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NONLINEAR HYPERBOLIC DIFFERENTIAL EQUATIONS RELATED WITH THE KLEIN-GORDON EQUATION BY DIFFERENTIAL SUBSTITUTIONS

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Abstract. We present a complete classification of nonlinear hyperbolic differential equations in two independent variables $u_{xy} = f(u, u_x, u_y)$ reduced to the Klein- Gordon equation $v_{xy} = F(v)$ by the substitutions of the special form $v = \varphi(u, u_x)$.

Keywords:nonlinear hyperbolic equations, differential substitution, Klein-Gordon equation.

1. Introduction

In the present paper we consider nonlinear hyperbolic equations of the form

$$u_{xy} = f(u, u_x, u_y). \tag{1.1}$$

The differential substitutions are widely used for studying the integrability of nonlinear differential equations. Sometimes, by the help of differential substitutions one succeeds to get a solution to an equation from a solution to another well-studied equation. The distinctive feature of the integrability of an equation is the existence of the symmetries. In the paper [1], it was proven that the nonlinear Klein-Gordon equation

$$v_{xy} = F(v) \tag{1.2}$$

possesses generalized symmetries if and only if it is equivalent either to Liouville equation

$$v_{xy} = \exp v, \tag{1.3}$$

or to Sine-Gordon equation

$$v_{xy} = \sin v, \tag{1.4}$$

or to Tzitzeica equation

$$v_{xy} = \exp v + \exp(-2v). \tag{1.5}$$

In the present paper we describe a class of nonlinear hyperbolic equations related to the Klein-Gordon equation by differential substitutions of a special form. In order to formulate the rigorous statements we mention the following. Since by u we denote any solution to equation (1.1), all the mixed derivatives are expressed via

$$u, \quad u_x, \quad u_y, \quad u_{xx}, \quad u_{yy}, \ldots \tag{1.6}$$

due to equation (1.1) and its differential consequences and are excluded from all the expressions. At that variables (1.6) are regarded as independent ones since they can not be expressed one via another by employing equation (1.1) and its differential consequences.

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Definition 1. *The relation*

$$v = \Phi\left(u, \frac{\partial u}{\partial x}, ..., \frac{\partial^n u}{\partial x^n}, \frac{\partial u}{\partial y}, ..., \frac{\partial^m u}{\partial y^m}\right) \tag{1.7}$$

is called a differential substitution from equation (1.1) *into the equation*

$$v_{xy} = g(v, v_x, v_y) \tag{1.8}$$

if for each solution $u(x, y)$ *of equation* (1.1) *function* (1.7) *satisfies equation* (1.8).

Before we proceed to the detailed description of the essence of this work, we mention briefly some publications devoted to the differential substitutions. As it is known ([2–4]), one of the integrability criteria for a nonlinear equations is the breaking on both sides the sequence of Laplace invariants of its linearization. Such equations are usually referred as the equations of Liouville type. In the works [5, 6] there were described the properties of the generalized Laplace invariants for nonlinear equations possessing differential substitutions. One of the most complete surveys devoted to the Liouville type equations is the work [7]. One should also mention the work [8], which was devoted to nonlinear hyperbolic equations possessing symmetries of the third order. We mention here exactly these works also because they contain a rather great number of the examples of the differential substitutions relating pairs of nonlinear equations.

The differential substitutions can be partial cases of Bäcklund transformation (see, for instance, [9]). In the paper [10] they described the pairs of nonlinear equations like (1.1), whose linearization are related by the first and second order Laplace transformation, and for each such pair the corresponding Bäcklund transformation was constructed.

The aim of the present work is to describe all nonlinear hyperbolic equations (1.1) which can be reduced by the differential substitutions

$$v = \varphi(u, u_x) \tag{1.9}$$

to Klein-Gordon equation (1.2). In other words, the problem is to determine the functions f, φ, and F.

The complete list of the required equations and differential substitutions is given in the second section. The third section is devoted to the proof of the main result. The last section is devoted to a in some sense "inverse" problem, which is the description of equations (1.2) being reduced to equation (1.1) by the differential substitutions

$$u = \psi(v, v_y) \tag{1.10}$$

Moreover, for partial pairs of the equations we construct Bäcklund transformations relating their solutions.

2. Classification of the equations being reduced to Klein-Gordon equation

The main result of the work is the following statement.

Theorem 1. *Let equation* (1.1) *be reduced to Klein-Gordon equation* (1.2) *by differential substitution* (1.9). *Then equations* (1.1), (1.2) *and substitution* (1.9) *up to point transformations* $u \to \theta(u)$, $v \to \kappa(v)$, $x \to \xi x$, $y \to \eta y$, *where* ξ *and* η *are constant cast into the form*

$$u_{xy} = uF'\big(F^{-1}(u_x)\big), \quad v_{xy} = F(v), \quad v = F^{-1}(u_x); \tag{2.1}$$

$$u_{xy} = \sin u\sqrt{1-u_x^2}, \quad v_{xy} = \sin v, \quad v = u + \arcsin u_x; \tag{2.2}$$

$$u_{xy} = \exp u\sqrt{1+u_x^2}, \quad v_{xy} = \exp v, \quad v = u + \ln\left(u_x + \sqrt{1+u_x^2}\right); \tag{2.3}$$

$$u_{xy} = \frac{\sqrt{2u_y}}{s'(u_x)}, \quad v_{xy} = F(v), \quad v = s(u_x); \tag{2.4}$$

$$u_{xy} = \frac{c - u_y\varphi_u(u,u_x)}{\varphi_{u_x}(u,u_x)}, \quad v_{xy} = 0, \quad v = \varphi(u,u_x); \tag{2.5}$$

$$u_{xy} = u_x\big(\psi(u,u_y) - u_y\alpha'(u)\big), \quad v_{xy} = \exp v, \quad v = \alpha(u) + \ln u_x; \tag{2.6}$$

