \xi^1(z,a)\,, \quad \xi^2 = \xi^2(z,\tau,a)\,, \quad \xi^3 = \xi^3(a)\,. \tag{45}$$

The solution of the last two equations (43) and (44), in virtue of (41) specifies the type of the coordinates ξ^1, ξ^2 and ξ^3, and, partially, η

$$\begin{aligned}
&\xi^1 = \beta_0(a) + \beta_1(a)z + \beta_2(a)z^2\,, \quad \xi^3 = \xi^3(a)\,,\\
&\xi^2 = c_0(a) + c_1(a)z + \frac{\tau}{2}\left(M(a) + \frac{\partial\xi^1}{\partial z}\right)\,,\\
&\eta = k(z,\tau,a)\exp\left(-\frac{aq}{\mu}\right) + c_2(a) - \frac{\tau}{2a}c_1(a) +\\
&\quad + \left(M(a) - \frac{\xi^3}{a}\right)q - \frac{1}{4a}\frac{\partial^2\xi^1}{\partial z^2}\left(\frac{\tau^2}{2} + \int \mu\,\mathrm{d}z\right)\,,
\end{aligned} \tag{46}$$

though there are two more conditions that should be satisfied

$$\frac{\partial k}{\partial z} - \mu\frac{\partial^2 k}{\partial\tau^2} + \frac{aq}{\mu^2}\mu_z k = 0\,, \tag{47}$$

$$(\mu_z/\mu)\xi^1 = M\,, \quad M = \mathrm{const}\,. \tag{48}$$

For the channel of the constant section $\mathrm{d}\mu/\mathrm{d}z = 0$ the first of these bounds results in the linear parabolic equation for the function $k(z,\tau,a)$, and the second bound gives $M = 0$. As a result we obtain an infinite dimensional group of symmetry (15), (19) and (20).

For the channels of the arbitrary variable cross-section $\mathrm{d}\mu/\mathrm{d}z \neq 0$ differentiating the first bound (47) by q gives $k(z,\tau,a) = 0$. The second bound coincides with the equation (13). As a result we obtain the group of symmetries (15), (16).

In the case of an acoustic channel with a slowly varying profile of the cross-section $\mu_z/\mu \ll 1$ analysed below the bounds (47), (48) can be considered as approximate, and contributions that are proportional to μ_z/μ are ignored in the zero-order limit. Therefore *in the zero order approximation with respect to this parameter* we obtain a group of approximate symmetries with the operator (37). Its coordinates are provided by the relationships (46) with $M = 0$ and $k(z,\tau,a) = k^{(0)}$, where $k^{(0)}$ follows the linear equation

$$\frac{\partial k^{(0)}}{\partial z} - \mu\frac{\partial^2 k^{(0)}}{\partial\tau^2} = 0\,, \tag{49}$$

and $\mu(z)$ is an *arbitrary* smoothly varying function z.

In *the next, first order in* the parameter μ_z/μ the second bound (48) coincides with the classifying relationship (13), and the first bound (47) is written in the form

$$\frac{\partial k^{(1)}}{\partial z} - \mu\frac{\partial^2 k^{(1)}}{\partial\tau^2} = -\frac{aq}{\mu^2}\mu_z k^{(0)}\,. \tag{50}$$

Here the functions $k^{(0)}$ and $k^{(1)}$ depend on z, τ and a, but do not depend on q. Therefore, differentiating (47) by q we obtain $k^{(0)} = 0$, i.e. *the first* order approximation destroys the symmetry of the *zero-order* approximation order. In other words, the infinite dimensional group given by the operator (37) with the coordinate $k^{(0)}$ is not inherited (in the classical meaning [21, 22]) already in the next order in the parameter μ_z/μ. To conserve an infinite dimensional subgroup we use the fact, that the symmetry of zero and the following approximations serves as an instrument for construction of the solution of the boundary-value problem (6), i.e. with the chosen function $k^{(0)}$ we can construct the solution $q^{(0)}$. Below, replacing in the right-hand side of the equation (50) the variable q by the solution of the zero approximation $q^{(0)}(z,\tau,a)$, we obtain an inhomogeneous parabolic equation with a source for $k^{(1)}$.

