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INTEGRAL ESTIMATES FOR DERIVATIVES OF ANALYTIC FUNCTIONS OUTSIDE CONVEX DOMAINS

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Abstract. In the present paper weight integral estimates are obtained for derivatives of functions which are analytic in the exterior of convex bounded domains. The estimates are obtained in terms of integrals of functions vanishing at infinity. This result generalizes the Hardy-Littlewood theorem for exteriors of convex bounded domains. Theorems of this kind have been earlier obtained by K. P. Isaev and R. S. Yulmukhametov for the power weight and for the first derivative of an analytic function of the first order belonging to L^2. N. M. Tkachenko and F. A. Shamoyan have generalized this result for all higher order derivatives belonging to the space L^p. In the present paper the class of weights under consideration is essentially enlarged.

Keywords: Analytic function, the Green function, the Laplace invariants, generalized Laplace invariants.

1. Introduction

Let G be a bounded convex domain and $D = \mathbb{C}\backslash\overline{G}$. We denote the distance from the point z to the boundary D by $d(z)$, $z \in \mathbb{C}$,: $d(z) = dist(z, \partial D)$.

Let us denote the space for the holomorphic functions

$$B^2_{w,n}(D) = \left\{ f \in H(D), f(\infty) = 0 : \|f\|^2_{B^2_{w,n}(D)} = \int\limits_D |f^{(n)}(z)|^2 w(d(z)) d\mu(z) < \infty \right\}, \quad (1)$$

$$(2)$$

where $d\mu(z)$ is the Lebesgue measure. In paper [1] we prove the following theorem

Theorem A. *There is an absolute constant $c > 0$ such that for any function $f \in B^2_{w,1}(D)$ the following relationship holds:*

$$c \int\limits_D |f''(z)|^2 d^2(z) d\mu(z) \leqslant \int\limits_D |f'(z)|^2 d\mu(z) \leqslant 2 \int\limits_D |f''(z)|^2 d^2(z) d\mu(z), \quad (3)$$

where $d(z) = dist(z, \partial G)$.

The analogue for the Smirnov space is described in the papers [2],[3].

Further advance in this direction is announced in the paper [4]. Namely, the following theorem has been proved.

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Theorem A′. *If $\alpha > -1/2$, then there is the constant $C(n,\alpha) > 0$, independent of the domain D and such that*

$$\sqrt{\frac{(\alpha+1)(2\alpha+1)}{2}}\int\limits_D |f^{(n)}(z)|^2 d^{2\alpha}(z)d\mu(z) \leqslant \int\limits_D |f^{(n+1)}(z)|^2 d^{2(\alpha+1)}(z)d\mu(z) \leqslant$$

$$\leqslant C(n,\alpha)\int\limits_D |f^{(n)}(z)|^2 d^{2\alpha}(z)d\mu(z).$$

The following estimate holds for $\alpha = -1/2$:

$$\frac{1}{2}\int\limits_D |f^{(n)}(z)|^2 d^{-1}(z)d\mu(z) \leqslant \int\limits_D |f^{(n+1)}(z)|^2 d(z)d\mu(z) \leqslant C\left(n,-\frac{1}{2}\right)\int\limits_D |f^{(n)}(z)|^2 d^{-1}(z)d\mu(z).$$

N. M. Tkachenko and F. A. Shamoyan generalized this result for derivatives of an arbitrary order from the space L^p with the degree weight (see [7]-[9]).

Let us note that importance of the problems under consideration is stipulated by their supposed application for generalization of results of the papers (see [10]-[15])

To prove these theorems the result (see [5], p.203) was used:

Theorem B. *Let A be an arbitrary closed set in $\mathbb{R}^n$. Then there is a finite differential function $\delta(x) = \delta(x, A)$, on $\mathbb{R}^n \backslash A$ having the following properties*

$$C_1\delta(x) \leqslant dist(x, A) \leqslant C_2\delta(x),$$

and for any α we have

$$\left|\frac{\partial^\alpha}{\partial x^\alpha}\delta(x)\right| \leqslant C_\alpha \left(dist(x, A)\right)^{1-|\alpha|},$$

where C_α, C_1, C_2 do not depend on A.