$$u_{xy} = u_x\big(\psi(u,u_y) - u_y\alpha'(u)\big), \quad v_{xy} = 0, \quad v = \alpha(u) + \ln u_x; \tag{2.7}$$

$$u_{xy} = u, \quad v_{xy} = v, \quad v = c_1u + c_2u_x; \tag{2.8}$$

$$u_{xy} = \delta(u_y), \quad v_{xy} = 1, \quad v = c_1u + c_2u_x. \tag{2.9}$$

Here c *is an arbitrary constant,* c_1 *and* c_2 *are so that* $(c_1,c_2) \neq (0,0)$, *the function* ψ *satisfies the condition* $(\psi_u, \psi_{u_y}) \neq (0,0)$. *In case* (2.4) *the functions* s *and* F *are connected by the relation* $s'(u_x)F\big(s(u_x)\big) = 1$; *in case* (2.6) *the functions* ψ *and* α *satisfy the identity*

$$\psi_u + \psi\psi_{u_y} - \alpha' u_y\psi_{u_y} = \exp\alpha,$$

and in case (2.7) *to the identity*

$$\psi_u + \psi\psi_{u_y} - \alpha' u_y\psi_{u_y} = 0;$$

in case (2.9) *the function* δ *is a solution to the ordinary differential equation* $\delta(c_1 + c_2\delta') = 1$.

Let us dwell on in detail on some of the obtained equations.

Case (2.1). As $F(v) = \exp v$, we get the equation

$$u_{xy} = uu_x, \tag{2.10}$$

which is reduced by the differential substitution $v = \ln u_x$ to Liouville equation (1.3). The third order symmetries, integrals and the general solution to equation (2.10) can be found, for instance, in [8].

As $F(v) = \sin v$, we obtain the equation

$$u_{xy} = u\sqrt{1-u_x^2}, \tag{2.11}$$

being reduced by the differential substitution $v = \arcsin u_x$ to Sine-Gordon equation (1.4). The symmetries of equation (2.11) are given in [8].

As $F(v) = \exp v + \exp(-2v)$, by point changes we arrive at the equations

$$u_{xy} = 3ub(u_x). \tag{2.12}$$

Here the function b is determined by the condition $(2u_x + b)^2(u_x - b) = 1$. The differential substitution $v = -\frac{1}{2}\ln\big(u_x - b(u_x)\big)$, reducing equation (2.12) to Tzitzeica equation (1.5), is known (see [7]).

Case (2.2). The equation $u_{xy} = \sin u\sqrt{1-u_x^2}$ possesses third order symmetries [8].

Case (2.3). The integrals and general solution to the equation $u_{xy} = \exp u\sqrt{1+u_x^2}$ can be found, for instance, in [8].

Case (2.4). As $F(v) = v$, we obtain a known Gürses equation

$$u_{xy} = 2\sqrt{u_xu_y}, \tag{2.13}$$

being reduced by the substitution $v = \sqrt{2u_x}$ to Helmholtz equation $v_{xy} = v$. Equation (2.13) possesses third order symmetries (see [8]).

As $F(v) = \sin v$, we arrive at the S–integrable equation [8]

$$u_{xy} = \sqrt{2u_y}\sqrt{1-u_x^2}, \tag{2.14}$$

being reduced by the substitution $v = \arccos(-u_x)$ to Sine-Gordon equation (1.4).

If $F(v) = \exp v$, then the equation

$$u_{xy} = u_x\sqrt{2u_y} \tag{2.15}$$

is reduced by the transformation $v = \ln u_x$ to Liouville equation (1.3). The symmetries, integrals, and general solution for (2.15) can be found in [8].

The equation obtained as $F(v) = \exp v + \exp(-2v)$, which after point change can be written as

$$u_{xy} = \sqrt{2u_y}a(u_x), \tag{2.16}$$

is also of interest. Here the function a is determined by the relation $2(a + 2u_x)^2(a - u_x) = 27$. The differential substitution

$$v = -\frac{1}{2}\ln\left(\frac{2a(u_x) - 2u_x}{3}\right)$$

transforms the solution to equation (2.16) to that of Tzitzeica equation (1.5). It should be noticed here that equation (2.16) and the last substitution are given in the work [11]. This substitution allows one to construct generalized symmetries of equation (2.16).

Case (2.5). The equation $u_{xy} = \frac{c - u_y\varphi_u(u,u_x)}{\varphi_{u_x}(u,u_x)}$ as $c = 0$ possesses x-integral $W = \varphi(u, u_x)$; as $c \neq 0$ it does x-integral $W = \varphi_{u_x}u_{xx} + \varphi_u u_x$.

Case (2.6). After the change $v \to v + \ln 2c_2$, $\alpha \to \alpha + \ln 2c_2$ as $\psi = c_1\exp(-u) + c_2\exp(u) + u_y$, $\alpha = u$ we obtain the equation

$$u_{xy} = u_x\big(c_1\exp(-u) + c_2\exp(u)\big), \tag{2.17}$$

which is reduced by the substitution $v = u + \ln u_x$ to the Liouvill equation $v_{xy} = 2c_2\exp v$. The symmetries, integrals, and general solution for (2.17) can be found in [8].

Next, as $\alpha(u) = u$, we arrive at the equation

$$u_{xy} = u_x\frac{\exp(u) - \psi_u(u,u_y)}{\psi_{u_y}(u,u_y)}$$

with y-integral $\bar{W} = \psi(u, u_y) - \exp u$.

