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LOCAL AND NONLOCAL CONSERVED VECTORS FOR THE NONLINEAR FILTRATION EQUATION

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Abstract. It is demonstrated that the nonlinear filtration equation is nonlinarly self-adjoint. Using this property, the conserved vectors associated with Lie point and nonlocal symmetries are constructed.

Keywords: nonlinear filtration equation, nonlinear self-adjointness, Lie point and nonlocal symmetries, conserved vectors.

1. Introduction

The present paper is a continuation of the Preprint [1], where we have applied the method of nonlinear self-adjointness [2] and constructed conservation laws

$$D_t\left(C^1\right) + D_x\left(C^2\right) = 0 \tag{1.1}$$

for the nonlinear heat and filtration equations associated with their *Lie point symmetries*.

In this introduction we revise and outline the results of [1] concerning the conservation laws for the nonlinear heat conduction equation

$$u_t = (k(u)u_x)_x\,. \tag{1.2}$$

It is well known that Eq. (1.2) with an arbitrary function $k(u)$ admits the three-dimensional Lie algebra L_3 with the basis

$$X_1 = \frac{\partial}{\partial t}\,, \quad X_2 = \frac{\partial}{\partial x}\,, \quad X_3 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} \tag{1.3}$$

and that this equation has a wider symmetry Lie algebra in the following special cases (see e.g. [3]):

if $k(u) = e^u$, the admitted Lie algebra L_3 extends by the operator

$$X_4 = x\frac{\partial}{\partial x} + 2\frac{\partial}{\partial u}\,; \tag{1.4}$$

if $k(u) = u^\sigma$, where $\sigma \neq 0, -\frac{4}{3}$, the algebra L_3 extends by the operator

$$X_4 = \sigma x\,\frac{\partial}{\partial x} + 2u\,\frac{\partial}{\partial u}\,; \tag{1.5}$$

finally, if $k(u) = u^{-4/3}$, the algebra L_3 extends by two operators

$$X_4 = -2x\,\frac{\partial}{\partial x} + 3u\,\frac{\partial}{\partial u}\,, \quad X_5 = -x^2\frac{\partial}{\partial x} + 3xu\frac{\partial}{\partial u}\,\cdot \tag{1.6}$$

Using the substitution

$$v = Ax + B, \quad A, B = \text{const.}, \tag{1.7}$$

found in the [2] from the equation

$$F^* \mid_{v=\varphi(t,x,u)} = \lambda\left[u_t - k(u)u_{xx} - k'(u)u_x^2\right]$$

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that connects Eq. (1.2) with its adjoint equation

$$F^* \equiv v_t + k(u)v_{xx} = 0,$$

and applying the general procedure from [2] to the Lie point symmetries (1.3)-(1.6), we have found in [1] the following conserved vectors for the nonlinear heat equation.

In the case of the arbitrary function $k(u)$ the symmetries X_2 and X_3 provide two linearly independent conserved vectors:

$$C^1 = u, \quad C^2 = -k(u)u_x \tag{1.8}$$

and

$$C^1 = xu, \quad C^2 = K(u) - xk(u)u_x\,, \tag{1.9}$$

respectively, where

$$K'(u) = k(u).$$

The time-translational symmetry X_1 leads to a trivial conserved vector (the similar result is proved in [4], Section 1.3 for the multi-dimensional case). The conservation law (1.1) for the vector (1.9) coincides with Eq. (1.1), whereas the vector (1.9) satisfies the conservation law (1.1) in the following form:

$$D_t\left(C^1\right) + D_x\left(C^2\right) = x[u_t - (k(u)u_x)_x].$$

In the special case $k(u) = e^u$ the additional symmetry X_4 given by Eq. (1.4) does not lead to a new conservation law. Indeed, one can verify that the conserved vector provided by this symmetry X_4 is equivalent to the conserved vector (1.9) with $k(u) = K(u) = e^u$.

In the special case $k(u) = u^\sigma$ the additional symmetry X_4 given by Eq. (1.5) also does not lead to a new conservation law. Indeed, the calculation shows that the conserved vector provided by this symmetry X_4 is a linear combination of the conserved vectors (1.8) and (1.9) with

$$k(u) = u^\sigma, \quad K(u) = \frac{1}{\sigma+1}\,u^{\sigma+1}.$$

Finally, in the case $k(u) = u^{-4/3}$ the conserved vector provided by the operator X_4 from (1.6) is a linear combination of the corresponding conserved vectors (1.8) and (1.9), whereas the operator X_5 lead again to the conserved vector (1.9).

Thus, the extended symmetries (1.4)-(1.6) do not give new conservation laws.

