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ITERATIVE LINEARIZATION OF THE EVOLUTION NAVIER-STOKES EQUATIONS

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Abstract. An iterative process is constructed and validated, which reduces the solution of nonlinear time-dependent Navier-Stokes equations to the solution of a sequence of linear problems. Using an priori estimates of solutions allows us to prove the convergence of the method with any initial approximation. It is shown that the proposed method can be used to prove the existence and uniqueness of the solution.

Keywords: Navier-Stokes equations, a priori estimates, the iterative process

1. Introduction

Let us consider an initial boundary-value problem for the generalized system of the Navier-Stokes equations

$$\mathbf{v}_t - \nu\Delta\mathbf{v} + v_i\mathbf{v}_{x_i} + grad\ p = \mathbf{f}(x,t), \tag{1}$$

$$\mathbf{v}\big|_{S_T} = 0, \quad \mathbf{v}\big|_{t=0} = \mathbf{a}(x), \tag{2}$$

$$div\ \mathbf{v} = 0 \tag{3}$$

in the domain $Q_T = \Omega \times [0,T]$, $S_T = S \times [0,T]$, S is a boundary of the domain Ω, $\mathbf{f} \in \overset{\circ}{\mathbf{J}}(Q_T)$, $\mathbf{L}_2(Q_T) = \mathbf{G}(Q_T) \oplus \overset{\circ}{\mathbf{J}}(Q_T)$ is an orthogonal extension on the gradient and solenoidal consistent parts of the space $\mathbf{L}_2(Q_T)$, $\mathbf{v} = (v_1, v_2, \dots, v_n)$,

$$div\ \mathbf{a} = 0, \quad \mathbf{a}\big|_S = 0. \tag{4}$$

Here and in what follows we, generally, use the notations used in paper [1]. For the single-valued definiteness of pressure we consider that $\int\limits_\Omega p(x,t)dx = 0$ almost everywhere on t on [0,T].

2. Construction of an iteration process

For construction of the iteration process we use an a priori estimate

$$\|\mathbf{v}_x(t)\| = \|\mathbf{v}_x(x,t)\| \leqslant M(t) \leqslant M_0, \quad \forall t \in [0,T]. \tag{5}$$

Let us introduce the following notations:

$$\alpha_R(t,\mathbf{v}_x) = min[1, R(t)/\|\mathbf{v}_x(t)\|], \tag{6}$$

where R(t) is a nondecreasing positive function on [0,T]. The operator $P_R\mathbf{v}_x = \alpha_R(t,\mathbf{v}_x)\mathbf{v}_x(t)$ is a projection operator on the sphere $\{\mathbf{v}_x(t) : \|\mathbf{v}_x(t)\| \leqslant R(t)\}$, therefore due to the property of the projection operator,

$$\left\|P_R\mathbf{v}_x^1(t) - P_R\mathbf{v}_x^2(t)\right\| \leqslant \left\|\mathbf{v}_x^1 - \mathbf{v}_x^2\right\| \forall t \in [0,T], \mathbf{v}^1(t), \mathbf{v}^2(t) \in \overset{\circ}{\mathbf{W}}{}_2^1(\Omega). \tag{7}$$

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If there is a solution for the problem (1) – (3) and the contingency (5) holds, then if $R(T) \geq M(t)$ $\alpha_R(t, \mathbf{v}_x) \equiv 1$, therefore $\mathbf{v}$ is also the solution of the equation

$$\mathbf{v}_t - \nu\Delta\mathbf{v} + \alpha_R(t, \mathbf{v}_x)v_i\mathbf{v}_{x_i} + grad\ p = \mathbf{f} \tag{8}$$

with the initial boundary conditions (2) and the condition (3).

For the solution of the problem (8), (2), (3) we construct an iteration process:

$$\mathbf{v}_t^{k+1} - \nu\Delta\mathbf{v}^{k+1} + \alpha_k v_i^k \mathbf{v}_{x_i}^{k+1} + grad\ p_{k+1} = \mathbf{f}, \tag{9}$$

$$\mathbf{v}^{k+1}\big|_{S_T} = 0, \quad \mathbf{v}^{k+1}\big|_{t=0} = \mathbf{a}(x), \tag{10}$$

$$div\ \mathbf{v}^{k+1} = 0, \tag{11}$$

where $\alpha_k = \alpha_k(t) = \alpha_R(t, \mathbf{v}_x^k)$.

Let us assume that the domain Ω is bounded, $S \in C^2$, n is equal to 2 or 3. Let us denote $a_i^k = \alpha_R(t, \mathbf{v}_x^k)v_i^k$ and show that

$$\left\|a_i^k\right\|_4 \leqslant c_1 R(t), \quad i = \overline{1,n},\ k = 0, 1, \dots,\ t \in [0, T]. \tag{12}$$

For this purpose we use the well known inequalities ([2] Ch. 2)

$$\|v\|_4 \leqslant c_2 \|v_x\|^{\frac{1}{2}} \|v\|^{\frac{1}{2}}, \quad \text{for } n = 2, \tag{13}$$

$$\|v\|_4 \leqslant c_3 \|v_x\|^{\frac{3}{4}} \|v\|^{\frac{1}{4}}, \quad \text{for } n = 3, \tag{14}$$

$$\|v\| \leqslant c_4 \|v_x\|, \tag{15}$$

which hold for $\forall v \in \overset{\circ}{W}{}_2^1(\Omega)$, where $c_2 = 2^{\frac{1}{4}}$, $c_3 = 2^{\frac{1}{2}}$, $c_4 = \lambda_1^{-\frac{1}{2}}$, λ_1 is the first eigenvalue of the Laplace operator with homogeneous boundary conditions of the first type.

We note for the further use that if the function v does not vanish on S, but satisfies the condition: $\int_\Omega v dx = 0$, the same holds for the inequality (13), (14), but with the constants $c_2 = c_2(\Omega)$, $c_3 = c_3(\Omega)$ depending on the domain.

It results from the inequalities (13) – (15) that

$$\|v\|_4 \leqslant c_2 c_4^{\frac{1}{2}} \|v_x\|, \quad n = 2, \tag{16}$$

$$\|v\|_4 \leqslant c_3 c_4^{\frac{1}{4}} \|v_x\|, \quad n = 3. \tag{17}$$

Taking into account the last inequalities and the relationship (6), we obtain

$$\left\|\alpha_R(t, \mathbf{v}_x^k)v_i^k\right\|_4 \leqslant c_5 \alpha_R(t, \mathbf{v}_x^k) \left\|\mathbf{v}_x^k\right\| \leqslant c_5 R(t), \tag{18}$$

where $c_5 = 2^{\frac{1}{4}} c_4^{\frac{1}{2}}$ when $n = 2$ and $c_5 = 2^{\frac{1}{2}} c_4^{\frac{1}{4}}$ when $n = 3$.