The infinite dimensional subgroup (37) with $k = k^{(1)}$ constructed in this way is not a symmetry of the initial *equation*, but it is a symmetry of *the solution* of the initial problem with the chosen boundary (initial) condition used for the construction of $q^{(0)}$. The obtained symmetry of the first approximation can be used for construction of the *improved* solution of the first approximation $q^{(1)}(z,\tau,a)$, and then to continue the described procedure for obtaining the function k in a given order of precision in μ_z/μ,

$$\frac{\partial k^{(i+1)}}{\partial z} - \mu\frac{\partial^2 k^{(i+1)}}{\partial \tau^2} = -A^i\,, \quad A^i = \frac{aq^{(i)}(z,\tau,a)}{\mu^2}\mu_z k^{(i)}\,, \quad i = 0,1,2,\ldots\,. \tag{51}$$

Let us illustrate the given procedure by the example of construction of the solution with the use of some first approximations for the function k. The needed for obtaining the solution of the boundary-value problem (6) operator of the renormgroup symmetry is constructed like in [20], from the linear combination of operators with the coordinates (46). We assume that the non-vanishing contributions are associated with $\xi^3(a) = 1$ generating the transformations of the parameter of nonlinearity a, and with the operator of an infinite subgroup with the function k which differs from zero and which is considered to be known

$$R = \frac{\partial}{\partial a} + \left(k(z,\tau,a)\exp\left(-\frac{aq}{\mu}\right) - \frac{q}{a}\right)\frac{\partial}{\partial q}\,. \tag{52}$$

Here the zero approximation $k = k^{(0)}$ of the function $k(z,\tau,a)$ solves the linear parabolic equation (49) with the initial condition $k^{(0)}(0,\tau,a) = Q(\tau)/a$, which follows from the invariance condition of the perturbation theory solution with $a \to 0$ with respect to the operator of the renormgroup symmetry (52). As a result we obtain

$$k^{(0)} = \frac{\nu}{a}K_a\,,\ K(a,x,\tau) = \int\limits_{-\infty}^{\infty} \mathrm{e}^{\frac{aQ(\xi)}{\nu}} G(x,\tau-\xi)\,\mathrm{d}\xi\,, \quad G(x,\tau) = \frac{1}{\sqrt{4\pi\nu x}}\mathrm{e}^{-\frac{\tau^2}{4\nu x}}\,. \tag{53}$$

Here the subscript denotes the partial derivative: $K_a \equiv \partial K/\partial a$.

Finite transformations of the group with the operator (52), in which we use the function (53) for k, result in the following approximate analytical solution of the initial problem (6):

$$q^{(0)} = \frac{\mu}{a}\ln\left[1 + \frac{\nu}{\mu}\left(K-1\right)\right]\,, \tag{54}$$

which holds in a medium with smoothly varying section $\mu_z/\mu \ll 1$. In fact, the difference of (54) from the solution in channels of a constant section appears in existence of the multiplier ν/μ, which is not equal to 1.

The advantage of the renormgroup approach in the whole and solution on the base of the operator (52), in particular, is an opportunity of subsequent improving of the obtained analytical approximations. With respect to the considered problem such an improvement in the *first order* μ_z/μ is achieved by using for the function $k(z,\tau,a)$ in the operator (52) the solution of the inhomogeneous parabolic equation (51) with $i = 0$, the right-hand side of which is proportional to the gradient of the cross-section μ_z/μ of the channel and is linearly dependent on the function $q^{(0)}$ of the zero approximation by this gradient,

