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ON INFINITESIMAL RECIPROCAL-TYPE TRANSFORMATIONS IN GASDYNAMICS. LIE GROUP CONNECTIONS AND NONLINEAR SELF-ADJOINTNESS

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Abstract. Bateman-type reciprocal transformations are represented as non-local infinitesimal symmetries of the governing equations of steady, two-dimensional, inviscid gasdynamics. In particular, this representation allows the construction of a novel non-local conservation law using the recently introduced concept of nonlinear self-adjointness.

Keywords: Bateman-type reciprocal transformations, gasdynamics, non-local symmetries and conservation laws.

Introduction

This paper is dedicated to Lev Ovsyannikov for his immense contributions to applications of Lie group theory in gasdynamics.

Reciprocal relations in steady two-dimensional gasdynamics have their origin in work of Bateman [1, 2]. These multi-parameter transformations leave invariant the governing equations, up to the equation of state. Loewner [3], in the context of irrotational planar gasdynamics undertook the systematic reduction of hodograph systems via finite matrix Bäcklund transformations to Cauchy-Riemann, Tricomi or classical wave canonical forms in subsonic, transonic and supersonic régimes respectively. Such reduction may be achieved for certain multi-parameter equations of state which may be used to approximate real gas behaviour. A detailed account of that work and reciprocal relations, along with other physical applications is given in [4]. Subsequently Loewner [5] extended his analysis to consider infinitesimal Bäcklund transformations which asymptotically reduce the hodograph equations to appropriate tractable canonical forms. In later work, a remarkable connection has been made in [6, 7] between the class of infinitesimal Bäcklund transformations introduced in a gasdynamics context in [4] and modern soliton theory of 2+1-dimensions [8, 9]. Thus, if the continuous parameter of the infinitesimal Bäcklund transformations of [4] is regarded as a third independent variable, then a triad of linear matrix equations is evident in the Loewner scheme and its generalisations. The compatibility conditions for the latter generate a broad class of novel 2+1-dimensional integrable non-linear evolution equations wherein the spatial variables have equal standing. The $\bar{\partial}$-dressing method was described for these systems in [7]. A characteristic feature of use 2+1-dimensional master systems is that they admit a compact representation in terms of a triad of eigenfunction matrices. The class includes, *inter alia* higher-dimensional

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integrable generalizations of the chiral-fields model, non-Abelian sine-Gordon and Toda lattice equations. A 2+1-dimensional integrable generalization of the classical sine-Gordon emerges naturally in this context (see e.g. [10, 11, 12, 13, 14, 15, 16]). With regard to the class of finite Bäcklund transformations introduced by Loewner in [4], it was established in [17] that these may represented as a composition of gauge and classical Darboux-type transformations. This result as used to construct iterated versions of Loewner-type transformations based on established procedures in soliton theory. It is noted that Bateman-type reciprocal transformations of steady rotational gasdynamics may be derived as a specialisation of the class of finite Loewner transformations [27].

Reciprocal transformations for non-steady physical systems were originally introduced in the setting of gasdynamics and magnetogasdynamics in [18, 19]. These constitute multi-parameter transformations which leave invariant the governing continuity and momentum equations but not, in general, the equation of state. The action of the reciprocal transformations on the gasdynamic or magneto-gasdynamic system is to provide a link to associated gasdynamic (magnetogasdynamic) systems with a new multi-parameter class of equations of state. Such reciprocal and related invariant transformations may be applied to classes of shock propagation problems in the manner described by Ustinov [20]. In recent work [21], the reciprocal transformations of 1+1-dimensional non-steady gasdynamics as derived in [18] were applied to derive a new integrable-differential version of the affinsphären equation. The latter describes the analogues of spheres in affine geometry (Jonas [22], Nomizu and Sasaki [23]). It represents an alternative avatar of the classical Tzitzeica equation [24]. It has been established that the affinsphären equation arises naturally out of a Lagrangian description of 1+1-dimensional anisentropic gasdynamics for a privileged class of constitutive laws [25, 26]. Integrability is preserved under the action of a reciprocal transformation as is supported by the construction of a Lax pair for the resultant deformed affinsphären equation.