In a general case the first equation (2.6) possesses y-integral

$$\bar{W} = \psi_{u_y}u_{yy} + \psi_u u_y - \frac{\psi^2}{2}$$

and x-integral

$$W = \frac{u_{xxx}}{u_x} - \frac{3}{2}\frac{u_{xx}^2}{u_x^2} + \left(\alpha''(u) - \frac{\alpha'^2(u)}{2}\right)u_x^2.$$

Case (2.7). Equation (2.7) possesses the integrals

$$W = \frac{u_{xx}}{u_x} + \alpha'(u)u_x, \quad \bar{W} = \psi(u, u_y).$$

All aforementioned equations possessing the integrals are contained in the list of Liouville type equation given in the survey [7].

3. Proof of the main result

In order to prove Theorem 1 we make the following transformations. We substitute function (1.9) into equation (1.2), taking into consideration formula (1.1),

$$\begin{aligned}\big(\varphi_{uu}u_x+\varphi_{uu_x}u_{xx}\big)u_y+\varphi_u f+\big(\varphi_{u_x u}u_x+\varphi_{u_x u_x}u_{xx}\big)f+\\ +\varphi_{u_x}\big(f_u u_x+f_{u_x}u_{xx}+f_{u_y}f\big)=F(\varphi).\end{aligned} \tag{3.1}$$

Since the variables u, u_x, u_y, and u_{xx} are independent and all the function appearing relation (3.1) are independent of u_{xx}, the last identity is equivalent to the following system

$$\begin{gathered}\varphi_{uu_x}u_y+\varphi_{u_x u_x}f+\varphi_{u_x}f_{u_x}=0,\\ \varphi_{uu}u_x u_y+\varphi_u f+\varphi_{uu_x}u_x f+\varphi_{u_x}f_u u_x+\varphi_{u_x}f_{u_y}f=F(\varphi).\end{gathered} \tag{3.2}$$

Integrating the first equation (3.2) w.r.t. the variable u_x, we arrive at the system

$$\varphi_u u_y+\varphi_{u_x}f=\psi(u,u_y),\quad u_x\psi_u+\big(\varphi_u+\varphi_{u_x}f_{u_y}\big)f=F(\varphi). \tag{3.3}$$

Thus, original problem (1.1), (1.2), (1.9) is reduced to the studying of system (3.3). By the first relation in (3.3) we determine the right hand side of equation (1.1),

$$f=\frac{\psi-u_y\varphi_u}{\varphi_{u_x}}. \tag{3.4}$$

We substitute function (3.4) into the second identity in (3.3),

$$u_x\varphi_{u_x}\psi_u+\psi\psi_{u_y}-u_y\psi_{u_y}\varphi_u=\varphi_{u_x}F(\varphi). \tag{3.5}$$

We apply the operator $\frac{\partial^2}{\partial u_x\partial u_y}$ to both sides of relation (3.5)

$$\psi_{uu_y}\big(\varphi_{u_x}u_x\big)_{u_x}-\varphi_{uu_x}\big(\psi_{u_y}u_y\big)_{u_y}=0. \tag{3.6}$$

Identity (3.6) is valid if one of the following conditions

$$\psi_{uu_y}=0,\quad \varphi_{uu_x}=0, \tag{3.7}$$

$$\psi_{uu_y}=0,\quad \big(\psi_{u_y}u_y\big)_{u_y}=0, \tag{3.8}$$

$$\varphi_{uu_x}=0,\quad \big(\varphi_{u_x}u_x\big)_{u_x}=0, \tag{3.9}$$

$$\psi_{uu_y}\varphi_{uu_x}\neq 0 \tag{3.10}$$

holds true.

We observe that if $\big(\varphi_{u_x}u_x\big)_{u_x}=0$ and $\big(\psi_{u_y}u_y\big)_{u_y}=0$, then

$$\varphi=c_1(u)\ln u_x+c_3(u),\quad \psi=c_2(u)\ln u_y+c_4(u).$$

It is easy to see that this case is partial for (3.8), (3.10).

Let us show now that the requirement (3.10) leads one to condition (3.7). Indeed, in accordance with relation (3.6) we have

$$\frac{\big(\varphi_{u_x}u_x\big)_{u_x}}{\varphi_{uu_x}}=\frac{\big(\psi_{u_y}u_y\big)_{u_y}}{\psi_{uu_y}}. \tag{3.11}$$

Since the variables u_x, u_y are independent, identity (3.11) is equivalent to the system

$$\frac{\big(\varphi_{u_x}u_x\big)_{u_x}}{\varphi_{uu_x}}=\alpha(u),\quad \frac{\big(\psi_{u_y}u_y\big)_{u_y}}{\psi_{uu_y}}=\alpha(u).$$

Let $\alpha\neq 0$, then

$$\big(\varphi_{u_x}u_x\big)_{u_x}=\alpha(u)\varphi_{uu_x},\quad \big(\psi_{u_y}u_y\big)_{u_y}=\alpha(u)\psi_{uu_y}. \tag{3.12}$$

Integrating each equation in system (3.12), we determine the functions φ and ψ,

$$\varphi = \lambda(u) + h\big(\kappa(u)u_x\big), \quad \psi = \mu(u) + H\big(\kappa(u)u_y\big). \tag{3.13}$$

Requirement (3.10) implies $\kappa' \neq 0$. Let us return now to formulas (1.1) and (3.4) which give

$$u_{xy} = \frac{\psi - u_y\varphi_u}{\varphi_{u_x}}$$

or

$$\varphi_{u_x}u_{xy} + u_y\varphi_u = \psi.$$

The last relation means that $\bar{D}(\varphi) = \psi$, where $\bar{D}$ denotes the operator of total differentiation w.r.t. the variable y. Substituting here functions (3.13), we obtain

$$\bar{D}\Big(\lambda(u) + h\big(\kappa(u)u_x\big)\Big) = \mu(u) + H\big(\kappa(u)u_y\big).$$

In the last identity we make the point change

$$\int \kappa(u)du = U,$$

which casts it into the form

$$\bar{D}\big(\chi(U) + h(U_x)\big) = \theta(U) + H(U_y).$$

Introducing the functions $\phi(U, U_x) = \chi(U) + h(U_x)$, $\Psi(U, U_y) = \theta(U) + H(U_y)$, we reduce this case to case (3.7).