Applying the Theorem 1′, ch.4, [1], we make sure that there is only one solution of the problem (9) – (11) in the class of functions $\mathbf{W}_2^{2,1}(Q_T)$, $\mathbf{p}_x \in \mathbf{L}_2(Q_T)$.

3. Boundedness of the iteration sequence

Let us further denote $|||\mathbf{v}|||_{0t} = \|\mathbf{v}\|_{\mathbf{L}_2(Q_T)}$, $Q_t = \Omega \times [0, t]$ and introduce an auxiliary norm $[\mathbf{v}]_{\lambda,t}^2 = \frac{1}{2} \underset{\tau \in [0,t]}{vraimax} \|\mathbf{v}_x(\tau)\|^2 + \nu |||\Delta\mathbf{v}|||_{0,t}^2 + \lambda |||\Delta\mathbf{v}_x|||_{0,t}^2$.

Let us show that in case of rather large $\lambda > 0$ the sequence $\{v_k\}_{k=0}^{\infty}$, determined by the iteration process (9) – (11) with any $\mathbf{v}_0 \in \overset{\circ}{\mathbf{W}}{}_2^{2,1}(Q_T)$ is bounded

$$[\mathbf{v}^k]_{\lambda t} \leqslant c_t e^{\lambda t} \leqslant c_6 \quad \forall t \in [0, T], k = 0, 1, \dots. \tag{19}$$

Let us assume that $\mathbf{v},\mathbf{w} \in \overset{\circ}{\mathbf{W}}{}_2^{2,1}(Q_T)$ and estimate the norm $|||\lambda v_i w_{x_i}|||_{0,t}$. Applying the inequalities $(16)-(18)$ and the second energetic inequality:

$$\|v_{xx}\| \leqslant c_7\|\Delta v\| \qquad \forall v \in \overset{\circ}{W}{}_2^1(\Omega)\cap \overset{\circ}{W}{}_2^2(\Omega), \tag{20}$$

we obtain

$$|||\alpha v_i w_{x_i}|||_{0,t} \leqslant \left(\int_0^t\int_\Omega |\alpha\mathbf{v}|^2|\mathbf{w}_x|^2\,dxd\tau\right)^{\frac12} \leqslant \left(\int_0^t \|\alpha\mathbf{v}\|_4^2\|\mathbf{w}_x\|_4^2\,d\tau\right)^{\frac12} \leqslant$$

$$\leqslant c_5\left(\int_0^t \|\alpha\mathbf{v}_x\|^2\|\mathbf{w}_x\|_4^2\,d\tau\right)^{\frac12} \leqslant c_5R(t)\left(\int_0^t \|\mathbf{w}_x\|_4^2\,d\tau\right)^{\frac12}. \tag{21}$$

To estimate the integral in the right-hand side of the inequality (21), we apply the inequality (20) and the inequalities (13), (14), in which c_2 and c_3 depend on the domain.

When $n=2$ we obtain

$$\left(\int_0^t \|\mathbf{w}_x\|_4^2\,d\tau\right)^{\frac12} \leqslant c_2c_7\left(\int_0^t \|\mathbf{w}_x\|\|\Delta\mathbf{w}\|\,d\tau\right)^{\frac12} \leqslant c_2c_7|||\mathbf{w}_x|||_{0,t}^{\frac12}|||\Delta\mathbf{w}|||_{0,t}^{\frac12}.$$

Whence, when $n=2$

$$|||\alpha v_i w_{x_i}|||_{0,t} \leqslant c_2c_5c_7R(t)|||\mathbf{w}_x|||_{0,t}^{\frac12}|||\Delta\mathbf{w}|||^{\frac12}. \tag{22}$$

When $n=3$, we apply the Holder inequality with the indices $\frac43, 4$ and obtain

$$\left(\int_0^t \|\mathbf{w}_x\|_4^2\,d\tau\right)^{\frac12} \leqslant c_3c_7\left(\int_0^t \|\mathbf{w}_x\|^{\frac12}\|\Delta\mathbf{w}\|^{\frac32}\,d\tau\right)^{\frac12} \leqslant c_3c_7|||\mathbf{w}_x|||_{0,t}^{\frac14}|||\Delta\mathbf{w}|||_{0,t}^{\frac34}.$$

Therefore, we obtain the estimate

$$|||\alpha v_i w_{x_i}|||_{0,t} \leqslant c_3c_5c_7R(t)|||\mathbf{w}_x|||_{0,t}^{\frac14}|||\Delta\mathbf{w}|||_{0,t}^{\frac34}, \text{ when } n=3.$$

It results from the estimates (21), (22) that for any $\varepsilon>0$ there is such $c(\varepsilon)$ that

$$|||\alpha v_i w_{x_i}|||_{0,t} \leqslant \varepsilon|||\Delta\mathbf{w}|||_{0,t} + c(\varepsilon)|||\mathbf{w}_x|||_{0,t}, \tag{23}$$

where $c(\varepsilon)$ depends on $c_i (i=2,3,4,7), R(t), \varepsilon$. Here when $n=2$ we used the Jung inequality: $ab \leqslant \frac{1}{m}\varepsilon_1^m a^m + \frac{m-1}{m}\varepsilon_1^{-\frac{m-1}{m}} b^{\frac{m}{m-1}}$, where $m=2$ when $n=2$, and when $n=3$ we assume that $m=\frac43$.