The following Theorem is also proved there.

Theorem C. *Let Ω be some open connected set on the complex plane $\mathbb{C}$, then there is such a set of squares*
$P = \{Q_1, Q_2, \ldots, Q_k, \ldots\}$, $(Q_k\backslash\partial Q_k) \cap (Q_m\backslash\partial Q_m) = \emptyset, k \neq m$, *that* $\bigcup\limits_k Q_k = \Omega$, *and*

$$c_1 diam(Q_k) \leqslant dist(Q_k, \partial\Omega) \leqslant c_2 diam(Q_k)$$

the constants c_1, c_2 do not depend on Ω. In brief, such relationships are written in the form $diam(Q_k) \asymp dist(Q_k, \partial\Omega)$.

In the paper [1] the following Lemma is also proved:

Lemma A.

1. The function of the distance $d(z)$ is convex (in particular, it is subharmonic) and satisfies the Lipschitz condition

$$|d(z_1) - d(z_2)| < |z_1 - z_2|, \forall z_1, z_2 \in G.$$

2. *Let G be a convex domain and $z_0 \notin G$. If the function $dist(z, G)$ is differentiated in the point z_0, then $|grad\ dist(z_0, G)| = 1$.*

3. *If D is a convex polygon, then $d(z)$ is continuously differentiated in G.*

We also use the technical lemma:

Lemma B. *Let $G \subseteq \mathbb{C}$ be a 1-connected domain, $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ be the Laplace operator, $f \in H(G)$, $2 \leqslant p < \infty$. Then*

$$\Delta|f^{(k)}(z)|^p = p^2|f^{(k)}(z)|^{p-2}|f^{(k+1)}(z)|^2.$$

Proof: Let us denote an integral function by $f^{(k)}(z) = u(x, y) + iv(x, y)$, consequently, $u'_y = -v'_x$, $v'_y = u'_x$ and $\Delta u = 0$, $\Delta v = 0$. In our notation

$$\Delta|f^{(k)}(z)|^p = p(p-2)\left(u^2+v^2\right)^{\frac{p}{2}-2}[(uu'_x + vv'_x)^2 + (uu'_y + vv'_y)^2]+$$

$$+p\left(u^2+v^2\right)^{\frac{p}{2}-1}\left[(u'_x)^2+(u'_y)^2+(v'_x)^2+(v'_y)^2+(u\Delta u+v\Delta v)\right]=$$
$$=p(p-2)\left(u^2+v^2\right)^{\frac{p}{2}-1}\left[(u'_x)^2+(v'_x)^2\right]+2p\left(u^2+v^2\right)^{\frac{p}{2}-1}\left[(u'_x)^2+(v'_x)^2\right]=$$
$$=p^2\left(u^2+v^2\right)^{\frac{p}{2}-1}\left[(u'_x)^2+(v'_x)^2\right]=|f^{(k+1)}(z)|^2.$$

We took into account that $f^{(k+1)}(z)=(u'_x+v'_y)/2+i(v'_x-u'_y)/2$, then the following equality holds:

$$f^{(k+1)}(z)=(u'_x+v'_y)/2+i(v'_x-u'_y)/2$$

and

$$|f^{(k+1)}(z)|^2=\frac{1}{4}((u'_x+v'_y))^2+((v'_x-u'_y))^2=(u'_x)^2+(v'_x)^2.$$

2. The main result

The following theorem holds:

Theorem 1. *Let G be a bounded convex domain and $D=\mathbb{C}\backslash\overline{G}$, $f\in B^p_w(D)$. Then the following estimates hold for $\forall n\in\mathbb{N}$, $2\leqslant p<\infty$,*

$$c_1\int\limits_D|f(z)|^p w(d(z))d\mu(z)\leqslant\int\limits_D|f^{(n)}(z)|^p d^{np}(z)w(d(z))d\mu(z)\leqslant c_2\int\limits_D|f(z)|^p w(d(z))d\mu(z),\quad(4)$$

where $c_1(n,p,\alpha)$, $c_2(n)$ are positive constants depending only on n, p, α, and the non-negative twice continuously differentiable function $w(t)$, satisfying the following conditions:

$$(t^2w(t))'\geqslant 0,$$
$$(t^2w(t))''\geqslant\alpha w(t),$$
$$w(2t)\leqslant\beta w(t),$$

for some positive constants α, β, and $\forall t>0$.