Here, infinitesimal versions of reciprocal-type invariance transformations in gasdynamics are presented along with novel Lie group and nonlinear self-adjointness connections.

1. The Bateman-type reciprocal transformations

Here, we consider invariance under reciprocal-type transformations of the governing equations of steady, two-dimensional, inviscid gasdynamics, namely

$$\begin{gathered}(\rho u)_x + (\rho v)_y = 0 \ , \\ \rho(uu_x + vu_y) + p_x = 0 \ , \quad \rho(uv_x + vv_y) + p_y = 0 \ , \\ us_x + vs_y = 0\end{gathered} \tag{1.1}$$

where

$$\mathbf{q} = u\mathbf{e}_x + v\mathbf{e}_y$$

is the gas velocity, while p, ρ and s denote, in turn, the gas pressure, gas density and specific entropy. To the system (1.1) must be adjoined an appropriate equation of state

$$p = p(\rho, s) \ , \quad \left.\frac{\partial p}{\partial \rho}\right|_s > 0 \tag{1.2}$$

The system (1.1) implies the pair of conservation laws

$$\begin{aligned}&(\rho uv)_x + (p + \rho v^2)_y = 0 \ , \\ &(p + \rho u^2)_x + (\rho uv)_y = 0\end{aligned} \tag{1.3}$$

whence, new independent variables may be introduced according to

$$\begin{aligned} dx' &= \beta_1^{-1}[\,(p+\beta_2+\rho v^2)dx - \rho uvdy\,]\ , \\ dy' &= \beta_1^{-1}[-\rho uvdx + (p+\beta_2+\rho u^2)dy\,] \end{aligned} \quad {}^{2} \tag{1.4}$$

subject to the requirement $0 < |J(x',y',x,y)| < \infty$ so that

$$0 < |(p+\beta_2)(p+\beta_2+\rho q^2)| < \infty\ . \tag{1.5}$$

It may be established that, the gasdynamic system (1.1) is invariant under the 4-parameter class of reciprocal transformations [28]

$$\begin{aligned} u' = \frac{\beta_1 u}{p+\beta_2}\ , \quad v' = \frac{\beta_1 v}{p+\beta_2}\ , \\ p' = \beta_4 - \frac{\beta_1^2\beta_3}{p+\beta_2}\ , \quad \rho' = \frac{\beta_3\rho(p+\beta_2)}{p+\beta_2+\rho q^2}\ , \quad s' = \Psi(s)\ . \end{aligned} \tag{1.6}$$

This result has its roots in work of Bateman on lift and drag functions in planar irrotational gasdynamics [1]. The subject was elaborated upon by Tsien [29]. Additional invariance properties of reciprocal transformations concerning the equation of state, were investigated in [30].

In more recent work, reciprocal transformations have been applied in [31] to uncover hidden soliton-type integrability in subsonic gasdynamics and to isolate periodic vortex motions which are reciprocally linked to hydrodynamic Stuart vortices [32]. In [33], reciprocal transformations were combined with the double action of an auto-Bäcklund transformation on an integrable elliptic Tzitzeica model in subsonic gasdynamics to isolate breather-type compressible vortex motions of a generalized Kármán-Tsien gas. Doubly periodic motions valid for such a gas were obtained via multi-parameter reciprocal transformations and the Hirota bilinear operator formalism in [34]. In [35], reciprocal transformations were adduced in conjunction with the action of an auto-Bäcklund transformation to isolate vortex train motions in super-Alfvénic magnetogasdynamics.