In the rest of the paper we dwell upon the nonlinear filtration equation

$$u_t = k(u_x)u_{xx} \tag{1.10}$$

and construct the conserved vectors associated not only with its Lie point symmetries, but also with the *nonlocal symmetries* found in [5]. Eq. (1.10) describes, in particular, a distribution of the pressure in a porous medium.

2. Nonlinear self-adjointness of the filtration equation

2.1. The general case. We will write Eq. (1.10) in the form

$$F \equiv -u_t + k(u_x)u_{xx} = 0. \tag{2.1}$$

Its adjoint equation has the form

$$F^* \equiv v_t + k(u_x)v_{xx} + k'(u_x)v_x u_{xx} = 0. \tag{2.2}$$

Let us find a function $\varphi(t,x,u)$ satisfying the nonlinear self-adjointness condition

$$F^*\,|_{v=\varphi(t,x,u)} = \lambda\left[u_t - k(u_x)u_{xx}\right]. \tag{2.3}$$

The expanded form of Eq. (2.3) is

$$\begin{gathered}\varphi_u u_t + \varphi_t + k(u_x)\left[\varphi_u u_{xx} + \varphi_{uu}u_x^2 + 2\varphi_{xu}u_x + \varphi_{xx}\right] + \\ + k'(u_x)\left[\varphi_u u_x + \varphi_x\right]u_{xx} = \lambda\left[u_t - k(u_x)u_{xx}\right].\end{gathered} \tag{2.4}$$

Equating the terms with u_t in both sides of Eq. (2.4) we obtain

$$\lambda = \varphi_u.$$

Taking this into account and equating the terms with u_{xx} in both sides of Eq. (2.4) we arrive at the equation

$$\varphi_u \left[2k(u_x) + u_x k'(u_x)\right] + \varphi_x k'(u_x) = 0. \tag{2.5}$$

Then Eq. (2.4) reduces to the following:

$$\varphi_t + k(u_x) \left[\varphi_{uu} u_x^2 + 2\varphi_{xu} u_x + \varphi_{xx}\right] = 0. \tag{2.6}$$

In the case of an arbitrary function $k(u_x)$ the *determining equations* (2.5)-(2.6) for $\varphi(t, x, u)$ are satisfied only if $\varphi = \text{const}$. We can let

$$\varphi = 1. \tag{2.7}$$

2.2. A special case. We will find now the particular form of $k(u_x)$ when Eqs. (2.5)-(2.6) are satisfied for a non-constant function $\varphi(t, x, u)$. Separating the variables in Eq. (2.5) we have:

$$\frac{2k(u_x)}{k'(u_x)} + u_x = -\frac{\varphi_x}{\varphi_u}.$$

It follows that

$$\frac{2k(u_x)}{k'(u_x)} + u_x = -a, \quad -\frac{\varphi_x}{\varphi_u} = -a, \quad a = \text{const}. \tag{2.8}$$

The first equation (2.8) written in the form

$$\frac{dk}{du_x} = -\frac{2k}{u_x + a}$$

gives

$$k(u_x) = \frac{m}{(u_x + a)^2}, \quad m = \text{const}. \tag{2.9}$$

The solution of the second equation (2.8), i.e. of the partial differential equation

$$a\frac{\partial \varphi}{\partial u} - \frac{\partial \varphi}{\partial x} = 0,$$

has the form

$$\varphi = \phi(t, z), \quad z = u + ax. \tag{2.10}$$

The substitution of (2.9) and (2.10) in Eq. (2.6) yields:

$$\phi_t + m\phi_{zz} = 0. \tag{2.11}$$

We further simplify Eqs. (2.9)-(2.11) by using the equivalence transformation

$$\bar{u} = u + ax \tag{2.12}$$

of Eq. (2.1). Applying this transformation and denoting $\bar{u}$ again by u we conclude that the nonlinear filtration equation

$$u_t = \frac{m}{u_x^2} u_{xx} \tag{2.13}$$

satisfies the nonlinear self-adjointness condition (2.3) with the function

$$\varphi = \phi(t, u), \tag{2.14}$$

where $\phi(t, u)$ is an arbitrary solution of the equation

$$\phi_t + m\,\phi_{uu} = 0. \tag{2.15}$$

3. Construction of conserved vectors

The nonlinear filtration equation (1.10) admits the four-dimensional Lie algebra L_4 with the basis

$$X_1 = \frac{\partial}{\partial t}\,, \quad X_2 = \frac{\partial}{\partial x}\,, \quad X_3 = \frac{\partial}{\partial u}\,, \quad X_4 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + u\frac{\partial}{\partial u}\,\cdot \tag{3.1}$$