If $\alpha = 0$, relations (3.12) give

$$\varphi = h(u)\ln u_x + \epsilon(u), \quad \psi = H(u)\ln u_y + \delta(u). \tag{3.14}$$

We substitute functions (3.14) into identity (3.5),

$$\big(H'\ln u_y + \delta'\big)h + \big(H\ln u_y + \delta\big)\frac{H}{u_y} - \big(h'\ln u_x + \epsilon'\big)H = \frac{h}{u_x}F(h\ln u_x + \epsilon).$$

It follows $H = 0$ that contradicts to the condition $\psi_{uu_y} \neq 0$. Case (3.10) is completed.

Let us proceed to the description of equations (1.1), (1.2) and the differential substitutions relating their solutions in cases (3.7) – (3.9). The following statement holds true.

Lemma 1. *Suppose condition* (3.7). *Then equations* (1.1), (1.2) *and substitution* (1.9) *up to the point transformations $u \to \theta(u)$, $v \to \kappa(v)$, $x \to \xi x$, $y \to \eta y$, where ξ and η are constant, cast into the form*

$$u_{xy} = \frac{c_1 - u_y q'(u)}{s'(u_x)}, \quad v_{xy} = 0, \quad v = q(u) + s(u_x); \tag{3.15}$$

$$u_{xy} = u_x\left(g(u) - u_y\frac{g''(u)}{g'(u)}\right), \quad v_{xy} = \exp v, \quad v = \ln g'(u) + \ln u_x; \tag{3.16}$$

$$u_{xy} = uF'\big(F^{-1}(u_x)\big), \quad v_{xy} = F(v), \quad v = F^{-1}(u_x); \tag{3.17}$$

$$u_{xy} = u, \quad v_{xy} = v, \quad v = c_1u + c_2u_x, ; \tag{3.18}$$

$$u_{xy} = \sin u\sqrt{1 - u_x^2}, \quad v_{xy} = \sin v, \quad v = u + \arcsin u_x; \tag{3.19}$$

$$u_{xy} = u_x\big(c_1\exp(-u) + c_2\exp(u)\big), \quad v_{xy} = 2c_2\exp v, \quad v = u + \ln u_x; \tag{3.20}$$

$$u_{xy} = \exp(u)\sqrt{1 + u_x^2}, \quad v_{xy} = \exp(v), \quad v = u + \ln\left(u_x + \sqrt{1 + u_x^2}\right); \tag{3.21}$$

$$u_{xy} = \frac{\sqrt{2u_y}}{S'(u_x)}, \quad v_{xy} = F(v), \quad v = S(u_x); \tag{3.22}$$

$$u_{xy}=0, \quad v_{xy}=0, \quad v=u+s(u_x); \tag{3.23}$$

$$u_{xy}=\frac{\big(p(u_y)-cu_y\big)c_3}{c_4}, \quad v_{xy}=c_3, \quad v=cu+\frac{c_4}{c_3}u_x. \tag{3.24}$$

Here c_1, c_2 are arbitrary, and c, c_3, c_4 are non-zero constant. The functions S and p satisfy the equations $S'(u_x)F\big(S(u_x)\big)=1$ and $p'(u_y)\big(p(u_y)-cu_y\big)=c_4$, respectively.

Proof. Suppose condition (3.7), then

$$\varphi=q(u)+s(u_x), \quad \psi=g(u)+p(u_y). \tag{3.25}$$

We substitute functions (3.25) into relation (3.5),

$$u_xs'(u_x)g'(u)+\big(g(u)+p(u_y)\big)p'(u_y)-u_yp'(u_y)q'(u)=s'(u_x)F\big(q(u)+s(u_x)\big). \tag{3.26}$$

Due to the independence of u_x and u_y, identity (3.26) is equivalent to the system

$$p'(u_y)\big(u_yq'(u)-g(u)-p(u_y)\big)=\lambda(u), \quad s'(u_x)\big(u_xg'(u)-F(q(u)+s(u_x))\big)=\lambda(u). \tag{3.27}$$

Let us consider the case

$$q''(u)\neq 0. \tag{3.28}$$

It follows from condition (3.28) that $q'(u)\neq 0$. Let

$$\lambda(u)=0, \tag{3.29}$$

then $p(u_y)=c_3$, where c_3 is an arbitrary constant. Moreover, addressing the second identity in (3.27), we have

$$F\big(q(u)+s(u_x)\big)=u_xg'(u). \tag{3.30}$$

We differentiate the last identity w.r.t. the variables u and u_x,

$$F'\big(q(u)+s(u_x)\big)q'(u)=u_xg''(u), \quad F'\big(q(u)+s(u_x)\big)s'(u_x)=g'(u). \tag{3.31}$$

As $F'=0$, employing relations (3.31), we get that $g'(u)=0$ and $F=0$. Then $\psi=c_4$ and $\varphi=q(u)+s(u_x)$, and we arrive to equations (3.15).