In the course of proving the convergence of the iteration process $(9)-(11)$ we require to estimate the integral $|||(\alpha_1v_i^1-\alpha_2v_i^2)\mathbf{v}_{x_i}|||_{0,t}$, where $\alpha_i=\alpha_i(t,\mathbf{v}_x^i)=min[1,R(t)/\|\mathbf{v}_x(t)\|]$ under the condition that

$$\underset{t\in[0,T]}{vraimax}\|\mathbf{v}_x(t)\| \leqslant c_8, \quad |||\Delta\mathbf{v}|||_{0,T} \leqslant c_8. \tag{24}$$

When $n=2$ we obtain the estimates

$$|||(\alpha_1v_i^1-\alpha_2v_i^2)\mathbf{v}_{x_i}|||_{0,t} \leqslant \left(\int_0^t \|\alpha_1\mathbf{v}^1-\alpha_2\mathbf{v}^2\|_4^2\ \|\mathbf{v}_x\|_4^2\ d\tau\right)^{\frac12} \leqslant$$

$$\leqslant \sqrt2 c_2c_7\left(\int_0^t \|\alpha_1\mathbf{v}^1-\alpha_2\mathbf{v}^2\|\|\alpha_1\mathbf{v}_x^1-\alpha_2\mathbf{v}_x^2\|\|\mathbf{v}_x\|\|\Delta\mathbf{v}\|\,d\tau\right)^{\frac12} \leqslant$$

$$\leqslant \sqrt2 c_2c_7\ \underset{\tau\in[0,t]}{vraimax}\|\mathbf{v}_x(\tau)\|^{\frac12}\ \underset{\tau\in[0,t]}{vraimax}\|\alpha_1\mathbf{v}_x^1-\alpha_2\mathbf{v}_x^2\|^{\frac12}\cdot$$

$$\cdot\left(\int_0^t \|\alpha_1\mathbf{v}^1-\alpha_2\mathbf{v}^2\|\ \|\Delta\mathbf{v}\|\ d\tau\right)^{\frac12}.$$

Taking into account the inequality (24) and the relationships

$$\|\alpha_1 \mathbf{v}_x^1 - \alpha_2 \mathbf{v}_x^2\| = \|P_R \mathbf{v}_x^1 - P_R \mathbf{v}_x^2\| \leqslant \|\mathbf{v}_x^1 - \mathbf{v}_x^2\|,$$

we obtain

$$|||(\alpha_1 v_i^1 - \alpha_2 v_i^2)\mathbf{v}_{x_i}|||_{0,t} \leqslant$$
$$\leqslant \sqrt{2} c_2 c_7 c_8^{\frac{1}{2}} \underset{\tau\in[0,t]}{vraimax} \|\mathbf{v}_x^1 - \mathbf{v}_x^2\|^{\frac{1}{2}} \left(\int_0^t \|\alpha_1 \mathbf{v}_x^1 - \alpha_2 \mathbf{v}_x^2\| \|\Delta \mathbf{v}\| \, d\tau \right)^{\frac{1}{2}} \leqslant$$
$$\leqslant \sqrt{2} c_2 c_7 c_8 \underset{\tau\in[0,t]}{vraimax} \|\mathbf{v}_x^1 - \mathbf{v}_x^2\|^{\frac{1}{2}} \|\mathbf{v}_x^1 - \mathbf{v}_x^2\|_{0,t}^{\frac{1}{2}}. \tag{25}$$

When $n = 3$ we likewise prove that

$$|||(\alpha_1 v_i^1 - \alpha_2 v_i^2)\mathbf{v}_{x_i}|||_{0,t} \leqslant 2 c_3 c_7 c_8 \underset{\tau\in[0,t]}{vraimax} \|\mathbf{v}_x^1 - \mathbf{v}_x^2\|^{\frac{3}{4}} |||\mathbf{v}_x^1 - \mathbf{v}_x^2|||_{0,t}^{\frac{1}{4}}. \tag{26}$$

Applying the Jung inequality again and taking into account the inequalities (25), (26) we obtain that

$$|||(\alpha_1 v_i^1 - \alpha_2 v_i^2)\mathbf{v}_{x_i}|||_{0,t} \leqslant \varepsilon \underset{\tau\in[0,t]}{vraimax} \|\mathbf{v}_x^1 - \mathbf{v}_x^2\| + c(\varepsilon) |||\mathbf{v}_x^1 - \mathbf{v}_x^2|||_{0,t}, \tag{27}$$

where $c(\varepsilon)$ depends on ε and $c_i (i = 2, 3, 7, 8)$.

In paper [1] we introduce the operator $\tilde{\Delta}$ as the Friedrichs extension of the operator $\mathbf{P_j}\Delta$, where $\mathbf{P_j}\Delta$ is a projection from $\mathbf{L}_2(\Omega)$ in $\overset{\circ}{\mathbf{J}}(\Omega)$ determined on $\mathbf{W}_2^2(\Omega) \cap \overset{\circ}{\mathbf{J}}(\Omega)$. The operator $\tilde{\Delta}$ has the same properties in the subspace $\overset{\circ}{\mathbf{J}}(\Omega)$ as the operator Δ in the space $\mathbf{L}_2(\Omega)$ (see §4, ch. 2 and §5, ch. 3, [1]).

Let us denote $\tilde{\mathbf{v}}^k = \mathbf{v}^k e^{-\lambda t}$, multiply the equation (9) by $-\tilde{\Delta}\mathbf{v}^{k+1} e^{-2\lambda t}$ and integrate it in the domain Q_t. Upon integrating by parts, we obtain the relationship

$$\tfrac{1}{2}\|\tilde{\mathbf{v}}_x^{k+1}(t)\|^2 + \int_0^t \left(\nu \|\tilde{\Delta}\tilde{\mathbf{v}}^{k+1}\|^2 + \lambda \|\tilde{\mathbf{v}}_x^{k+1}\|^2 \right) d\tau + \int_0^t \alpha_k v_i^k \tilde{\mathbf{v}}_{x_i}^{k+1} \Delta \tilde{\mathbf{v}}^{k+1} \, dx d\tau =$$
$$= -\int_{Q_t} \tilde{\mathbf{f}} \tilde{\Delta} \mathbf{v}^{k+1} \, dx dt, \tag{28}$$

where $\tilde{\mathbf{f}} = \mathbf{f} e^{-\lambda t}$.