Remark: A partial case of the weights in the theorem is $w(t)=t^\gamma$, when $\gamma>-1$.

Proof: Let us prove the left-hand side inequality (4).

Let $B(r)$ be a circle of the radius r with the centre in the origin of coordinates, $R_0=diam(G)$. We consider that $0\in G$ without loss of generality. Let us construct a convex polygon M such that $\overline{G}\subset M\subset B(2R_0)$. Let us determine the domain $U=\mathbb{C}\backslash\overline{M}\cap B(R)$ for the arbitrary $R>2R_0$. Since the boundary U is piecewise-smooth, then we can apply the Green formula:

$$\int\limits_U(h(z)\Delta g(z)-g(z)\Delta h(z))d\mu(z)=\int\limits_{\partial U}\left(h(z)\frac{\partial g(z)}{\partial n}-g(z)\frac{\partial h(z)}{\partial n}\right)ds(z),\quad(5)$$

where $ds(z)$ is an element of the length of the boundary ∂U.

Let us determine the functions h and g.

Let $\eta(z)$ be a non-negative smooth function of the "hat" type (see [5]): $\eta(z)\equiv 0$ when $|z|\geqslant 1$, $\int\limits_{\mathbb{C}}\eta(z)d\mu(z)=1$. We prolong the function $d(z,\partial M)$ by zero on ∂M, and consider the smooth function

$$d_\varepsilon(z)=\frac{1}{\varepsilon^2}\int\limits_{\mathbb{C}}\eta\left(\frac{\theta-z}{\varepsilon}\right)d(\theta,\partial M)d\mu(\theta)$$

for the arbitrary $\varepsilon>0$.

Since the function $d(z,\partial M)$ is convex, in particular, subharmonic, then the family of functions $d_\varepsilon(z,\partial M)$ is also subharmonic and decrease when $\varepsilon\to 0$. They converge to $d(z,\partial M)$ and, moreover, the inequalities $\Delta d_\varepsilon(z)\geqslant 0$ hold (see[6]). Let us assume that $h(z)=d_\varepsilon^{kp+2}(z)w(d_\varepsilon(z))$, and $g(z) = |f^{(k)}(z)|^p$ $(k \in \mathbb{Z}_+), k \leqslant n$. Let R be so large that when $|z| > R$ the inequality $|z|/2 < R < |z|$ holds and that

$$\int\limits_{|z|>R} |f^{(k+1)}(z)|^p d_\varepsilon^{(k+1)p}(z) w(d_\varepsilon(z)) d\mu(z) < \infty \tag{6}$$

according to the condition. Then the formula (5) takes the form:

$$\int\limits_U (d_\varepsilon^{kp+2}(z) w(d_\varepsilon(z)) \Delta |f^{(k)}(z)|^p - |f^{(k)}(z)|^p \Delta (d_\varepsilon^{kp+2}(z) w(d_\varepsilon(z))) d\mu(z) =$$

$$= \int\limits_{\partial U} \left(d_\varepsilon^{kp+2}(z) w(d_\varepsilon(z)) \frac{\partial |f^{(k)}(z)|^p}{\partial n} - |f^{(k)}(z)|^p \frac{\partial (d_\varepsilon^{kp+2}(z) w(d_\varepsilon(z)))}{\partial n} \right) ds(z).$$

On the boundary of the polygon we have M $d_\varepsilon(z) = 0$, $\frac{d_\varepsilon(z)}{\partial n} = 0$. Taking into account (6) and Lemma B, we obtain that the integrals on the boundary of the circle $B(R)$ tend to zero with the increase of R. Therefore the formula is transformed to the form

$$\begin{gathered} \int\limits_{\mathbb{C}\setminus\overline{M}} d_\varepsilon^{kp+2}(z) w(d_\varepsilon(z)) \Delta |f^{(k)}(z)|^p d\mu(z) = \\ = \int\limits_{\mathbb{C}\setminus\overline{M}} |f^{(k)}(z)|^p \Delta (d_\varepsilon^{kp+2}(z) w(d_\varepsilon(z))) d\mu(z). \end{gathered} \tag{7}$$