If $F'\neq 0$, then it follows from (3.31) that

$$\frac{1}{s'(u_x)u_x}=\frac{g''(u)}{q'(u)g'(u)}=c\neq 0.$$

It yields

$$s(u_x)=\frac{1}{c}\ln(c_1u_x), \quad g'(u)=\exp\big(cq(u)+c_2\big).$$

We substitute the functions s and g' into formula (3.30),

$$\begin{aligned} F\big(q(u)+s(u_x)\big)&=u_x\exp\big(cq(u)+c_2\big)= \\ &=\exp\left(c\left(q(u)+\frac{1}{c}\ln(c_1u)-\frac{1}{c}\ln c_1\right)+c_2\right)= \\ &=\exp\Big(c\big(q(u)+s(u_x)\big)+c_2-\ln c_1\Big). \end{aligned}$$

The last relation means that

$$F(v)=\exp\big(cv+c_2-\ln c_1\big).$$

Hence, we arrive at the equations

$$u_{xy}=cu_x\left(g(u)-\frac{1}{c}u_y\frac{g''(u)}{g'(u)}\right), \quad v=\frac{1}{c}\ln g'(u)-\frac{c_2}{c}+\frac{1}{c}\ln(c_1u_x),$$
$$v_{xy}=\exp\big(cv+c_2-\ln c_1\big).$$

After the change $cv + c_2 - \ln c_1 \to v$ the obtained equations become

$$u_{xy} = u_x \left(cg(u) - u_y \frac{g''(u)}{g'(u)} \right), \quad v = \ln g'(u) + \ln u_x, \quad \frac{1}{c} v_{xy} = \exp v.$$

The change $cg(u) \to g(u)$ transforms the last system to

$$u_{xy} = u_x \left(g(u) - u_y \frac{g''(u)}{g'(u)} \right), \ v = \ln g'(u) - \ln c + \ln u_x, \ v_{xy} = \exp(v + \ln c).$$

And, finally, the shift transformation $v + \ln c \to v$ lead us to equations (3.16).

It is easy to show that the case $\lambda \neq 0$ is not realized.

Consider now the case $q'(u) = c$ that implies

$$q(u) = cu + c_3. \tag{3.32}$$

We substitute function (3.32) into the last relation (3.27),

$$p'(u_y)\big(cu_y - g(u) - p(u_y)\big) = \lambda(u). \tag{3.33}$$

As $g'(u) \neq 0$, identity (3.33) follows

$$p'(u_y) = c_1. \tag{3.34}$$

If $c_1 = 0$, then by (3.33) we get $\lambda(u) = 0$ and $p(u_y) = c_2$. And addressing the second identity in (3.27), we obtain

$$u_x g'(u) = F\big(cu + s(u_x) + c_3\big).$$

The change $s(u_x) + c_3 \to s(u_x)$ gives

$$u_x g'(u) = F\big(cu + s(u_x)\big). \tag{3.35}$$

Let $c = 0$, then functions (3.25) and relation (3.35) cast into the form

$$\psi = g(u) + c_2, \quad \varphi = s(u_x), \quad u_x g'(u) = F\big(s(u_x)\big).$$

The change $g(u) + c_2 \to g(u)$ leads to the formulas

$$\psi = g(u), \quad \varphi = s(u_x), \quad u_x g'(u) = F\big(s(u_x)\big). \tag{3.36}$$

Due to the independence of the variables u, u_x and the restriction $g'(u) \neq 0$, by the latter identity (3.36) we conclude that $g'(u) = c_4 \neq 0$, which yields

$$g(u) = c_4 u + c_5. \tag{3.37}$$

We substitute function (3.37) into the last relation in (3.36),

$$F\big(s(u_x)\big) = c_4 u_x. \tag{3.38}$$

Employing relation (3.38), we determine the function s,

$$s(u_x) = F^{-1}(c_4 u_x).$$

In this way we arrive at the equations

$$u_{xy} = \frac{c_4 u + c_5}{\big(F^{-1}(c_4 u_x)\big)' c_4}, \quad v = F^{-1}(c_4 u_x), \quad v_{xy} = F(v).$$

By the scaling transformation $c_4 u \to u$ and the shift of the variable $u + c_5 \to u$ we reduce the equations to (3.17).

Next we suppose that $c \neq 0$. Differentiate identity (3.35) w.r.t. the variables u and u_x independently,

$$u_x g''(u) = cF'\big(cu + s(u_x)\big), \tag{3.39}$$

$$g'(u) = s'(u_x) F'\big(cu + s(u_x)\big). \tag{3.40}$$

We exclude the function F' from relations (3.39) and (3.40),

$$\frac{g''(u)}{g'(u)} = \frac{c}{u_x s'(u_x)}.$$

Due to the independence of u and u_x, the last identity is equivalent to the system

$$\frac{g''(u)}{g'(u)} = \alpha, \qquad \frac{c}{u_x s'(u_x)} = \alpha, \qquad \alpha \neq 0. \tag{3.41}$$

Integrating equations (3.41) w.r.t. the variables u, u_x, respectively, we obtain

$$g(u) = \frac{1}{\alpha}\exp(\alpha u + \beta) + \delta, \qquad s(u_x) = \frac{c}{\alpha}\ln(\gamma u_x). \tag{3.42}$$

We substitute functions (3.42) into (3.35),

$$F\big(cu + s(u_x)\big) = u_x \exp(\alpha u + \beta) = \exp\left(\frac{\alpha}{c}\left(cu + \frac{c}{\alpha}\ln(\gamma u_x) - \frac{c}{\alpha}\ln\gamma\right) + \beta\right) =$$
$$= \exp\left(\frac{\alpha}{c}\big(cu + s(u_x)\big) - \ln\gamma + \beta\right).$$

The last relation means that

$$F(v) = \exp\left(\frac{\alpha}{c}v - \ln\gamma + \beta\right).$$

Thus, we arrive at the equation

$$u_{xy} = u_x\left(\frac{1}{c}\exp(\alpha u + \beta) + \frac{\alpha}{c}\delta - \alpha u_y\right), \qquad v = cu + \frac{c}{\alpha}\ln(\gamma u_x),$$
$$v_{xy} = \exp\left(\frac{\alpha}{c}v - \ln\gamma + \beta\right).$$

The scaling and shift $\alpha u \to u$, $\alpha v/c \to v$ give

$$u_{xy} = u_x\left(\frac{1}{c}\exp(u + \beta) + \frac{\alpha}{c}\delta - u_y\right),$$
$$v = u + \ln u_x + \ln\gamma - \ln\alpha, \qquad v_{xy} = \frac{\alpha}{c}\exp(v - \ln\gamma + \beta).$$

After the transformations $u + \beta - \ln c \to u$, $v - \ln\gamma + \beta + \ln\alpha - \ln c \to v$ the obtained equations become

$$u_{xy} = u_x\left(\exp u + \frac{\alpha}{c}\delta - u_y\right), \qquad v = u + \ln u_x, \qquad v_{xy} = \exp v.$$

Therefore, we have obtained the case which partial for (3.16).