Applying the estimate (23) we obtain

$$J_1(t) \equiv \left| \int_0^t \alpha_k v_i^k \tilde{\mathbf{v}}_{x_i}^{k+1} \tilde{\Delta} \tilde{\mathbf{v}}^{k+1} \, dx d\tau \right| \leqslant |||\alpha_k v_i^k \tilde{\mathbf{v}}_{x_i}^{k+1}|||_{0,t} \; |||\tilde{\Delta}\tilde{\mathbf{v}}^{k+1}|||_{0,t} \leqslant$$
$$\leqslant \varepsilon \left(|||\tilde{\Delta}\tilde{\mathbf{v}}^{k+1}|||_{0,t} + \varepsilon^{-1} c(\varepsilon) |||\tilde{\mathbf{v}}_x^{k+1}|||_{0,t} \right) |||\tilde{\Delta}\tilde{\mathbf{v}}^{k+1}|||_{0,t} \leqslant$$
$$\leqslant \nu^{-1}\varepsilon \left(\nu^{\frac{1}{2}} |||\tilde{\Delta}\tilde{\mathbf{v}}^{k+1}|||_{0,t} + \nu^{\frac{1}{2}} c(\varepsilon)\varepsilon^{-1} |||\tilde{\mathbf{v}}_x^{k+1}|||_{0,t} \right) |||\nu^{\frac{1}{2}} \tilde{\Delta}\mathbf{v}^{k+1}|||_{0,t} \leqslant$$
$$\leqslant \nu^{-1}\varepsilon \left(\tfrac{3}{2}\nu |||\tilde{\Delta}\mathbf{v}^{k+1}|||_{0,t}^2 + \tfrac{1}{2}\nu^{-2}\varepsilon c^2(\varepsilon) |||\tilde{\mathbf{v}}_x^{k+1}|||_{0,t}^2 \right).$$

We used here the inequalities $(a+b)c \leqslant \frac{1}{2}(a+b)^2 + \frac{1}{2}c^2 \leqslant a^2 + b^2 + \frac{1}{2}c^2$. Further we choose $\varepsilon = \nu/6, \lambda > 36\nu^{-1}c^2(\varepsilon)$, and obtain:

$$J_1(t) \leqslant \frac{1}{4}\nu |||\tilde{\Delta}\mathbf{v}^{k+1}|||_{0,t}^2 + \frac{1}{2}\lambda |||\tilde{\mathbf{v}}_x^{k+1}|||_{0,t}^2. \tag{29}$$

Taking into account the relationship (28), the inequality (29) and the inequality

$$\left| \int_{Q_t} \tilde{\mathbf{f}} \tilde{\Delta} \tilde{\mathbf{v}}^{k+1} \, dx dt \right| \leqslant \tfrac{1}{4}\nu |||\tilde{\Delta}\tilde{\mathbf{v}}^{k+1}|||_{0,t}^2 + 4\nu^{-1} |||\tilde{\mathbf{f}}|||_{0,t}^2,$$

we obtain

$$\tfrac{1}{2}\left\|\tilde{\mathbf{v}}_x^{k+1}(t)\right\|^2 + \tfrac{1}{2}\nu |||\tilde{\Delta}\tilde{\mathbf{v}}^{k+1}|||_{0,t}^2 + \tfrac{1}{2}\lambda |||\tilde{\mathbf{v}}_x^{k+1}|||_{0,t}^2 \leqslant 4\nu^{-1} |||\tilde{\mathbf{f}}|||_{0,t}^2.$$

Hence it follows that

$$\nu|||\tilde{\Delta}\tilde{\mathbf{v}}^{k+1}|||_{0,t}^2 + \lambda|||\tilde{\mathbf{v}}_x^{k+1}|||_{0,t}^2 \leqslant 8\nu^{-1}|||\tilde{\mathbf{f}}|||_{0,t}^2$$

and

$$\tfrac{1}{2} \underset{\tau\in[0,t]}{vraimax} \|\tilde{\mathbf{v}}_x(\tau)\|^2 \leqslant 4\nu^{-1}|||\tilde{\mathbf{f}}|||_{0,t}.$$

Considering further that $|||\tilde{\mathbf{f}}|||_{0,t} \leqslant |||\mathbf{f}|||_{0,t}$, $|||\tilde{\mathbf{v}}_x^{k+1}|||_{0,t} \geq e^{-\lambda t}|||\mathbf{v}^{k+1}|||_{0,t}$, $|||\tilde{\Delta}\tilde{\mathbf{v}}^{k+1}|||_{0,t} \geq$ $\geq e^{-\lambda t}|||\tilde{\Delta}\tilde{\mathbf{v}}^{k+1}|||_{0,t}$ and $\underset{\tau\in[0,t]}{vraimax}\|\tilde{\mathbf{v}}_x(\tau)\| \geq e^{-\lambda t} \underset{\tau\in[0,t]}{vraimax}\|\mathbf{v}_x(\tau)\|$, we obtain the inequality:

$$[\mathbf{v}^{k+1}]_{\lambda,t}^2 \leqslant 12\nu^{-1}|||\mathbf{f}|||_{0,t}^2\, e^{2\lambda t}. \tag{30}$$

Let us note that in the last inequality the parameter λ is determined by the values ν, $R(t)$ and constants c_i $\left(i=\overline{2,8}\right)$. Therefore, the inequality (19), where $c_t = 2\sqrt{3}\nu^{-\frac{1}{2}}|||\mathbf{f}|||_{0,t}$ and $c_6 = c_T e^{2\lambda T}$ has been proved.

4. Convergence of the iteration sequence

Let us introduce the notation $\mathbf{w}^k = \mathbf{v}^k - \mathbf{v}^{k-1}$, $\delta p_k = p_k - p_{k-1}$ and note that $\mathbf{w}^{k+1} \in \overset{\circ}{\mathbf{J}}(Q_T)$ satisfies the equation

$$\mathbf{w}_t^{k+1} - \nu\Delta\mathbf{w}^{k+1} + \alpha_k v_i^k \mathbf{w}_{x_i}^{k+1} + grad\ \delta p_k = -\left(\alpha_k v_i^k - \alpha_{k-1}v_i^{k-1}\right)\mathbf{v}_{x_i}^k, \tag{31}$$

of the homogeneous and boundary conditions.

Let us multiply the equation (31) by $-\tilde{\Delta}\mathbf{w}^{k+1}e^{-2\lambda t}$ and integrate it by parts. Then we obtain the relationship:

$$\tfrac{1}{2}\left\|\tilde{\mathbf{w}}^{k+1}(t)\right\|^2 + \nu|||\tilde{\Delta}\tilde{\mathbf{w}}^{k+1}|||_{0,t}^2 + \lambda|||\tilde{\mathbf{w}}_x^{k+1}|||_{0,t}^2 + \int\limits_{Q_t}\alpha_k v_i^k\tilde{\mathbf{w}}_{x_i}^{k+1}\tilde{\Delta}\tilde{\mathbf{w}}^{k+1}\,dxdt =$$

$$= \int\limits_{Q_t}\left(\alpha_k\tilde{v}_i^k - \alpha_{k-1}\tilde{v}_i^{k-1}\right)\mathbf{v}_{x_i}^k\tilde{\Delta}\tilde{\mathbf{w}}^{k+1}\,dxdt, \tag{32}$$

where $\tilde{\mathbf{w}}^{k+1} = \mathbf{w}^{k+1}e^{-\lambda t}$.