$$A^0 = (\mu_z/\mu)(\nu/a)K_a\ln\left[1 + (\nu/\mu)(K-1)\right]\,. \tag{55}$$

The solution of the equation (51) subject to (55) provides a modified (due to the contribution with the gradient of the cross-section) expression for the function $k(a,x,\tau)$ in (52), i.e. for $k^{(1)}(a,x,\tau)$

$$\begin{gathered} k^{(1)} = \frac{\nu}{a}K_a - \int\limits_0^x \mathrm{d}x' \int\limits_{-\infty}^{\infty} G(x-x',\tau-\tau')\frac{\nu\mu'_{x'}}{a\mu'}K'_a\ln\left(1 + \frac{\nu}{\mu'}\left(K'-1\right)\right)\mathrm{d}\tau'\,, \\ \mu' \equiv \mu(x')\,, \quad K' \equiv K(a,x',\tau')\,. \end{gathered} \tag{56}$$

The substitution of $k^{(1)}$ instead of k in the infinitesimal operator (52) and the following solution of the Lie equation results in an improved approximation for the needed solution,

$$q^{(1)} = \frac{\mu}{a} \ln \left\{ 1 + \frac{\nu}{\mu}(K-1) - \frac{\nu}{\mu} \int\limits_0^x \frac{\mu'_{x'}}{\mu'} \mathrm{d}x' \int\limits_{-\infty}^{\infty} G(x-x', \tau-\tau') \times \right.$$
$$\left. \times \left[1 - K' + \left(K' - 1 + \frac{\mu'}{\nu} \right) \ln \left(1 + \frac{\nu}{\mu'} (K'-1) \right) \right] \mathrm{d}\tau' \right\}. \tag{57}$$

This procedure can be continued if we use $k^{(1)}$ and $q^{(1)}$ for calculations of A^1 in the right-hand side of the equation (51) and the further substitution of the solution of the equation for $k^{(1)}$ instead of k into the operator (52) to find solution of the second approximation. However the result of such calculations is rather lengthy and is not presented here.

Since the constructed solution is an approximate one, the natural question arises on its precision in case of different values of the parameter μ_z/μ. The answer to this question also depends on the value of a/ν: with small values of a/ν the first two terms of the series expansion of the solution (57) with respect to the parameter of nonlinearity a has the form:

$$q^{pt} = \nu K_a^{(0)} + \frac{\nu a}{2} \left[K_{aa}^{(0)} - \frac{\nu}{\mu} (K_a^{(0)})^2 - \right.$$
$$\left. - \int\limits_0^x \frac{\nu \mu'_{x'}}{(\mu')^2} \mathrm{d}x' \int\limits_{-\infty}^{\infty} G(x-x', \tau-\tau') ((K')_a^{(0)})^2 \, \mathrm{d}\tau' \right] + O(a^2), \tag{58}$$

where $K_a^{(0)}$ and $K_{aa}^{(0)}$ denote the values of the partial derivatives of the function K, calculated within the limit $a \to 0$. The direct substitution of the periodic initial condition $Q(\xi) = \cos\xi$ into (53), for which the function K is given by

$$K = I_0(a/\nu) + 2 \sum_{k=1}^{\infty} I_k(a/\nu) \cos k\tau \, \mathrm{e}^{-\nu k^2 x}, \tag{59}$$

and calculation of the occurring integrals shows, that the expression for q^{pt} coincides with the result obtained in [10] for the harmonic initial perturbation without a supplementary assumption on the value of μ_z/μ.

For non-small values of a/ν comparison of the approximate solution of the first approximation (57) with the numerical one [23] shows a good matching of the results (with precision to several percents) even for high values of the acoustic Reynolds number $a/\nu = 10$.