As $c_1 \neq 0$, addressing to formulas (3.34), we get

$$p(u_y) = c_1 u_y + c_2. \tag{3.43}$$

Substituting function (3.43) into (3.33), after the change $g(u) + c_2 \to g(u)$ we have

$$c_1\big(cu_y - g(u) - c_1 u_y\big) = \lambda(u).$$

Since the varibles u, u_y are independent, by the last identity we conclude that $c = c_1$ and

$$\lambda(u) = -c_1 g(u). \tag{3.44}$$

We substitute function (3.44) into the second relation in (3.27),

$$s'(u_y)\big(u_x g'(u) - F(c_1 u + c_3 + s(u_x))\big) = -c_1 g(u).$$

After the change $c_3 + s(u_x) \to s(u_x)$ the last identity casts into the form

$$F\big(c_1 u + s(u_x)\big) = u_x g'(u) + \frac{c_1 g(u)}{s'(u_x)}. \tag{3.45}$$

We differentiate (3.45) w.r.t. the variables u and u_x independently,

$$c_1 F'\big(c_1 u + s(u_x)\big) = u_x g''(u) + \frac{c_1 g'(u)}{s'(u_x)}, \tag{3.46}$$

$$s'(u_x) F'\big(c_1 u + s(u_x)\big) = g'(u) - c_1 g(u) \frac{s''(u_x)}{s'^2(u_x)}. \tag{3.47}$$

We exclude the function F' from relations (3.46) and (3.47),

$$\frac{g''(u)}{g(u)} = -c_1^2 \frac{s''(u_x)}{u_x s'^3(u_x)}. \tag{3.48}$$

since u, u_x are independent, identity (3.48) is equivalent to the system

$$\frac{g''(u)}{g(u)} = -c_1^2\alpha^2, \quad \frac{c_1^2 s''(u_x)}{u_x s'^3(u_x)} = c_1^2\alpha^2,$$

where α is an arbitrary constant. Or

$$g''(u) + c_1^2\alpha^2 g(u) = 0, \quad \frac{s''(u_x)}{s'^3(u_x)} = \alpha^2 u_x. \tag{3.49}$$

If $\alpha = 0$, then $g(u) = \epsilon u + \delta$, $s(u_x) = \gamma u_x + d$, $\epsilon\gamma \neq 0$. In addition, relation (3.45) yields

$$F\big(c_1 u + s(u_x)\big) = \epsilon u_x + c_1 \frac{\epsilon u + \delta}{\gamma} = \frac{\epsilon}{\gamma}(c_1 u + \gamma u_x + d) - \frac{\epsilon d}{\gamma} + \frac{c_1\delta}{\gamma} =$$
$$= \frac{\epsilon}{\gamma}\big(c_1 u + s(u_x)\big) - \frac{\epsilon d}{\gamma} + \frac{c_1\delta}{\gamma}.$$

The last relation means that

$$F(v) = \frac{\epsilon}{\gamma} v - \frac{\epsilon d}{\gamma} + \frac{c_1\delta}{\gamma}.$$

Thus, we have obtained the equations

$$u_{xy} = \frac{\epsilon u + \delta}{\gamma}, \quad v = c_1 u + \gamma u_x + d, \quad v_{xy} = \frac{\epsilon}{\gamma} v - \frac{\epsilon d}{\gamma} + \frac{c_1\delta}{\gamma}.$$

The transformations $y \to \epsilon y/\gamma$, $u + \frac{\delta}{\epsilon} \to u$, $v - d + c_1\delta/\epsilon \to v$ imply (3.18).

If $\alpha \neq 0$, then by equations (3.49) the functions g and s' are determined as follows,

$$g(u) = A\exp(ic_1\alpha u) + B\exp(-ic_1\alpha u), \tag{3.50}$$

$$s'(u_x) = \frac{1}{\sqrt{\beta - \alpha^2 u_x^2}}. \tag{3.51}$$

Let $\beta \neq 0$. Integrating (3.51) w.r.t. the variable u_x, we determine the function s,

$$s(u_x) = \frac{1}{i\alpha} \ln\left(i\alpha u_x + \sqrt{\beta - \alpha^2 u_x}\right) + \gamma. \tag{3.52}$$

Then relation (3.45) can be written as

$$F\left(c_1 u + s(u_x)\right) = c\exp\big(i\alpha(c_1 u + s(u_x))\big) + D\exp\big(-i\alpha(c_1 u + s(u_x))\big).$$

Thus, we arrive at the formulas

$$u_{xy} = \frac{A\exp(ic_1\alpha u) + B\exp(-ic_1\alpha u)}{s'(u_x)}, \tag{3.53}$$

$$v = c_1 u + s(u_x), \tag{3.54}$$

$$v_{xy} = C\exp(i\alpha v) + D\exp(-i\alpha v), \tag{3.55}$$

where s satisfy (3.51) and $CD = ABc_1^2\beta$. The following cases are possible,

$$CD \neq 0, \tag{3.56}$$

$$C = D = 0, \tag{3.57}$$
$$C = 0, \quad D \neq 0, \tag{3.58}$$
$$D = 0, \quad C \neq 0. \tag{3.59}$$