2. Infinitesimal Bateman-type reciprocal transformations

It is observed that the one-parameter class of reciprocal relations (1.6) with

$$\beta_1 = \beta_2 = \beta_4 = \beta\ , \quad \beta_3 = 1 \tag{2.1}$$

reduces to the identity transformation in the limit $\beta \to \infty$. The reciprocally associated gasdynamic flows may be regarded as a *deformation* of the original motions. Action of this class on a seed hydrodynamic vortex street was used in [31] to construct a compressible, subsonic version of the Stuart vortex street, valid for a generalised Kármán-Tsien model.

If one sets $\epsilon \equiv \beta^{-1}$ then the one-parameter class of reciprocal transformations becomes

$$\begin{aligned} u' = \frac{u}{1+\epsilon p}\ , \quad v' = \frac{v}{1+\epsilon p}\ , \\ p' = \frac{p}{1+\epsilon p}\ , \quad \rho' = \frac{\rho(1+\epsilon p)}{1+\epsilon(p+\rho q^2)}\ , \quad s' = \Psi(s) \end{aligned} \tag{2.2}$$

together with

$$\begin{aligned} dx' &= \epsilon[\,(p+\rho v^2)dx - \rho uvdy\,] + dx\ , \\ dy' &= \epsilon[\,-\rho uvdx + (p+\rho u^2)dy\,] + dy\ . \end{aligned} \tag{2.3}$$

[2]The reciprocal variables x' and y' are up to scaling, the drag and lift functions of Bateman [1]

On expansion and retention of terms, to $0(\epsilon^2)$ the relations (2.2) produce the class of infinitesimal transformations

$$\begin{gathered} u' = u(1-\epsilon p)\ , \quad v' = v(1-\epsilon p)\ , \\ p' = p(1-\epsilon p)\ , \quad \rho' = \rho(1-\epsilon\rho q^2)\ , \quad s' = \Psi(s) \end{gathered} \tag{2.4}$$

where ϵ is now deemed to be a small deformation parameter. It is readily shown, *ab initio* that the gasdynamic system is invariant under this class of infinitesimal transformations augmented by (2.3), up to the equation of state.

Thus, with regard to the continuity equations

$$\begin{aligned} -\rho' v' dx' + \rho' u' dy' = \\ &= -\rho v[\,1-\epsilon(p+\rho q^2)+0(\epsilon^2)\,]\cdot[\,\epsilon(\,(p+\rho v^2)dx - \rho uvdy) + dx\,] + \\ &\quad + \rho u[\,1-\epsilon(p+\rho q^2)+0(\epsilon^2)\,]\,[\,\epsilon(-\rho uvdx + (p+\rho u^2)dy) + dy\,] = \\ &= -\rho vdx + \rho udy + \\ &\quad + \epsilon[\,-\rho v(\,(p+\rho v^2)dx - \rho uvdy) + (p+\rho q^2)\rho vdx\,] + \\ &\quad + \epsilon[\,\rho u(-\rho uvdx + (p+\rho u^2)dy) - \\ &\quad - \epsilon(p+\rho q^2)\rho udy\,] + 0(\epsilon^2) = \\ &= -\rho vdx + \rho udy + 0(\epsilon^2)\ . \end{aligned} \tag{2.5}$$

Moreover,

$$\begin{aligned} -(p' + \rho' v'^2)dx' + \rho' u' v' dy' = \\ &= -[\,p(1-\epsilon p) + \rho v^2(1-\epsilon\rho q^2 - 2\epsilon p) + 0(\epsilon^2)\,]dx' + \\ &\quad + \rho uv(1-\epsilon\rho q^2 - 2\epsilon p + 0(\epsilon^2)\,)dy' = \\ &= -(p+\rho v^2)dx + \\ &\quad + \epsilon[\,p^2 + \rho^2 v^2 q^2 + 2\rho v^2 p - (p+\rho v^2)^2\,]dx + \\ &\quad + \epsilon\rho uv(p+\rho v^2)dy + \rho uvdy - \epsilon\rho^2 u^2 v^2 dx + \\ &\quad + \epsilon\rho uvdy[\,-(\rho q^2 + 2p) + p + \rho u^2\,] + 0(\epsilon^2) = \\ &= -(p+\rho v^2)dx + \rho uvdy + 0(\epsilon^2)\ \ . \end{aligned}$$