As we have proved the sequence $\left\{\mathbf{v}^k\right\}$ satisfies the condition (24) if we assume that $c_8 = c_6$.

Taking into account the inequality (23) we obtain

$$J_2(t) = \left|\int\alpha_k v_i^k\tilde{\mathbf{w}}_{x_i}^{k+1}\tilde{\Delta}\tilde{\mathbf{w}}^{k+1}\,dxdt\right| \leqslant |||\alpha_k v_i^k\tilde{\mathbf{w}}_{x_i}^{k+1}|||_{0,t}\ |||\tilde{\Delta}\mathbf{w}^{k+1}|||_{0,t} \leqslant$$

$$\leqslant \varepsilon\nu^{-1}\left(\nu^{\frac{1}{2}}|||\Delta\tilde{\mathbf{w}}^{k+1}|||_{0,t} + \nu^{\frac{1}{2}}c_1(\varepsilon)|||\tilde{\mathbf{w}}_x^{k+1}|||_{0,t}\right)|||\nu^{\frac{1}{2}}\tilde{\Delta}\mathbf{w}^{k+1}|||_{0,t} \leqslant$$

$$\leqslant 2\varepsilon\nu^{-1}\left(\nu|||\Delta\tilde{\mathbf{w}}^{k+1}|||_{0,t}^2 + \nu c_1^2(\varepsilon)|||\tilde{\mathbf{w}}_x^{k+1}|||_{0,t}^2\right). \tag{33}$$

We apply the inequality (27) to estimate the integral in the right-hand side of the relationship (32) and obtain

$$J_3(t) = \left|\int\limits_{Q_t}\left(\alpha_k\tilde{v}_i^k - \alpha_{k-1}\tilde{v}_i^{k-1}\right)\mathbf{v}_{x_i}^k\tilde{\Delta}\tilde{\mathbf{w}}^{k+1}\,dxd\tau\right| \leqslant$$

$$\leqslant \sqrt{2}\varepsilon\nu^{-\frac{1}{2}}\left(\tfrac{1}{\sqrt{2}}\underset{\tau\in[0,t]}{vraimax}\left\|\tilde{\mathbf{w}}_x^k(\tau)\right\| + \tfrac{1}{\sqrt{2}}c_2(\varepsilon)|||\tilde{\mathbf{w}}_x^k|||_{0,t}\right)\ |||\nu^{\frac{1}{2}}\tilde{\Delta}\tilde{\mathbf{w}}^{k+1}|||_{0,t} \leqslant$$

$$\leqslant 2\varepsilon\nu^{\frac{1}{2}}\left(\frac{1}{2}\underset{\tau\in[0,t]}{vraimax}\left\|\tilde{\mathbf{w}}_x^k(\tau)\right\|^2 + \frac{1}{2}c_2^2(\varepsilon)|||\tilde{\mathbf{w}}^k|||_{0,t}^2\right)^{\frac{1}{2}}\ |||\nu^{\frac{1}{2}}\tilde{\Delta}\tilde{\mathbf{w}}^k|||_{0,t}. \tag{34}$$

Note that here $c_1(\varepsilon)$, $c_2(\varepsilon)$ depend on ε, c_i $\left(i=\overline{2,7}\right)$, $R(t)$.

We choose λ satisfying the condition

$$\lambda > max\left[\nu c_1^2(\varepsilon),\ \frac{1}{2}c_2(\varepsilon)\right], \tag{35}$$

then it results from the relationship (32) and the inequalities (33), (34) that

$$\frac{1}{2}\left\|\tilde{\mathbf{w}}^{k+1}(t)\right\|^2+\left(1-2\varepsilon\nu^{-1}\right)\left(\nu|||\tilde{\Delta}\tilde{\mathbf{w}}^{k+1}|||_{0,t}^2+\lambda|||\tilde{\mathbf{w}}_x^{k+1}|||_{0,t}^2\right)\leqslant 2\varepsilon\nu^{-\frac{1}{2}}\left[\tilde{\mathbf{w}}^k\right]_{\lambda,t}\left[\tilde{\mathbf{w}}^{k+1}\right]_{\lambda,t}. \quad (36)$$

Hence it follows that

$$\nu|||\tilde{\Delta}\tilde{\mathbf{w}}^{k+1}|||_{0,t}^2+\lambda|||\tilde{\mathbf{w}}_x^{k+1}|||_{0,t}^2\leqslant 2\varepsilon\nu^{-\frac{1}{2}}\left(1-2\varepsilon\nu^{-1}\right)\left[\tilde{\mathbf{w}}^k\right]_{\lambda,t}\left[\tilde{\mathbf{w}}^{k+1}\right]_{\lambda,t};$$

$$\tfrac{1}{2}\ \underset{\tau\in[0,t]}{vraimax}\left\|\tilde{\mathbf{w}}^{k+1}(\tau)\right\|^2\leqslant 2\varepsilon\nu^{-\frac{1}{2}}\left[\tilde{\mathbf{w}}^k\right]_{\lambda,t}\left[\tilde{\mathbf{w}}^{k+1}\right]_{\lambda,t}.$$

It results from the last two inequalities that

$$\left[\tilde{\mathbf{w}}^{k+1}\right]_{\lambda,t}\leqslant q(\varepsilon)\left[\tilde{\mathbf{w}}^k\right]_{\lambda,t}, \quad (37)$$

where $q(\varepsilon)=2\varepsilon\nu^{-\frac{1}{2}}\left(\left(1+2\varepsilon\nu^{-1}\right)^{-1}+1\right)=4\varepsilon\nu^{-\frac{1}{2}}\left(\nu+\varepsilon\right)\left(\nu+2\varepsilon\right)^{-1}$. Since $\lim\limits_{\varepsilon\to 0}q(\varepsilon)=0$, then in any $q\in(0,1)$ we can find such $\varepsilon>0$ that $q\in(0,1)$, and find λ in $\varepsilon>0$ which satisfies the condition (35).