Suppose identity (3.56) holds true. Then in equations (3.51) — (3.55), having made the change $\alpha u \to u$, $\alpha v \to v$, $\alpha s(u_x) \to s(u_x)$, we arrive at the formulas

$$u_{xy} = \big(A\exp(ic_1u) + B\exp(-ic_1u)\big)\sqrt{\beta - u_x^2},$$
$$v = c_1u + s(u_x), \quad s'(u_x) = \frac{1}{\sqrt{\beta - u_x^2}},$$
$$v_{xy} = C\exp(iv) + D\exp(-iv), \quad CD = ABc_1^2\beta \neq 0.$$

The change $u - b \to u$, $v - a \to v$ transforms the last system to

$$u_{xy} = \big(A\exp(ic_1b)\exp(ic_1u) + B\exp(-ic_1b)\exp(-ic_1u)\big)\sqrt{\beta - u_x^2},$$
$$v + a = c_1(u + b) + s(u_x), \quad s'(u_x) = \frac{1}{\sqrt{\beta - u_x^2}},$$
$$v_{xy} = C\exp(ia)\exp(iv) + D\exp(-ia)\exp(-iv), \quad CD = ABc_1^2\beta \neq 0.$$

We choose a and b so that $A\exp(ic_1b) = B\exp(-ic_1b)$ and $C\exp(ia) = D\exp(-ia)$, then

$$\frac{1}{A\exp(ic_1b)2i}u_{xy} = \sin(c_1u)\sqrt{\beta - u_x^2},$$
$$v = c_1u + s(u_x), \quad s'(u_x) = \frac{1}{\sqrt{\beta - u_x^2}},$$
$$\frac{1}{C\exp(ia)2i}v_{xy} = \sin v, \quad C^2\exp(2ia) = A^2\exp(2ic_1b)c_1^2\beta \neq 0.$$

The scaling of the variable $yA\exp(ic_1b)2i \to y$ leads us to the equations

$$c_1\sqrt{\beta}u_{xy} = \sin(c_1u)\sqrt{\beta - u_x^2}$$
$$v = c_1u + s(u_x), \quad s'(u_x) = \frac{1}{\sqrt{\beta - u_x^2}},$$
$$v_{xy} = \sin v.$$

Next, we make the change $uc_1 \to u$, $s(u_x/c_1) \to s(u_x)$, then

$$\sqrt{\beta}u_{xy} = \sin(u)\sqrt{\beta - \frac{u_x^2}{c_1^2}},$$
$$v = u + s(u_x), \quad s'(u_x) = \frac{1}{\sqrt{\beta c_1^2 - u_x^2}},$$
$$v_{xy} = \sin v.$$

We introduce the notation $c_1\sqrt{b} = a$ and after the scaling of the variables $ax \to x$, $y/a \to y$ we arrive at equations (3.19).

Assume condition (3.57) holds true. We substitute (3.54) into (3.55), bearing in mind (3.53),

$$c_1\big(A\exp(ic_1\alpha u)+B\exp(-ic_1\alpha u)\big)\sqrt{\beta - \alpha^2u_x^2}+\big(A\exp(ic_1\alpha u)ic_1\alpha - Bic_1\alpha\exp(-ic_1\alpha u)\big)u_x = 0,$$

which implies $\beta = 0$, $A = 0$. Equations (3.51) – (3.55) become

$$u_{xy} = Bi\alpha u_x\exp(-ic_1\alpha u), \quad v = c_1u - \frac{i}{\alpha}\ln u_x + \gamma, \quad v_{xy} = 0.$$

By the change $-ic_1\alpha u \to u$ we reduce the last system to

$$u_{xy} = Bi\alpha \exp(u) u_x, \quad v = -\frac{1}{i\alpha} u - \frac{i}{\alpha} \ln u_x + \frac{i}{\alpha} \ln(-ic_1\alpha) + \gamma, \quad v_{xy} = 0.$$

Then the transformations $u + \ln(Bi\alpha) \to u$, $-i\alpha v + \ln(-ic_1\alpha) - \frac{\alpha\gamma}{i} + \ln(Bi\alpha) \to v$ give

$$u_{xy} = \exp(u) u_x, \quad v = u - \ln u_x, \quad v_{xy} = 0. \tag{3.60}$$

Thus, we have obtained the equation which are partial cases of equations (3.20).

Suppose condition (3.58) holds. Substituting (3.54) into (3.55), taking into consideration (3.53), we arrive at the identities

$$Ac_1\sqrt{\beta - \alpha^2 u_x^2} + ic_1\alpha u_x A = 0. \tag{3.61}$$

$$c_1 B\sqrt{\beta - \alpha^2 u_x^2} - Bic_1\alpha u_x = D\exp\bigl(-s(u_x)i\alpha\bigr). \tag{3.62}$$

It follows from (3.61) that $A = 0$. And also $B \neq 0$, since $D \neq 0$. We rewrite relation (3.62), bearing in mind (3.52) as follows,

$$\frac{B}{D} c_1 \bigl(\sqrt{\beta - \alpha^2 u_x^2} - i\alpha u_x\bigr) = \frac{1}{i\alpha u_x + \sqrt{\beta - \alpha^2 u_x^2}}.$$

The last identity holds only under the condition $Bc_1\beta/D = 1$. Thus, employing formula (3.52), system (3.51) — (3.55) can be represented as

$$u_{xy} = B\exp(-ic_1\alpha u)\sqrt{\beta - \alpha^2 u_x^2},$$
$$v = c_1 u - \frac{1}{i\alpha} \ln\bigl(-i\alpha u_x + \sqrt{\beta - \alpha^2 u_x^2}\bigr) + c,$$
$$v_{xy} = D\exp(-i\alpha v), \quad \frac{B}{D} c_1\beta = 1.$$