Similarly,

$$\begin{aligned} -\rho' u' v' dx' + (p' + \rho' u'^2)dy' = \\ &= -\rho uv(1-\epsilon(\rho q^2 + 2p) + 0(\epsilon^2)\,)dx' + \\ &\quad + [\,p(1-\epsilon p) + \rho u^2(1-\epsilon(\rho q^2+2p)\,] + 0(\epsilon^2)dy' = \\ &= -\rho uvdx + \epsilon\rho uv(\rho q^2 + 2p)dx - \\ &\quad - \rho uv\epsilon(p+\rho v^2)dx - \epsilon\rho^2 u^2 v^2 dy + \\ &\quad + (p+\rho u^2)dy - \epsilon[\,p^2 + \rho u^2(\rho q^2 + 2p)\,]dy - \\ &\quad - \epsilon\rho uv(p+\rho u^2)dx + \epsilon(p+\rho u^2)^2 dy + 0(\epsilon^2) = \\ &= -\rho uvdx + (p+\rho u^2)dy + 0(\epsilon^2)\ . \end{aligned}$$

Hence, to $0(\epsilon^2)$, the gasdynamic system determined by the conservation laws consisting of the continuity equation augmented by the pair (1.3) is invariant under the class of infinitesimal transformations (2.3)–(2.4).

Further,

$$\begin{aligned}
d\Psi' &= -\rho' v' dx' + \rho' u' dy' = \\
&= -\rho v[\,1 - \epsilon(p + \rho q^2) + 0(\epsilon^2)\,]dx' + \\
&\quad + \rho u[\,1 - \epsilon(p + \rho q^2) + 0(\epsilon^2)\,]dy' = \\
&= -\rho v dx + \epsilon[\,\rho v(p + \rho q^2) - (p + \rho v^2)\rho v\,]dx + \epsilon \rho u v^2 dy + \\
&\quad + \rho u dy + \epsilon[\,-\rho u(p + \rho q^2)\,]dy - \epsilon \rho^2 u^2 v dx + \\
&\quad + \epsilon \rho u(p + \rho u^2)dy + 0(\epsilon^2) = \\
&= -\rho v dx + \rho u dy + 0(\epsilon^2) = \\
&= d\Psi + 0(\epsilon^2)
\end{aligned}$$

so that the isentropic condition $(1.1)_3$ is also preserved under the class of infinitesimal transformations.

3. Lie group connection. Reconstruction of the Bateman relations via initial value problems

The class of reciprocal transformations (2.2) is readily reconstructed from (2.4) via the usual Lie group procedure involving the solution of a system of initial value problems (see e.g. [36]) which, in this case yields

$$\begin{aligned}
&\frac{dp'}{d\epsilon} = -p'^2\,, \quad p'|_{\epsilon=0} = p\,, \\
&\frac{du'}{d\epsilon} = -u'p'\,, \quad u'|_{\epsilon=0}\,, \quad \frac{dv'}{d\epsilon} = -v'p'\,, \quad v'|_{\epsilon=0} = v\,, \\
&\frac{d\rho'}{d\epsilon} = -\rho'^2 q'^2\,, \quad \rho'|_{\epsilon=0} = \rho\,,
\end{aligned} \tag{3.1}$$

together with

$$\begin{aligned}
&d(dx') = [\,(p + \rho v^2)dx - \rho u v dy\,]d\epsilon\,, \quad dx'|_{\epsilon=0} = dx\,, \\
&d(dy') = [\,-\rho u v dx + (p + \rho v^2)dy\,]d\epsilon\,, \quad dy'|_{\epsilon=0} = dy\,.
\end{aligned} \tag{3.2}$$