Let us write the Laplace operator in detail.

$$\Delta (d_\varepsilon^{kp+2}(z) w(d_\varepsilon(z))) = \frac{\partial (d_\varepsilon^{kp+2}(x,y) w(d_\varepsilon(x,y)))}{\partial x^2} + \frac{\partial (d_\varepsilon^{kp+2}(x,y) w(d_\varepsilon(x,y)))}{\partial y^2},$$

where $z = x + iy$. For the sake of brevity we omit parameters in partial derivatives. We obtain the following formula:

$$\begin{gathered} (d^{kp} d^2 w(d))'' = [(kp+2) d^{kp+1} w(d) d' + d^{kp+2} w'(d) d']' = \\ = d^{kp} [(kp+2)(kp+1) w(d) + 2(kp+2) d w'(d) + d^2 w''(d)] \,(d')^2 + \\ + d^{kp+1} [(kp+2) w(d) + d w'(d)] d''. \end{gathered}$$

Let us gather all the partial derivatives

$$\begin{gathered} \Delta (d_\varepsilon^{kp+2}(z) w(d_\varepsilon(z))) = d_\varepsilon^{kp}(z) [(kp+2)(kp+1) w(d_\varepsilon(z)) + \\ + 2(kp+2) d_\varepsilon(z) w'(d_\varepsilon(z)) + d_\varepsilon^2(z) w''(d_\varepsilon(z))] |grad\, d_\varepsilon(z)|^2 + \\ + d_\varepsilon^{kp+1}(z) [(kp+2) w(d_\varepsilon(z)) + d_\varepsilon(z) w'(d_\varepsilon(z))] \Delta d_\varepsilon(z). \end{gathered} \tag{8}$$

Having regrouped the summands, we obtain

$$\begin{gathered} \Delta (d_\varepsilon^{kp+2}(z) w(d_\varepsilon(z))) = d_\varepsilon^{kp}(z) kp(kp-1) w(d_\varepsilon(z)) |grad\, d_\varepsilon(z)|^2 + \\ + d_\varepsilon^{kp}(z) [kp\{2(2w(d_\varepsilon(z)) + d_\varepsilon(z) w'(d_\varepsilon(z)))\}] |grad\, d_\varepsilon(z)|^2 + \\ + d_\varepsilon^{kp}(z) [2w(d_\varepsilon(z)) + 4 d_\varepsilon(z) w'(d_\varepsilon(z)) + d_\varepsilon^2(z) w''(d_\varepsilon(z))] |grad\, d_\varepsilon(z)|^2 + \\ + d_\varepsilon^{kp+1}(z) [kp w(d_\varepsilon(z))] \Delta d_\varepsilon(z) + \\ + d_\varepsilon^{kp+1}(z) [2w(d_\varepsilon(z)) + d_\varepsilon(z) w'(d_\varepsilon(z))] \Delta d_\varepsilon(z). \end{gathered} \tag{9}$$

Since according to the condition $w(t)t^2$ is an increasing convex function, then $2w(t) + tw' \geqslant 0$ and $2w(t) + 4tw' + t^2 w'' \geqslant 0$. If we also take into account that $\Delta d_\varepsilon(z) \geqslant 0$ and that all the lines of the formula (2) are non-negative, we obtain the inequality

$$\Delta(d_\varepsilon^{kp+2}(z)w(d_\varepsilon(z))) \geqslant d_\varepsilon^{kp}(z)kp(kp-1)w(d_\varepsilon(z))|grad\, d_\varepsilon(z)|^2. \tag{10}$$