3. Conservation laws for the generalized Webster equation

In this section we consider possibility of construction of conservation laws for the equation (6) using the method developed in [24, 25]. According to this method, every symmetry of the equation (or the system of differential equations) leads to certain conservation law provided that this equation (the system of differential equations) is nonlinearly self-adjoint in the sense of [24, 25]. Verification of nonlinear self-adjointness of (6) consists in to introducing the formal Lagrangian

$$\mathcal{L} = wF \equiv w \left(\frac{\partial q}{\partial z} - a \left(\frac{\partial q}{\partial \tau} \right)^2 - \mu \frac{\partial^2 q}{\partial \tau^2} \right), \tag{60}$$

calculating the adjoint equation to (6) by the formula

$$F^* \equiv \frac{\delta \mathcal{L}}{\delta q} = 0, \quad \frac{\delta}{\delta q} = \frac{\partial}{\partial q} + \sum_{s=1}^{\infty} (-1)^s D_{i_1} \dots D_{i_s} \frac{\partial}{\partial q_{i_1 \dots i_s}}, \tag{61}$$

and solving the equation

$$F^* \big|_{w=\varphi(z,\tau,q)} = \lambda F, \tag{62}$$

where w is a new dependent variable and λ is an undetermined coefficient. Substitution $w = \varphi(z, \tau, q)$ in the equation (62) and solution of the resulting overdetermined system of equations for φ concretize the form of w and λ.

For the generalized Webster equation (6) the adjoint equation (61) has the form

$$F^* = -w_z + 2a(w_\tau q_\tau + w q_{\tau\tau}) - \mu w_{\tau\tau}\,. \tag{63}$$

Substitution of (63) into (62) with substitution $w = \varphi$, $w_z = D_z(\varphi)$, $w_\tau = D_\tau^2(\varphi)$ and $w_{\tau\tau} = D_\tau(\varphi)$ provides an exponential dependence of the variable w on q:

$$\lambda = -\frac{\partial\varphi}{\partial q}\,, \quad \varphi \equiv w = \Phi(z,\tau)\exp\left(\frac{aq}{\mu}\right)\,, \tag{64}$$

where the function Φ of two variables z and τ satisfies the equation

$$\frac{\partial\Phi}{\partial z} + \mu\frac{\partial^2\Phi}{\partial\tau^2} - aq\frac{\mathrm{d}\mu}{\mathrm{d}z}\frac{\Phi}{\mu^2} = 0\,. \tag{65}$$

In case of the arbitrary function $\mu(z)$ this equation holds only with zero values Φ. For the channels of constant section $\mathrm{d}\mu/\mathrm{d}z = 0$ we obtain the linear parabolic equation for Φ. In case of low gradients $(1/\mu)\mathrm{d}\mu/\mathrm{d}z \ll 0$ this equation can be considered as approximate, which solution can be constructed by the method of subsequent approximations (see below section 3.2).

According to [24, 25] the conservation law for (6) is written in the form

$$D_z(C^1) + D_\tau(C^2) = 0\,, \tag{66}$$

where the components of the conserved vector are expressed by (60), (64) and coordinates (12) of the operator of the admitted symmetry group as follows:

$$C^i = W\left[\frac{\partial\mathcal{L}}{\partial q_i} - D_j\left(\frac{\partial\mathcal{L}}{\partial q_{ij}}\right)\right] + D_j(W)\frac{\partial\mathcal{L}}{\partial q_{ij}}\,, \quad W = \eta - \xi^j q_j\,, \quad i,j = 1,2\,. \tag{67}$$