Thus, solution of the initial value problem $(3.1)_1$ yields the reciprocal pressure

$$p' = \frac{p}{1 + \epsilon p}\,,$$

while insertion, in turn of this relation into $(3.1)_2$ produces

$$\frac{du'}{d\epsilon} = -\frac{u'p}{1 + \epsilon p}\,,$$

whence, on integration and use of the initial condition $u'|_{\epsilon=0} = u$ it is seen that

$$u' = \frac{u}{1 + \epsilon p}$$

and, in a similar manner

$$v' = \frac{v}{1 + \epsilon p}\,.$$

Accordingly, $(3.1)_3$ yields

$$\frac{d\rho'}{d\epsilon} = -\frac{\rho'^2 q^2}{(1 + \epsilon p)^2}$$

whence, on integration and use of the initial condition $\rho'|_{\epsilon=0}$ we retrieve the reciprocal density relation

$$\rho' = \frac{\rho(1+\epsilon p)}{1+\epsilon(p+\rho q^2)} .$$

The reciprocal variable relations (2.3) are likewise retrieved via (3.2).

Thus, the one-parameter class (2.2) of Bateman-type reciprocal transformations is seen to have Lie group origin. It is natural to investigate to what extent this class may be additional boosted by more general Lie group action. In particular, Prim-type substitution invariance principles of gasdynamics and magneto-gasdynamics [37, 38, 39] may be generated by Lie group methods (see [40]−[42] and work cited therein). The imbedding of the one-parameter class of one-parameter reciprocal transformations in a more general Lie-group investigation of invariance of the gasdynamic system (1.1) will be investigated subsequently in Section 5.

It is remarked that, Lie group methods as described in the authoritative monograph by Ovsiannikov [47] have extensive applications in hydrodynamics (Andreev *et al* [48]). Importantly, such applications are not restricted to the construction of exact solutions. Thus, in particular, key properties such as the time evolution of moments of inertia as described by Ball [49, 50] may, in fact, be generated via Lie-group methods [51, 52]. Variants of such theorems may be shown to be key to isolating hidden integrable Hamiltonian structure of Ermakov-Ray-Reid type in rotating shallow water theory [53], gas cloud evolution [54] and magnetogasdynamics [55].

4. Infinitesimal reciprocal transformations in non-steady gasdynamics

The governing equations of one-dimensional, anisentropic non-steady gasdynamics, neglecting heat conduction and radiation are

$$\begin{gathered} \rho_t + (\rho u)_x = 0\ , \quad \rho(u_t + u u_x) + p_x = 0\ , \\ s_t + u s_x = 0 \end{gathered} \tag{4.1}$$

together with a prevailing gas law of the type (1.2).

In [18] it was established that the gasdynamic system is invariant, up to the equation of state, under the 4-parameter class of reciprocal transformations

$$\begin{gathered} u' = \frac{\beta_1 u}{p+\beta_2}\ , \\ p' = \beta_4 - \frac{\beta_1^2\beta_3}{p+\beta_2}\ , \quad \rho' = \frac{\beta_3\rho(p+\beta_2)}{p+\beta_2+\rho u^2}\ , \\ s' = \Psi(s)\ , \\ dt' = \beta_1^{-1}[\,-\rho u dx + (p+\rho u^2+\beta_2)dt\,]\ , \quad dx' = dx\ , \\ 0 < |p+\rho u^2+\beta_2| < \infty\ . \end{gathered} \tag{4.2}$$

The specialisation of the parameters β_1 as in (2.1) with $\epsilon = \beta^{-1}$ leads to the one-parameter subclass of invariant transformations

$$\begin{gathered} u' = \frac{u}{1+\epsilon p}\ , \\ p' = \frac{p}{1+\epsilon p}\ , \quad \rho' = \frac{\rho(1+\epsilon p)}{1+\epsilon(p+\rho u^2)} \end{gathered} \tag{4.3}$$

together with

$$dt' = \epsilon[\,-\rho u dx + (p+\rho u^2)dt\,] + dt\ , dx' = dx\ . \tag{4.4}$$