The estimate (7) takes the form

$$\frac{p^2}{4}\int\limits_{\mathbb{C}\backslash\overline{M}} |f^{(k)}(z)|^{p-2}|f^{(k+1)}(z)|^2 d_\varepsilon^{kp+2}(z)w(d_\varepsilon(z))d\mu(z) \geqslant$$

$$\geqslant kp(kp-1)\int\limits_{\mathbb{C}\backslash\overline{M}} |f^{(k)}(z)|^p d_\varepsilon^{kp}(z)w(d_\varepsilon(z))|grad\, d_\varepsilon(z)|^2 d\mu(z)$$

due to Lemma B. When $k=0$, the third line of the formula (7) provides the estimate

$$\frac{p^2}{4}\int\limits_{\mathbb{C}\backslash\overline{M}} |f^{(k)}(z)|^{p-2}|f^{(k+1)}(z)|^2 d_\varepsilon^{kp+2}(z)w(d_\varepsilon(z))d\mu(z) \geqslant$$

$$\geqslant \alpha\int\limits_{\mathbb{C}\backslash\overline{M}} |f^{(k)}(z)|^p d_\varepsilon^{kp}(z)w(d_\varepsilon(z))|grad\, d_\varepsilon(z)|^2 d\mu(z).$$

Let us tend ε to zero, then due to the continuous differentiability of the function $d(z,\partial M)$, we have

$$\frac{\partial d_\varepsilon(z)}{\partial x} \to \frac{\partial d(z,\partial M)}{\partial x}, \quad \frac{\partial d_\varepsilon(z)}{\partial y} \to \frac{\partial d(z,\partial M)}{\partial y}, z\in\mathbb{C}\backslash\overline{M}.$$

Consequently,

$$\lim_{\varepsilon\to 0}|grad\, d_\varepsilon(z)|^2 = |grad\, d(z,\partial M)|^2$$

and taking into account Lemma A, we obtain the following estimate within the limit:

$$C(k,p,\alpha)\int\limits_{\mathbb{C}\backslash\overline{M}} |f^{(k)}(z)|^{p-2}|f^{(k+1)}(z)|^2 d^{kp+2}(z)w(d(z))d\mu(z) \geqslant$$

$$\geqslant \int\limits_{\mathbb{C}\backslash\overline{M}} |f^{(k)}(z)|^p d^{kp}(z)w(d(z))d\mu(z),$$

where

$$C(k,p,\alpha) = \begin{cases} \dfrac{p^2}{4\alpha}, & \text{for } k=0,\\ \dfrac{p^2}{4kp(kp-1)}, & \text{for } k>0. \end{cases}$$

Let us choose a sequence of convex polygons M_n, such that:

$$M_n\subset B(2R_0),\quad \overline{G}\subset M_n,\quad \overline{M_{n+1}}\subset M_n,\quad \bigcap_{n=1}^{\infty} M_n = \overline{G}.$$

Since the last inequality with one and the same constant holds for each of these polygons, then we obtain within the limit that:

$$\int\limits_D |f^{(k)}(z)|^p d^{kp}(z)w(d(z))d\mu(z) \leqslant$$

$$\leqslant C(k,p,\alpha)\int\limits_D |f^{(k)}(z)|^{p-2}|f^{(k+1)}(z)|^2 d^{kp+2}(z)w(d(z))d\mu(z).$$

Let us apply the Hölder inequality with the indexes $\frac{p}{p-2}$ and $\frac{p}{2}$ to the right-hand side integral.

$$\int\limits_D |f^{(k)}(z)|^{p-2}|f^{(k+1)}(z)|^2 d^{kp+2}(z)w(d(z))d\mu(z) =$$

$$= \int\limits_D |f^{(k)}(z)|^{p-2} d^{k(p-2)}(z) w^{\frac{(p-2)}{p}}(d(z))\ |f^{(k+1)}(z)|^2 d^{2(k+1)}(z) w^{\frac{2}{p}}(d(z))d\mu(z) \leqslant$$

$$\leqslant \left(\int\limits_D |f^{(k)}(z)|^p d^{kp}(z)w(d(z))d\mu(z)\right)^{\frac{p-2}{p}} \times$$

$$\times \left(\int\limits_D |f^{(k+1)}(z)|^p d^{(k+1)p}(z)w(d(z))d\mu(z)\right)^{\frac{2}{p}}.$$