The algebra L_4 extends by one additional admitted operator X_5 in the following cases ([3], Sect. 10.3):

if $k(u_x) = e^{u_x}$, then

$$X_5 = t\frac{\partial}{\partial t} - x\frac{\partial}{\partial u}\,;$$

if $k(u_x) = u_x^n$ $(n \geq -1, n \neq 0)$, then

$$X_5 = nt\frac{\partial}{\partial t} - u\frac{\partial}{\partial u}\,;$$

if $k(u_x) = \dfrac{e^{(n \arctan u_x)}}{u_x^2 + 1}$ $(n \geq 0)$, then

$$X_5 = nt\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} - x\frac{\partial}{\partial u}\,\cdot$$

Let us construct the conservation laws

$$D_t\left(C^1\right) + D_x\left(C^2\right) = 0$$

for the operators $X_1, \ldots, X_7$ using the algorithm given in [2]. Namely, writing the formal Lagrangian in the form

$$\mathcal{L} = v\left[u_t - k(u_x)u_{xx}\right] \tag{3.2}$$

we have the following expressions for the components of the conserved vectors:

$$\begin{aligned} C^1 &= W\frac{\partial \mathcal{L}}{\partial u_t} = Wv, \\ C^2 &= W\left[\frac{\partial \mathcal{L}}{\partial u_x} - D_x\left(\frac{\partial \mathcal{L}}{\partial u_{xx}}\right)\right] + D_x(W)\frac{\partial \mathcal{L}}{\partial u_{xx}} = \\ &= Wk(u_x)v_x - D_x(W)k(u_x)v, \end{aligned} \tag{3.3}$$

where we should make the substitution $v = \varphi(t, x, u)$.

In the general case we have $\varphi = 1$ (see Eq. (2.7)). One can verify that X_1, X_2 and X_3 provide only trivial conserved vectors whereas X_4 yields the following conserved vector:

$$C^1 = u, \quad C^2 = -\mathcal{K}(u_x), \tag{3.4}$$

where

$$\mathcal{K}'(u_x) = k(u_x).$$

In the case

$$k(u_x) = e^{u_x}$$

the operator X_5 provides the conserved vector

$$C^1 = -x - te^{u_x}u_{xx}, \qquad C^2 = e^{u_x} + te^{2u_x}(u_{xx}^2 + u_{xxx}).$$

In the case

$$k(u_x) = u_x^n$$

the operator X_5 yields

$$\begin{aligned} &\text{at } \ n > -1, n \neq 0 \qquad C^1 = -u, \quad C^2 = \frac{u_x^{n+1}}{n+1}, \\ &\text{at } \ n = -1 \qquad\qquad\;\; C^1 = -u, \quad C^2 = \ln u_x. \end{aligned}$$

In the case

$$k(u_x) = \frac{e^{n \arctan \mathrm{u_x}}}{u_x^2 + 1}, \quad n \geq 0,$$

the operator X_5 yields the trivial conserved vector

$$C^1 = -x, \quad C^2 = 0.$$

Remark 1. *The conservation law for the conserved vector (3.4) coincides with Equation (1.10). The other conserved vectors obtained in this section can be reduced to the trivial conserved vector.*

4. Conservation laws in the special case

Let us turn to Eq. (2.13). In this case $\varphi(t,x,u)$ given by Eq (2.14). The symmetries of Eq.(2.13) are given by (3.1).
Let us begin with Consider the symmetry X_3. We have $W = 1$, and Eqs. (3.3) give the infinite set of conserved vectors

$$C^1 = \phi, \quad C^2 = \frac{m}{u_x}\,\phi_u \tag{4.1}$$

involving an arbitrary solution $\phi = \phi(t,u)$ of Eq. (2.15). We have:

$$D_t(C^1) + D_x(C^2) = \phi_t + m\phi_{uu} + \left[u_t - \frac{m}{u_x^2}\,u_{xx}\right]\phi_u.$$

Hence, invoking Eq. (2.15), we obtain the conservation equation

$$D_t(C^1) + D_x(C^2) = \left[u_t - \frac{m}{u_x^2}\,u_{xx}\right]\phi_u.$$

Consider the symmetry X_1. Eqs.(3.3) give

$$C^1 = -\phi u_t, \quad C^2 = -\frac{m}{u_x}\phi_u u_t + \frac{m}{u_x^2}\phi u_{tx}.$$

Since

$$-\phi u_t = -\frac{m}{u_x^2}\phi u_{xx} = mD_x\left(\frac{\phi}{u_x}\right) - m\phi_u$$

we can write the above conserved vector in the form

$$C^1 = \phi_u, \quad C^2 = -\frac{1}{u_x}\phi_t. \tag{4.2}$$

This vector satisfies the conservation equation due Eq.(2.15) because

$$D_t(C^1) + D_x(C^2) = (\phi_t + m\phi_{uu})\frac{u_{xx}}{u_x^2} + \left(u_t - \frac{m}{u_x^2}u_{xx}\right)\phi_{uu}.$$