The equation (66) should hold for solutions of the equation (6). This can be checked by the direct substitution of expressions (67) for C^1 and C^2 subject to (64),

$$\begin{aligned} C^1 &= W\Phi\exp\left(\frac{aq}{\mu}\right)\,, \quad W = \eta - \xi^1 q_z - \xi^2 q_\tau\,, \\ C^2 &= \mu W\frac{\partial\Phi}{\partial x}\exp\left(\frac{aq}{\mu}\right) - \mu\Phi D_x\left(W\exp\left(\frac{aq}{\mu}\right)\right)\,, \end{aligned} \tag{68}$$

into the equation (66), which takes the form:

$$\begin{aligned} D_z(C^1) + D_\tau(C^2) = \Phi\exp\left(\frac{aq}{\mu}\right)&\left[\left(D_z - \mu D_\tau^2 - 2aq_\tau D_\tau\right)W\right] + \\ &+ W\frac{a\Phi}{\mu}\exp\left(\frac{aq}{\mu}\right)\left[q_z - aq_\tau^2 - \mu q_{\tau\tau}\right] + \\ &+ W\exp\left(\frac{aq}{\mu}\right)\left[\frac{\partial\Phi}{\partial z} + \mu\frac{\partial^2\Phi}{\partial\tau^2} - aq\frac{\mathrm{d}\mu}{\mathrm{d}z}\frac{\Phi}{\mu^2}\right] = 0\,. \end{aligned} \tag{69}$$

The equation (69) is valid because the expression in the first square bracket coincides with the defining equation of the group for (6), the expression in the second square bracket coincides with the equation (6) itself, and finally the expression in the third square bracket vanishes due to the equation (65).

3.1. Conservation laws in the case $\mathrm{d}\mu/\mathrm{d}z = 0$**.** The results of this section are presented in Table 5, where explicit expressions for the components C^1 and C^2 of the conserved vector are given for each operator (15), (19), (14). Here $\mu = \nu = \mathrm{const}$, when the function $\Phi(z,\tau)$ solves the linear parabolic equation. Due to the arbitrary way in the choice of $\Phi(z,\tau)$ we have chosen the value $\Phi = 1$ when constructing Table 5. Not that the table contains only symmetries providing "non-trivial" conservation laws with $C^i \neq 0$.

Note that according to (69) and Table 5 the Webster equation (6) with the arbitrary coefficient $\mu(z)$ does not have a form of the conservation law. The potential Burgers' equation (the equation (6) when $\mu = \nu = \mathrm{const}$) formally also does not have the form of the conservation law. However it can be

rewritten in the form of the conservation law on an infinite number of ways. For example, using the second line of Table 5 we obtain

$$D_z\left(\mathrm{e}^{aq/\nu}\right)+D_\tau\left(-aq_\tau\mathrm{e}^{aq/\nu}\right)=0\,. \tag{70}$$

3.2. Approximate conservation laws in the case $\mathrm{d}\mu/\mathrm{d}z\neq 0$**.** For the channels of the variable section with $\mu(z)\neq$ const it follows from the equation (65) that $\Phi=0$. Hence there is no non-vanishing function w depending on z, τ and q, for which the condition of nonlinear self-adjointness of the equation (6) would hold. Ignoring the question of a more complex form of w, for example, depending not only on dependent and independent variables, but also on derivatives of high orders and/or on non-local variables, we show the possibility to construct *an approximate* conservation law for the equation (6). Let us assume, that the area of the cross-section smoothly varies with the varying of the coordinate z of the axis of the wave conductor, $(1/\mu)\mathrm{d}\mu/\mathrm{d}z\ll 1$. Then, like for the approximate symmetries, in constructing the zero approximation $\Phi^{(0)}$ the last contribution in the equation for the function Φ in (65) is not substantial, and while calculating the following approximation $\Phi^{(1)}$ it is considered already as a known source in the right-hand side of the parabolic equation (like in the equation (51)),

$$\frac{\partial\Phi^{(0)}}{\partial z}+\mu\frac{\partial^2\Phi^{(0)}}{\partial\tau^2}=0\,,\quad \frac{\partial\Phi^{(1)}}{\partial z}+\mu\frac{\partial^2\Phi^{(1)}}{\partial\tau^2}-aq^{(0)}\frac{\mathrm{d}\mu}{\mathrm{d}z}\frac{\Phi^{(0)}}{\mu^2}=0\,. \tag{71}$$