Finally we obtain

$$\int\limits_D |f^{(k)}(z)|^p d^{kp}(z)w(d(z))d\mu(z) \leqslant$$

$$\leqslant C(k,p,\alpha) \left(\int\limits_D |f^{(k)}(z)|^p d^{kp}(z)w(d(z))d\mu(z)\right)^{\frac{p-2}{p}} \times$$

$$\times \left(\int\limits_D |f^{(k+1)}(z)|^p d^{(k+1)p}(z)w(d(z))d\mu(z)\right)^{\frac{2}{p}}.$$

The reduction of the left-hand and the right-hand sides to the factor

$$\left(\int\limits_D |f^{(k)}(z)|^p d^{kp}(z)w(d(z))d\mu(z)\right)^{\frac{p-2}{p}}$$

results in the inequality

$$\left(\int\limits_D |f^{(k)}(z)|^p d^{kp}(z)w(d(z))d\mu(z)\right)^{\frac{2}{p}} \leqslant$$

$$\leqslant C(k,p,\alpha) \left(\int\limits_D |f^{(k+1)}(z)|^p d^{(k+1)p}(z)w(d(z))d\mu(z)\right)^{\frac{2}{p}}$$

or

$$\int\limits_D |f^{(k)}(z)|^p d^{kp}(z)w(d(z))d\mu(z) \leqslant$$

$$\leqslant C^{\frac{p}{2}}(k,p,\alpha) \int\limits_D |f^{(k+1)}(z)|^p d^{(k+1)p}(z)w(d(z))d\mu(z).$$

Considering $k = n-1, \ldots, k = 2, k = 1, k = 0$ sequentially we obtain the inequality

$$\int\limits_D |f(z)|^p w(d(z))d\mu(z) \leqslant$$

$$\leqslant c(n,p,\alpha)\int\limits_D |f^{(n)}(z)|^p d^{np}(z)w(d(z))d\mu(z).$$

Let us now prove the right-hand side inequality of Theorem 1, applying Theorem C. Let $P=\{Q_1,Q_2,\dots,Q_k,\dots\}$ be the specified above set of squares, $G=\bigcup\limits_k Q_k$, then

$$\int\limits_G |f^{(n)}(z)|^p d^{np}(z)w(d(z))d\mu(z)=\sum_k\int\limits_{Q_k}|f^{(n)}(z)|^p d^{np}(z)w(d(z))d\mu(z)\leqslant$$
$$\leqslant\sum_k \max_{z\in Q_k}[|f^{(n)}(z)|^p d^{np}(z)w(d(z))]diam^2(Q_k)\leqslant$$
$$\leqslant c_2^2\sum_k \max_{z\in Q_k}[|f^{(n)}(z)|^p d^{np+2}(z)w(d(z))]\leqslant$$
$$\leqslant c_2^2\sum_k |f^{(n)}(z_k)|^p d^{np+2}(z_k)w(d(z_k)),$$

where $z_k\in Q_k$.

Let us assume now Q_k^* is a square that has a general centre with Q_k and it is increased by $1+\varepsilon$-times, where $0<\varepsilon<0.25$. Let us us the notation

$$B_r(z_k)=\{z:|z-z_k|<r\},\ \text{where } 0<k<dist(Q_k,\partial Q_k^*)/2.$$

Since

$$f^{(n)}(z_k)=\frac{n!}{2\pi i}\int\limits_{\partial B_r(z_k)}\frac{f(z)dz}{(z-z_k)^{n+1}},\text{then}$$

$$\left|f^{(n)}(z_k)\right|=\frac{n!}{2\pi}\frac{1}{r^n}\max_{\partial B_r(z_k)}|f(z)|\leqslant\frac{c_1}{d^n(\tilde z_k,\partial G)}|f(\tilde z_k)|,$$

where $\tilde z_k\in\partial B_r(z_k)$. It is known due to the squares construction that

$$diam(Q_k)\asymp d(z_k,\partial G), diam(Q_k^*)\asymp d(\tilde z_k,\partial G),$$

but

$$diam(Q_k)<diam(Q_k^*)<\frac{5}{4}diam(Q_k),$$

therefore

$$d(\tilde z_k,\partial G)\asymp d(z_k,\partial G).$$

Moreover, from the conditions imposed on the function w we obtain the equivalence

$$A_1 w(d(\tilde z_k,\partial G))\leqslant w(d(z_k,\partial G))\leqslant A_2 w(d(\tilde z_k,\partial G)).$$