For X_2 we obtain

$$C^1 = -\phi u_x, \quad C^2 = -m\phi_u + \frac{m}{u_x^2}\phi u_{xx}.$$

We have

$$\phi u_x = D_x[\Phi(t,u)],$$

where $\Phi(t,u)$ is defined by the equation

$$\Phi_u = \phi(t,u).$$

Therefore the above conserved vector is equivalent to

$$C^1 = 0, \quad C^2 = -m\phi_u(t,u) - \Phi_t(t,u). \tag{4.3}$$

The conservation equation for this vector is satisfied due to Eq. (2.15). Namely, we have:

$$D_t(C^1) + D_x(C^2) = -(\phi_t + m\phi_{uu})u_x.$$

For X_4 we obtain

$$C^1 = (u - 2tu_t + xu_x)\phi,$$

$$C^2 = \frac{m}{u_x}(u - 2tu_t - xu_x)\phi_u + \frac{m}{u_x^2}(2tu_{tx} + xu_{xx})\phi$$

We have

$$-2t\phi u_t = -2t\phi\frac{m}{u_x^2}u_{xx} = D_x\left(2mt\frac{\phi}{u_x}\right) - 2mt\phi_u,$$

and

$$-x\phi u_x = -D_x(x\Phi) + \Phi,$$

where $\Phi = \Phi(t,u)$ has been defined in the previous case. Therefore the above conserved vector is equivalent to

$$\begin{aligned} C^1 &= u\phi - 2mt\phi_u + \Phi, \\ C^2 &= -x\Phi_t + \frac{2m}{u_x}(\phi + t\phi_t) + \frac{m\phi_u}{u_x}(u - xu_x). \end{aligned} \tag{4.4}$$

The conservation equation for this vector is satisfied in the following form:

$$\begin{aligned} D_t(C^1) + D_x(C^2) &= \left(u - xu_x - \frac{2m}{u_x^2}u_{xx}\right)(\phi_t + m\phi_{uu}) + \\ &+ (2\phi + u\phi_u - 2mt\phi_{uu})\left(u_t - \frac{m}{u_x^2}u_{xx}\right). \end{aligned}$$

5. Nonlocal symmetries and conserved vectors

The nonlinear filtration equation (1.10) has nonlocal symmetries (see [5]) in the case when the function $k(u_x)$ has the form

$$k(u_x) = u_x^{\sigma-1} \tag{5.1}$$

with $\sigma = 1/3$ and $\sigma = -1/3$.

In the case $\sigma = 1/3$ the corresponding equation (1.10) is written

$$u_t = u_x^{-2/3}\, u_{xx}\,. \tag{5.2}$$

It has the nonlocal symmetry

$$X_6 = w\frac{\partial}{\partial x} - u^2\frac{\partial}{\partial u}\,, \tag{5.3}$$

where w is a nonlocal variable defined by the equations

$$w_x = u, \quad w_t = 3\,(w_{xx})^{\frac{1}{3}}\,. \tag{5.4}$$

The application of the general method to the nonlocal symmetry (5.3) gives the conserved vector

$$C^1 = u^2 + wu_x, \quad C^2 = -3uu_x^{1/3} - wu_x^{-2/3}u_{xx}\,. \tag{5.5}$$

The conservation law for the vector (5.5) is satisfied in the following form:

$$\begin{aligned} &D_t(C^1) + D_x(C^2) = \\ &= 2u\left(u_t - u_x^{-2/3}\,u_{xx}\right) + wD_x\left(u_t - u_x^{-2/3}\,u_{xx}\right) + u_x\left(w_t - 3w_{xx}^{1/3}\right). \end{aligned} \tag{5.6}$$

In the case $\sigma = -1/3$ the corresponding equation (1.10) is

$$u_t = u_x^{-4/3}\, u_{xx}\,. \tag{5.7}$$

It has the nonlocal symmetry

$$X_7 = x^2\frac{\partial}{\partial x} + (w - xu\frac{\partial}{\partial u}),$$

where w solves the equations

$$w_x = u, \quad w_t = -3\,(w_{xx})^{-\frac{1}{3}}\,. \tag{5.8}$$

In this case the conserved vector has the form

$$C^1 = w - xu - x^2u_x, \quad C^2 = u_x^{-\frac{4}{3}}(3xu_x + x^2u_{xx})$$

and satisfies the conservation equation

$$D_t(C^1) + D_x(C^2) = w_t + 3w_{xx}^{-\frac{1}{3}} - x(u_t - u_x^{-\frac{4}{3}}u_{xx}) + (u_t - u_x^{-\frac{4}{3}}u_{xx})_x.$$

Remark 2. *The nonlocal conserved vectors obtained in this section can be reduced to the trivial conserved vector.*

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