Let us denote $\mathbf{w}^{k,l}=\mathbf{v}^{k+l}-\mathbf{v}^k$, $\tilde{\mathbf{w}}^{k,l}=\tilde{\mathbf{v}}^{k+l}-\tilde{\mathbf{v}}^k$, and considering the inequality (37), we obtain

$$\left[\tilde{\mathbf{w}}^{k,l}\right]_{\lambda,t}\leqslant\sum_{j=k}^{k+l}\left[\tilde{\mathbf{w}}^j\right]_{\lambda,t}\leqslant(1-q)^{-1}\left[\mathbf{w}^k\right]_{\lambda,t}\leqslant q^{k-1}\left(1-q\right)^{-1}\left[\tilde{\mathbf{w}}^1\right]_{\lambda,t}, \quad (38)$$

where $q=q(\varepsilon)$. Hence it follows that

$$\left[\mathbf{w}^{k,l}\right]_{\lambda,t}\leqslant e^{\lambda t}q\left(1-q\right)^{-1}\left[\mathbf{w}^1\right]_{\lambda,t}. \quad (39)$$

We note further that $\mathbf{w}^{k,l}$ satisfies the equation

$$\mathbf{w}_t^{k,l}-\nu\tilde{\Delta}\mathbf{w}^{k,l}+grad\ \delta p_{k,l}+\alpha_{k+l-1}v_i^{k+l-1}\mathbf{w}_{x_i}^{k,l}=-\left(\alpha_{k+l-1}v_i^{k+l-1}-\alpha_{k-1}v_i^{k-1}\right)\mathbf{v}_{x_i}^k, \quad (40)$$

here $p_{kl}=p_{k+l}-p_k$.

Let us multiply the last equality by $\mathbf{w}_t^{k,l}$.Integrating by parts, we obtain

$$|||\mathbf{w}_t^{k,l}|||_{0,t}^2+\tfrac{1}{2}\left\|\mathbf{w}_x^{k,l}(t)\right\|^2=-\int\limits_{Q_t}\alpha_{k+l-1}v_i^{k+l-1}\mathbf{w}_{x_i}^{k,l}\mathbf{w}_t^{k,l}\,dxd\tau-$$

$$-\int\limits_{Q_t}\left(\alpha_{k+l-1}v_i^{k+l-1}-\alpha_{k-1}v_i^{k-1}\right)\mathbf{v}_{x_i}^k\mathbf{w}_t^{k,l}\,dxd\tau. \quad (41)$$

Taking into account the inequalities (23), (27), we obtain

$$|||\alpha_{k+l-1}v_i^{k+l-1}\mathbf{w}_{x_i}^{k,l}|||_{0,t}\leqslant\varepsilon\left(|||\Delta\mathbf{w}^{k,l}|||_{0,t}+\varepsilon^{-1}c(\varepsilon)|||\mathbf{w}_x^{k,l}|||_{0,t}\right), \quad (42)$$

$$|||\left(\alpha_{k+l-1}v_i^{k+l-1}-\alpha_{k-1}v_i^{k-1}\right)\mathbf{v}_{x_i}^k|||_{0,t}\leqslant\varepsilon\left(\underset{\tau\in[0,t]}{vraimax}\left\|\mathbf{w}^{k-1,l}(\tau)\right\|+\varepsilon^{-1}c(\varepsilon)|||\mathbf{w}_x^{k-1,l}|||_{0,t}\right). \quad (43)$$

Consider that λ satisfies the condition (35) and $\lambda>\varepsilon^{-1}c(\varepsilon)$, where $c(\varepsilon)$ is taken from the inequalities (23), (27). Taking into account the equality (41) and the inequalities (42), (43), (39), we obtain

$$|||\mathbf{w}_t^{k,l}|||_{0,t}\leqslant\sqrt{2}\varepsilon\left(\left[\mathbf{w}^{k,l}\right]_{\lambda,t}+\left[\mathbf{w}^{k-1,l}\right]_{\lambda,t}\right)\leqslant 2\sqrt{2}q^{k-2}\left(1-q\right)^{-1}e^{\lambda,t}\left[\mathbf{w}^1\right]_{\lambda,t}. \quad (44)$$

Note that the equation (40) is the solution of the Stokes system

$$-\Delta\mathbf{w}^{k,l}+grad\ \delta p_{k,l}=-\alpha_{k+l-1}v_i^{k+l-1}\mathbf{v}_{x_i}^{k+l}-\left(\alpha_{k+l-1}v_i^{k+l-1}-\alpha_{k-1}v_i^{k-1}\right)\mathbf{v}_{x_i}^k-\mathbf{w}_t^{k,l}\equiv g_k(t)$$

almost everywhere on t on [0,T].

By virtue of the known inequality for the Stokes operator(see, for example, [3] ch.1)

$$\left\|\mathbf{w}^{k,l}\right\|_{\mathbf{w}_2^2(\Omega)}+\left\|\delta_{k,l}\right\|_{\mathbf{w}_2^1(\Omega)}\leqslant c_0\left\|g_k(t)\right\|_{\mathbf{L}_2^n(\Omega)}.$$

Therefore, taking into account the estimates (39), (42) − −(44), we obtain

$$|||p_{k,l}|||_{\mathbf{w}_2^{1,0}(Q_t)} \leqslant c_0|||g_k|||_{0,t} \leqslant 4c_0\varepsilon q^{k-1}\left(1-q\right)^{-1} e^{\lambda,t}\left[\mathbf{w}^1\right]_{\lambda t}. \tag{45}$$

It results from the estimates (39), (44), (45) that the sequence $\left\{\mathbf{v}^k\right\}_{k=0}^{\infty}$ converges to the norm $|||\mathbf{v}_t|||_{0,T}+[\mathbf{v}]_{\lambda,T}$, and the sequence $\{p_k\}_{k=1}^{\infty}$ converges to the norm $\mathbf{W}_2^{1,0}(Q_t)$. Proceeding to the limit when $k\to\infty$ in the iteration process (9) − (11), we make sure that the functions $\mathbf{v},p$ are the solutions of the problem (8), (2), (3).