Expansion and retention of terms to $0(\epsilon^2)$ in (4.3) leads to the class of infinitesimal transformations

$$\begin{gathered} u' = u(1-\epsilon p)\ , \quad \rho' = \rho(1-\epsilon\rho u^2)\ , \quad p' = p(1-\epsilon p)\ , \\ dt' = \epsilon[\,-\rho u dx + (p+\rho u^2)dt\,] + dt\ , \quad dx' = dx \end{gathered} \tag{4.5}$$

augmented by $s' = \Psi(s)$. It is readily shown that the gasdynamic system (4.1) is invariant to $0(\epsilon^2)$ under this class of infinitesimal transformations and that the original reciprocal transformations (4.3) may be reconstructed via the solution of a system of initial value problems analogous to that in (3.1).

5. Representation of reciprocal transformations as non-local symmetries

It has been shown in [43], using the transition from Lagrange to Euler coordinates in gasdynamics, that the equations

$$\begin{aligned} & v_t + v v_x + \frac{1}{\rho} p_x = 0, \\ & \rho_t + v\rho_x + \rho v_x = 0, \\ & p_t + v p_x - p v_x = 0 \end{aligned} \tag{5.1}$$

describing the Chaplygin gas have two *nonlocal* symmetries:

$$\begin{aligned} X_1 &= \sigma\frac{\partial}{\partial x} - \frac{\partial}{\partial p} + \frac{\rho}{p}\frac{\partial}{\partial \rho}, \\ X_2 &= \left(\frac{t^2}{2}+s\right)\frac{\partial}{\partial x} + t\frac{\partial}{\partial v} - \tau\frac{\partial}{\partial p} + \frac{\rho\tau}{p}\frac{\partial}{\partial \rho}, \end{aligned} \tag{5.2}$$

where τ, s, σ are *non-local variables* defined by the compatible over-determined systems

$$\begin{aligned} & \tau_x = \rho, \qquad \tau_t = -v\rho, \\ & s_x = -\frac{\tau}{p}, \quad s_t = \frac{v\tau}{p}, \\ & \sigma_x = -\frac{1}{p}, \quad \sigma_t = \frac{v}{p}\cdot \end{aligned} \tag{5.3}$$

Equations (5.3) can be equivalently written as the following conservation laws:

$$\begin{aligned} & D_t(\rho) + D_x(v\rho) = 0, \\ & D_t\left(\frac{\tau}{p}\right) + D_x\left(\frac{v\tau}{p}\right) = 0, \\ & D_t\left(\frac{1}{p}\right) + D_x\left(\frac{v}{p}\right) = 0. \end{aligned} \tag{5.4}$$

The second conservation equation (5.4) contains the non-local variable τ and therefore it is termed a non-local conservation law. The first, second and third conservation equations (5.4) provide the compatibility conditions for the over-determined equations (5.3) for the non-local variables τ, s and σ, respectively.

If we apply the same approach to the conservation laws (1.3),

$$\begin{aligned} & (\rho u v)_x + (p+\rho v^2)_y = 0, \\ & (p+\rho u^2)_x + (\rho u v)_y = 0, \end{aligned} \tag{1.3}$$

we arrive at the non-local variables α and σ determined by the equations

$$\alpha_x = p + \rho v^2, \quad \alpha_y = -\rho uv \tag{5.5}$$

and

$$\sigma_x = -\rho uv, \quad \sigma_y = p + \rho u^2, \tag{5.6}$$

respectively. Using these variables, we can replace the infinitesimal reciprocal transformations (3.2)-(3.3) with the following non-local symmetry generator

$$X = \alpha\frac{\partial}{\partial x} + \sigma\frac{\partial}{\partial y} - p^2\frac{\partial}{\partial p} - pu\frac{\partial}{\partial u} - pv\frac{\partial}{\partial v} - (u^2+v^2)\rho^2\frac{\partial}{\partial \rho}\cdot \tag{5.7}$$