Table 5. The components C^1, C^2 for the symmetries (12) when $\mathrm{d}\mu/\mathrm{d}z=0$

	C^1	C^2
X_∞	k	$-\nu k_\tau$
X_2	$\exp(aq/\nu)$	$-aq_\tau\exp(aq/\nu)$
X_3	$-\tau/(2a)\exp(aq/\nu)$	$(\tau q_\tau/2-\nu/(2a))\exp(aq/\nu)$
X_{42}	$(-z(aq_\tau^2+\nu q_{\tau\tau})+\nu/(2a))\exp(aq/\nu)$	$(z\,(3a\nu q_\tau q_{\tau\tau}+\nu^2q_{\tau\tau\tau}+a^2q_\tau^3)+\nu q_\tau/2)\exp(aq/\nu)$
X_{43}	$(-(\tau^2-2\nu z)/(4a)$ $-z^2\,(aq_\tau^2+\nu q_{\tau\tau}))\exp(aq/\nu)$	$(\tau^2q_\tau/4-\nu\tau/(2a)+3\nu zq_\tau/2$ $+z^2\,(3a\nu q_\tau q_{\tau\tau}+\nu^2q_{\tau\tau\tau}+a^2q_\tau^3))\exp(aq/\nu)$

As an example we write an approximate conservation law associated with the renormgroup symmetry with the generator (52). Though this symmetry acts in an expanded space of the independent variables $\{z,\tau,a\}$, the conservation law in this case is written in the form (66), the conjugated equation still has the form (63), and the formulae for w are provided by (64). The difference appears only in the formula for W, which in this case has the form $W=\eta-\xi^1q_z-\xi^1q_\tau-\xi^3q_a$. For the operator (52) with $\xi^3=1$, $\xi^1=\xi^2=0$ and $\eta=k\exp(-aq/\mu)-q/a$ this provides the following expressions for C^i:

$$\begin{aligned}
&C^1=\Phi\left[k-\left(q_a+\frac{q}{a}\right)\exp\left(\frac{aq}{\mu}\right)\right],\\
&C^2=\mu\left(k\frac{\partial\Phi}{\partial\tau}-\Phi\frac{\partial k}{\partial\tau}\right)+\left[\left(q_a+\frac{q}{a}\right)\left(aq_\tau\Phi-\mu\frac{\partial\Phi}{\partial\tau}\right)+\mu\Phi\left(q_{a\tau}+\frac{q_\tau}{a}\right)\right]\exp\left(\frac{aq}{\mu}\right),\\
&C^3=0\,.
\end{aligned} \tag{72}$$

Precision of the conservation law is determined by precision of the equations (71).

Conclusion

In conclusion, let us note that in the present paper we have found a three-dimensional group of point symmetries for the generalized Webster equation (6) describing nonlinear acoustic waves in channels of an arbitrary variable section with absorption. It was shown that for the profiles of a cross-section of a special type extension of an admitted three-parameter group of point transformations occurred, and the problem of group classifications by different types of S was solved. Optimal systems of one-dimensional subalgebras of the admitted Lie algebra have been constructed and examples of invariant

solutions have been given. We have also formulated an algorithm of finding a group of the approximate renormgroup symmetries of the solution of the boundary-value problem for the generalized Webster equation for channels with an arbitrary smoothly varying cross-section in the case with arbitrary initial conditions. We have constructed approximate analytical solutions. We have also constructed conservation laws for the generalized Webster equation for the case of an acoustic waveguide with an invariable cross-section, and also for the case of a waveguide with a smoothly varying cross-section.

The obtained results can be applied not only in acoustics but also in other spheres, for example, in medical research in describing nonlinear pulse waves [26]. Investigation of conditions of sound waves propagation on the basis of the Webster equation with the variable function $S(x)$ can be also applied to construction of a classical analogue of processes of perturbations propagation on diversities of different topological dimension with the help of the Klein–Gordon equation [27].

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