Let us denote $\mathbf{z}^k=\mathbf{v}^k-\mathbf{v}$ and $\delta\overline{p}_k=p_k-p$, then it is easy to obtain the following equality instead of the relationship (31):

$$\mathbf{z}_t^{k+1}-\nu\Delta\mathbf{z}^{k+1}+\alpha_k v_i^k\mathbf{z}_{x_i}^{k+1}=\left(\alpha_k v_i^k-\alpha_{k-1}v_i^{k-1}\right)\mathbf{v}_{x_i},$$

from which we obtain both the inequality (37) and

$$\left[\tilde{\mathbf{z}}^{k+1}\right]_{\lambda,t}\leqslant q\left(\varepsilon\right)\left[\tilde{\mathbf{z}}^{k}\right]_{\lambda,t},$$

where $\tilde{\mathbf{z}}^k=\mathbf{z}^k e^{-\lambda t}$. Taking into account the last inequality we find that

$$\left[\mathbf{z}^k\right]_{\lambda,t}\leqslant e^{\lambda t}q^k\left[\mathbf{z}^0\right]_{\lambda,t}. \tag{46}$$

Further instead of the inequality (44), we sequentially obtain the estimate

$$|||\mathbf{z}_t^k|||_{0,t}\leqslant 2\varepsilon e^{\lambda t}q^k\left[\mathbf{z}^0\right]_{\lambda,t} \tag{47}$$

and instead of the estimate (45) we obtain the inequality

$$\|\delta\overline{p}_k\|_{\mathbf{W}_2^{1,0}(Q_t)}\leqslant 4c_0e^{\lambda t}q^k\left[\mathbf{z}^0\right]_{\lambda,t}. \tag{48}$$

Let us denote the space $\mathbf{W}^{2,1}(Q_T)\cap\mathbf{L}_\infty(0,T;\mathbf{W}_2^1(\Omega))$ via $\mathbf{V}_2$ with the norm

$$\|\mathbf{v}\|_{\mathbf{V}_2}=\|\mathbf{v}\|_{\mathbf{W}^{2,1}(Q)}+\underset{t\in[0,T]}{vraimax}\|\mathbf{v}_x\| \tag{49}$$

and note that the norm determined by the formula $\|\mathbf{v}\|_{\lambda,T}=|||\mathbf{v}_t|||_{0,T}+[\mathbf{v}]_{\lambda,T}$ is equivalent to the norm (49).

As it was specified above, there is ε for any $q\in(0,1)$, and for $\varepsilon-\lambda$, under which the estimates (46) − (48) hold. Therefore it results from these estimates that

$$\left\|\mathbf{z}^k\right\|_{\mathbf{V}_2}\leqslant c\left(q\right)q^k\left\|\mathbf{z}^0\right\|_{\mathbf{V}_2}, \tag{50}$$

$$\|\delta\overline{p}_k\|_{\mathbf{W}_2^{1,0}(Q_t)}\leqslant c\left(q\right)q^k\left\|\mathbf{z}^0\right\|_{\mathbf{V}_2}. \tag{51}$$

Let us show that the solution of the problem (8), (2), (3) is unique. Indeed, assume that $\mathbf{v}^1$, $\mathbf{v}^2$; p_1, p_2 are two solutions of this problem. Then $\mathbf{w}=\mathbf{v}^1-\mathbf{v}^2$ is the solution of the equation

$$\mathbf{w}_t-\nu\Delta\mathbf{w}+\alpha_1v_i^1\mathbf{w}_{x_i}+\left(\alpha_1v_i^1-\alpha_2v_i^2\right)\mathbf{v}_{x_i}^2=grad\left(p_1-p_2\right),$$

where $\alpha_i=\alpha_i\left(t,\mathbf{v}_x^i\right)=min\left[1,R(t)\left\|\mathbf{v}_x^i\right\|^{-1}\right]\left(i=1,2\right)$.

The last equation has the form of the equation (31). Repeating estimates similarly to deducing the inequality (37), we obtain the inequality:

$$\left[\tilde{\mathbf{w}}\right]_{\lambda,t}\leqslant q\left(\varepsilon\right)\left[\tilde{\mathbf{w}}\right]_{\lambda,t},$$

where $\tilde{\mathbf{w}}=\mathbf{w}e^{\lambda t}$, $q\left(\varepsilon\right)\in(0,1)$. Therefore $\tilde{\mathbf{w}}=0$ and, consequently, $\mathbf{v}^1\equiv\mathbf{v}^2$, $p_1=p_2$.

Thus, we have proved **Theorem.** Assume that $\mathbf{f}\in\overset{\circ}{\mathbf{J}}(Q_T)$, Ω is a bounded domain with the boundary $S\in C^2$, $\mathbf{a}(x)$ satisfies the conditions (4); then the problem (8), (2), (3) has a unique solution $\mathbf{v}$, p with $\mathbf{v}_{xx}$, $\mathbf{v}_t$, p_x from $\mathbf{L}_2(Q_T)$, the sequences $\left\{\mathbf{v}^k\right\}_{k=0}^{\infty}$, $\left\{p^k\right\}_{k=1}^{\infty}$, determined by the iteration process (9) − (11), where $\alpha_k=min\left[1,R(t)\left\|\mathbf{v}_x^k\right\|^{-1}\right]$, $R(t)$ is a bounded nondecreasing function. The following estimates converge to the solution of the problem (8), (2), (3), and hold:

$$\left\|\mathbf{v}^k - \mathbf{v}\right\|_{\mathbf{V}_2} \leqslant c(q)q^k \left\|\mathbf{v}^0 - \mathbf{v}\right\|_{\mathbf{V}_2}, \tag{52}$$

$$\|p_k - p\|_{\mathbf{W}_2^{1,0}(Q_T)} \leqslant c(q)q^k \left\|\mathbf{v}^0 - \mathbf{v}\right\|_{\mathbf{V}_2} \tag{53}$$

with any $q \in (0,1)$.

4. Corollaries, remarks, other variants of iteration processes

Remark 2. The proved theorem guarantees the existence of the solution of the problem (8), (2), (3) and the convergence of the iteration process (9) $-$ (11) at any interval $[0,T]$, where $\mathbf{f} \in \overset{\circ}{\mathbf{J}}(Q_T)$.

Corollary 1. If the solution $\mathbf{v}^*$, p_* of the problem (8), (2), (3) satisfies the inequality

$$\|\mathbf{v}_x^*\| \leqslant R(t) \quad \forall t \in [0, T_1], \tag{54}$$

then the solution of the problem (1) $--$(3) at the interval $[0,T_1](T_1 \leqslant T)$ also exists at this interval for $\mathbf{v} = \mathbf{v}^*, p = p_*$.

Indeed, if the inequality (54) holds, then $\alpha\left(t, \mathbf{v}_x^*\right) = 1$, therefore the equations (1) and (8) coincide.

Corollary 2. If the a priori estimate (5) holds at the interval $[0,T_1]$ for the solution of the problem (1) $-$ (3) and the problem (8), (2), (3) when $R(t) \geq M(t)$, then at the same interval there is the solution of the problem (1) $-$ (3), which coincides with the solution of the problem (8), (2), (3).