Let us verify that the operator (5.7) is admitted by the steady two-dimensional equations of gasdynamics (1.1),

$$\begin{aligned} &(\rho u)_x + (\rho v)_y = 0,\\ &p_x + \rho(uu_x + vu_y) = 0,\\ &p_y + \rho(uv_x + vv_y) = 0,\\ &us_x + vs_y = 0, \end{aligned} \tag{1.1}$$

with an arbitrary equation of state. The usual prolongation formula

$$\zeta_i^k = D_i(\eta^k) - u_j^k D_i(\xi^j)$$

gives the following prolongation of the operator (5.7) to the derivatives involved in the equations (1.1):

$$\widetilde{X} = X + \zeta_x^p\frac{\partial}{\partial p_x} + \zeta_y^p\frac{\partial}{\partial p_y} + \zeta_x^u\frac{\partial}{\partial u_x} + \zeta_y^u\frac{\partial}{\partial u_y} + \zeta_x^v\frac{\partial}{\partial v_x} + \zeta_y^v\frac{\partial}{\partial v_y} + \zeta_x^\rho\frac{\partial}{\partial \rho_x} + \zeta_y^\rho\frac{\partial}{\partial \rho_y} + \zeta_x^s\frac{\partial}{\partial s_x} + \zeta_y^s\frac{\partial}{\partial s_y}\,, \tag{5.8}$$

where

$$\begin{aligned} &\zeta_x^p = \rho uvp_y - (3p + \rho v^2)p_x, \quad \zeta_y^p = \rho uvp_x - (3p + \rho u^2)p_y,\\ &\zeta_x^u = \rho uvu_y - (2p + \rho v^2)u_x - up_x, \quad \zeta_x^u = \rho uvu_x - (2p + \rho u^2)u_y - up_y,\\ &\zeta_x^v = \rho uvv_y - (2p + \rho v^2)v_x - vp_x, \quad \zeta_x^v = \rho uvv_x - (2p + \rho u^2)v_y - vp_y,\\ &\zeta_x^\rho = \rho uv\rho_y - (p + 2\rho u^2 + 3\rho v^2)\rho_x - 2\rho^2(uu_x + vv_x),\\ &\zeta_y^\rho = \rho uv\rho_x - (p + 2\rho v^2 + 3\rho u^2)\rho_y - 2\rho^2(uu_y + vv_y),\\ &\zeta_x^s = \rho uvs_y - (p + \rho v^2)s_x, \quad \zeta_y^s = \rho uvs_x - (p + \rho u^2)s_y. \end{aligned} \tag{5.9}$$

The calculation shows that

$$\begin{aligned} \widetilde{X}[(\rho u)_x + (\rho v)_y] =& -2[p + \rho(u^2+v^2)][(\rho u)_x + (\rho v)_y]-\\ & - \rho u[p_x + \rho(uu_x + vu_y)] - \rho v[p_y + \rho(uv_x + vv_y)],\\ \widetilde{X}[p_x + \rho(uu_x + vu_y)] =& -[3p + \rho(u^2+v^2)][p_x + \rho(uu_x + vu_y)],\\ \widetilde{X}[p_y + \rho(uv_x + vv_y)] =& -[3p + \rho(u^2+v^2)][p_y + \rho(uv_x + vv_y))]. \end{aligned} \tag{5.10}$$

Hence, the infinitesimal test for the invariance of Equations (1.1) is satisfied.