Indeed, assume that $A^{-1}t_1\leqslant t_2\leqslant At_1$, then $\exists m\in\mathbb{Z}:2^{m-1}<A\leqslant 2^m$, and $2^{-m}t_2\leqslant t_1\leqslant 2^m t_2$. Since $t^2w(t)$ is increasing, then

$$t_1^2w(t_1)\leqslant 2^{2m}t_2^2w(2^mt_2)\leqslant 2^{2m}t_2^2\beta^m w(t_2)\leqslant 2^{2m}(2^mt_1)^2\beta^m w(t_2)\Rightarrow$$
$$\Rightarrow w(t_1)\leqslant 2^{4m}\beta^m w(t_2).$$

Likewise, taking into account that $t_2\leqslant 2^mt_1\Rightarrow w(t_2)\leqslant 2^{4m}\beta^m w(t_1)$, i.e.

$$2^{-4m}\beta^{-m}w(t_2)\leqslant w(t_1)\leqslant 2^{4m}\beta^m w(t_2).$$

Consequently, if $t_1\asymp t_2\Rightarrow w(t_1)\asymp w(t_2)$.

$$\sum_k |f^{(n)}(z_k)|^p d^{np+2}(z_k) w(d(z_k)) \leqslant c_3 \sum_k |f(\tilde{z}_k)|^p \frac{d^{np+2}(z_k)w(d(z_k))}{d^{np}(\tilde{z}_k)} \leqslant$$
$$\leqslant c_4 \sum_k |f(\tilde{z}_k)|^p d^2(\tilde{z}_k) w(d(z_k)) \leqslant c_5 \sum_k |f(\tilde{z}_k)|^p d^2(\tilde{z}_k) w(d(\tilde{z}_k)).$$

Let us take $r_0 : 0 < r_0 < dist(Q_k, \partial Q_k^*)/2$. It is obvious that $B_{r_0}(\tilde{z}_k) \subset Q_k^*$. Taking into account that $|f(z)|^p$ as subharmonic for all values $p : 0 < p < \infty$, we have

$$|f(\tilde{z}_k)|^p \leqslant \frac{1}{\pi r_0^2} \int\limits_{B_{r_0}(\tilde{z}_k)} |f(z)|^p d\mu(z) \leqslant \frac{c_6}{d^2(\tilde{z}_k, \partial G)} \int\limits_{Q_k^*} |f(z)|^p d\mu(z)$$

Then,

$$|f(\tilde{z}_k)|^p d^2(\tilde{z}_k, \partial G) \leqslant c_7 \int\limits_{Q_k^*} |f(z)|^p d\mu(z),$$

$$|f(\tilde{z}_k)|^p d^2(\tilde{z}_k, \partial G) w(d(\tilde{z}_k, \partial G)) \leqslant c_8 \int\limits_{Q_k^*} |f(z)|^p w(d(\tilde{z}_k, \partial G)) d\mu(z).$$

Since $d(\tilde{z}_k, \partial G)$ and $d(z, \partial G)$ are equivalent in the area of Q_k^*, and hence $w(d(\tilde{z}_k, \partial G))$ and $w(d(z, \partial G))$ are also equivalent we obtain the inequality

$$|f(\tilde{z}_k)|^p d^2(\tilde{z}_k, \partial G) w(d(\tilde{z}_k, \partial G)) \leqslant c_9 \int\limits_{Q_k^*} |f(z)|^p w(d(z, \partial G)) d\mu(z).$$

Taking into account that we can single out a finite subsystem covering the domain G from the system $\{Q_k^*\}$, we have

$$\int\limits_G |f^{(n)}(z)|^p d^{np}(z) w(d(z)) d\mu(z) \leqslant c_{10} \sum_k \int\limits_{Q_k^*} |f(z)|^p w(d(z, \partial G)) d\mu(z) \leqslant$$
$$\leqslant c_{11} \int\limits_G |f(z)|^p w(d(z, \partial G)) d\mu(z).$$

The theorem has been proved.

Remark: The upper estimate in Theorem 1 is obtained for arbitrary domains.

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