Let us note that as a rule (see, for example, Lemma 9, ch. 6, [1]) the a priori estimate (5) for the problem (1) $-$ (3) is, obtained from the estimate of the integral $J_1 = \left|\int\limits_\Omega v_i \mathbf{v}_{x_i} \tilde{\Delta}\mathbf{v}\, dx\right| \leqslant c\nu^{-\frac{1}{2}} \left\|\tilde{\Delta}\mathbf{v}\right\|^{\frac{3}{2}} \|\mathbf{v}_x\|^{\frac{3}{2}}$ (in the case $n = 3$). Whereas $|\alpha(t, \mathbf{v}_x)| \leqslant 1$ and does not depend on x, then one can readily see that the integral $\left|\int\limits_\Omega \alpha(t, \mathbf{v}_x) v_i \mathbf{v}_{x_i} \tilde{\Delta}\mathbf{v}\, dx\right|$ has the same estimate. Therefore the a priori estimate for the problem (1) $-$ (3) obtained also holds for the problem (8), (2), (3).

Remark 3. Taking into account the corollaries 1, 2 it is easy to construct an iteration process, converging to the solution of the problem (1) $-$ (3) without application of the estimate of the form (5) under the condition that the solution of the problem (1) $-$ (3) does exist and satisfies the limit of the form (5). Indeed, we set some positive, bounded and nondecreasing function $R_1(t)$ and solve the problem (8), (2), (3) when $R(t) = R_1(t)$. Further we verify the condition (54) when $R(t) = R_1(t)$. If the condition holds, then the problem (1) $-$ (3) is solved. If the condition does not hold, then we assume that $R_2(t) = R_1(t) + K$ (K is the parameter of the method) and repeat the iteration process. It is clear that after the finite number of steps the condition (54) holds and, consequently, the problem (1) $-$ (3) is solved.

Remark 4. If the condition (5) holds, then $\|\mathbf{v}(t)\|_4 \leqslant cM(t)$ as it results from the inequalities (16), (17). Analysing the proof of the theorem it is easy to see that the statement of the theorem and the remarks specified above hold if we determine α_k by the formula

$$\alpha_k = \alpha_k(t, \mathbf{v}^k) = min\left[1, R(t)/\left\|\mathbf{v}^k\right\|_4\right]. \tag{55}$$

In this case $\alpha_k \mathbf{v}^k = P\mathbf{v}^k$ is a projection of the vector $\mathbf{v}^k$ on the sphere $\{\mathbf{v} \in L_4(\Omega) : \|\mathbf{v}\|_4 \leqslant R(t)\}$.

The key inequalities in the proof of the Theorem are the inequalities (23), (27), which proof is even simplified in this case. For example, the inequality (23) is obtained from the inequality

(21), which in this case can be rewritten in the form:

$$|||\alpha v_i \mathbf{w}_{x_i}|||_{0,t} \leqslant \left(\int_0^t \|\alpha \mathbf{v}\|_4^2 \|\mathbf{w}_x\|_4^2 \, d\tau \right)^{\frac{1}{2}} \leqslant R(t) \left(\int_0^t \|\mathbf{w}_x\|_4^2 \, d\tau \right)^{\frac{1}{2}} . \tag{56}$$

Taking into account the fact that in case of the estimate of the integral $\int_0^t \|\mathbf{w}_x\|_4^2 \, d\tau$ the a priori estimate is not used. It is easy to see that the inequality (23) holds.

In the course of proving the inequalities (27) we use the fact that $\|\alpha_1 \mathbf{v}^1 - \alpha_2 \mathbf{v}^2\|_4 \leqslant \leqslant \|\mathbf{v}^1 - \mathbf{v}^2\|_4$, then $|||\left(\alpha_1 v_i^1 - \alpha_2 v_i^2\right) \mathbf{v}_{x_i}|||_{0,t} \leqslant \left(\int_0^t \|\mathbf{v}^1 - \mathbf{v}^2\|_4^2 \|\mathbf{v}_x\|_4^2 \, d\tau \right)^{\frac{1}{2}}$. Simplifying the calculations in the course of proving the inequalities (27), we make sure that it holds when α_k is determined by the formula (55).

Remark 5. In the case when the uniform estimate is known

$$R_1(t) \leqslant |\mathbf{v}(x,t)| \leqslant R_2(t) \quad \forall \, (x,t) \in Q_t \tag{57}$$

we consider instead of the equation (8) the equation

$$\mathbf{v}_t - \nu \Delta \mathbf{v} + P v_i \mathbf{v}_{x_i} + grad \; p = \mathbf{f}, \tag{58}$$

where

$$P v_i(x,t) = \begin{cases} R_1(t), & \text{if } v_i(x,t) < R_1(t), \\ v_i(x,t), & \text{if } R_1(t) \leqslant v_i(x,t) \leqslant R_2(t), \\ R_2(t), & \text{if } v_i(x,t) > R_2(t) \end{cases} \tag{59}$$

and we substitute the equation (9) by the equation

$$\mathbf{v}_t^{k+1} - \nu \Delta \mathbf{v}^{k+1} + P v_i^k \mathbf{v}_{x_i}^{k+1} + grad \; p_{k+1} = \mathbf{f}. \tag{9$'$}$$

One can readily see that the statements of the Theorem and also corollaries and remarks, which are similar to the corollaries 1, 2 and remarks 2, 3 hold for the problem (58), (2), (3). The sequence $\left\{\mathbf{v}^k\right\}$ is determined here by the iteration process (9′), (10), (11).

Indeed, the inequality (23) obviously holds when $\varepsilon = 0$, $c(\varepsilon) = R_2(t)$. The validity of the inequality of the form (27) can be easily verified if we take into account that $|Pv_i^1 - Pv_i^2| \leqslant$ $|v_i^1 - v_i^2|$
$\forall \, (x,\tau) \in Q_t$, therefore $\|P\mathbf{v}^1 - P\mathbf{v}^2\|_4 \leqslant \|\mathbf{v}^1 - \mathbf{v}^2\|_4$.

In conclusion we note that the present paper allows one to reduce the solution of the non-linear Navier-Stokes system to the solution of the sequence of linear problems. There are different approaches to the solution of linear problems. We can find among them an approach based on gradient methods of minimisation of the functional $J(\mathbf{v}) = \int_{Q_T} |div \; \mathbf{v}|^2 \, dxdt$, in which the pressure p is considered as control (see, for example, [4], [5]). In paper [6] we introduce the Theorem on convergence of one of the variants of the gradient method for the solution of the non-linear Navier-Stokes problem. We have constructed an iteration method of the fastest descent for the solution of a linear problem. The method parameters are found explicitly, which makes the preliminary linearisation of the problem expedient.

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