We will use the non-local symmetry (5.7) for constructing a conservation law by the method of nonlinear self-adjointness [44] (see also [45]). According to this method, we take the *formal Lagrangian* of Equations (1.1) in the form

$$\begin{aligned} \mathcal{L} = & R(\rho u_x + \rho v_y + u\rho_x + v\rho_y) + S(us_x + vs_y)+\\ & + U[p_x + \rho(uu_x + vu_y)] + V[p_y + \rho(uv_x + vv_y)] \end{aligned} \tag{5.11}$$

and write the adjoint system

$$\frac{\delta \mathcal{L}}{\delta u} = 0, \quad \frac{\delta \mathcal{L}}{\delta v} = 0, \quad \frac{\delta \mathcal{L}}{\delta p} = 0, \quad \frac{\delta \mathcal{L}}{\delta \rho} = 0, \quad \frac{\delta \mathcal{L}}{\delta s} = 0, \tag{5.12}$$

to Equations (1.1), where $\delta\mathcal{L}/\delta u$ is the variational derivative of the formal Lagrangian with respect to the dependent variable u, etc.

The nonlinear self-adjointness implies that the adjoint equations (5.12) are satisfied for all solutions of the system (1.1). In our case this property follows, e.g. from the conservation form of the first equation of the system (1.1). Namely, it is well known that if a function of independent and dependent variables together with partial derivatives of any order has a divergence form, then the variational derivatives of this function vanish (see, e.g. [46]). Using this statement and noting that the expression

$$\rho u_x + \rho v_y + u\rho_x + v\rho_y$$

has a divergence form,

$$\rho u_x + \rho v_y + u\rho_x + v\rho_y = D_x(\rho u) + D_y(\rho v),$$

we see that the formal Lagrangian (5.11) with

$$R = 1, \quad U = V = S = 0 \tag{5.13}$$

solves the equations (5.12).

Now we apply to the non-local symmetry (5.7) the general formula from [45]) for constructing conserved vectors associated with symmetries and obtain

$$\begin{aligned} C^1 &= -W^1 \frac{\partial \mathcal{L}}{\partial u_x} - W^2 \frac{\partial \mathcal{L}}{\partial v_x} - W^3 \frac{\partial \mathcal{L}}{\partial p_x} - W^4 \frac{\partial \mathcal{L}}{\partial \rho_x} - W^5 \frac{\partial \mathcal{L}}{\partial s_x}, \\ C^2 &= -W^1 \frac{\partial \mathcal{L}}{\partial u_y} - W^2 \frac{\partial \mathcal{L}}{\partial v_y} - W^3 \frac{\partial \mathcal{L}}{\partial p_y} - W^4 \frac{\partial \mathcal{L}}{\partial \rho_y} - W^5 \frac{\partial \mathcal{L}}{\partial s_y}, \end{aligned} \tag{5.14}$$

where

$$\begin{aligned} &W^1 = -pu - \alpha u_x - \sigma u_y, \quad W^2 = -pv - \alpha v_x - \sigma v_y, \\ &W^3 = -p^2 - \alpha p_x - \sigma p_y, \quad W^4 = -(u^2 + v^2)\rho^2 - \alpha\rho_x - \sigma\rho_y, \\ &W^5 = -\alpha s_x - \sigma s_y. \end{aligned} \tag{5.15}$$

Inserting in (5.14) the expressions (5.15) for $W^1, \dots W^5$ and invoking Equations (5.13) we obtain the following *non-local conserved vector*:

$$\begin{aligned} C^1 &= \rho p u + u(u^2 + v^2)\rho^2 + \alpha(\rho u)_x + \sigma(\rho u)_y, \\ C^2 &= \rho p v + v(u^2 + v^2)\rho^2 + \alpha(\rho v)_x + \sigma(\rho v)_y. \end{aligned} \tag{5.16}$$

The vector (5.16) satisfies the conservation law in the following form:

$$\begin{aligned} D_x(C^1) + D_y(C^2) = 2[p + \rho(u^2 + v^2)][(\rho u)_x + (\rho v)_y] + \rho u[p_x + \rho(u u_x + v u_y)] + \\ + \rho v[p_y + \rho(u v_x + v v_y)] + \alpha[(\rho u)_x + (\rho v)_y]_x + \sigma[(\rho u)_x + (\rho v)_y]_y. \end{aligned} \tag{5.17}$$

A BIOGRAPHICAL NOTE

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