The scaling of the variable $-i\alpha u \to u$ leads us to the equations

$$u_{xy} = -i\alpha B\exp(c_1 u)\sqrt{\beta + u_x^2},$$
$$v = -\frac{c_1 u}{i\alpha} - \frac{1}{i\alpha} \ln\bigl(u_x + \sqrt{\beta + u_x^2}\bigr) + c,$$
$$v_{xy} = D\exp(-i\alpha v), \quad \frac{B}{D} c_1\beta = 1.$$

After the change $-i\alpha v \to v$ we obtain

$$u_{xy} = -i\alpha B\exp(c_1 u)\sqrt{\beta + u_x^2},$$
$$v = c_1 u + \ln\bigl(u_x + \sqrt{\beta + u_x^2}\bigr) + c,$$
$$v_{xy} = -i\alpha D\exp(v).$$

Or

$$u_{xy} = B\exp(c_1 u)\sqrt{\beta + u_x^2},$$
$$v = c_1 u + \ln\bigl(u_x + \sqrt{\beta + u_x^2}\bigr) + c,$$
$$v_{xy} = D\exp(v).$$

Substituting the function v into the last equation, we obtain that $c_1 B = D\exp(c)$, and the obtained equations cast into the form

$$u_{xy} = B\exp(c_1 u)\sqrt{\beta + u_x^2}, \quad v = c_1 u + \ln\left(u_x + \sqrt{\beta + u_x^2}\right) + c, \quad v_{xy} = c_1 B\exp(v - c).$$

The shift $v - c \to v$ gives

$$u_{xy} = B\exp(c_1 u)\sqrt{\beta + u_x^2}, \quad v = c_1 u + \ln\left(u_x + \sqrt{\beta + u_x^2}\right), \quad v_{xy} = c_1 B\exp(v).$$

The change $c_1u \to u$, the scaling of the variable $ax \to x$ for a so that $\beta c_1^2/a^2 = 1$, the shift $v + \ln c_1 - \ln a \to v$, and, finally, the transformation $u + \ln B \to u$, $v + \ln B \to v$ lead us to equations (3.21).

Let us consider now case (3.59). Equations (3.51)– (3.55) become

$$u_{xy} = \frac{A\exp(ic_1\alpha u) + B\exp(-ic_1\alpha u)}{s'(u_x)},$$
$$v = c_1u + s(u_x), \quad s'(u_x) = \frac{1}{\sqrt{\beta - \alpha^2 u_x^2}},$$
$$v_{xy} = C\exp(i\alpha v).$$

We substitute the function v into the last equation,

$$c_1\big(A\exp(ic_1\alpha u) + B\exp(-ic_1\alpha u)\big)\sqrt{\beta - \alpha^2u_x^2} +$$
$$+ ic_1\alpha\big(A\exp(ic_1\alpha u) - B\exp(-ic_1\alpha u)\big)u_x = C\exp\big(i\alpha(c_1u + s(u_x))\big).$$

It yields

$$c_1A\sqrt{\beta - \alpha^2u_x^2} + ic_1\alpha u_x A = C\exp(i\alpha),$$
$$c_1B\sqrt{\beta - \alpha^2u_x^2} - ic_1\alpha u_x B = 0.$$

Since $\beta \neq 0$, then $B = 0$, and therefore

$$u_{xy} = A\exp(ic_1\alpha u)\sqrt{\beta - \alpha^2u_x^2},$$
$$v = c_1u + \frac{1}{i\alpha}\ln\left(\alpha iu_x + \sqrt{\beta - \alpha^2u_x^2}\right),$$
$$v_{xy} = C\exp(i\alpha v).$$

It is reduced to the previous case.

Let us consider the case $\beta = 0$. By formula (3.51) we get that

$$s(u_x) = \frac{1}{i\alpha}\ln(c_2u_x).$$

We substitute the function s into system (3.53) — (3.55)

$$u_{xy} = \big(A\exp(ic_1\alpha u) + B\exp(-ic_1\alpha u)\big)i\alpha u_x, \tag{3.63}$$
$$v = c_1u + \frac{1}{i\alpha}\ln(c_2u_x), \tag{3.64}$$
$$v_{xy} = C\exp(i\alpha v) + D\exp(-i\alpha v). \tag{3.65}$$

We substitute function (3.64) into (3.65), taking into consideration (3.63),

$$c_1\big(A\exp(ic_1\alpha u) + B\exp(-ic_1\alpha u)\big)i\alpha u_x +$$
$$+ ic_1\alpha u_x\big(A\exp(ic_1\alpha u) - B\exp(-ic_1\alpha u)\big) = Cc_2u_x\exp(i\alpha c_1u) + \frac{D}{c_2u_x}\exp(-i\alpha c_1u).$$

It follows that $D = 0$, $Cc_2 = 2c_1\alpha Ai$, and equations (3.63)–(3.65) can be represented as

$$u_{xy} = \big(A\exp(ic_1\alpha u) + B\exp(-ic_1\alpha u)\big)i\alpha u_x,$$
$$v = c_1u + \frac{1}{i\alpha}\ln(c_2u_x),$$
$$v_{xy} = \frac{2c_1\alpha Ai}{c_2}\exp(i\alpha v).$$

We apply the change of the variables $i\alpha v \to v$, $uc_1i\alpha \to u$ and after the shift $v - \ln(c_2) + \ln(c_1i\alpha) \to v$ we arrive at the equations like (3.20),

$$u_{xy} = \big(A\exp(u) + B\exp(-u)\big)u_x, \quad v = u + \ln u_x, \quad v_{xy} = 2A\exp(v).$$

Suppose now that $g(u) = c_1$, where c_1 is an arbitrary constant. In this case, recalling (3.32), we rewrite (3.27),

$$p'(u_y)\big(cu_y - c_1 - p(u_y)\big) = \lambda(u), \quad -s'(u_x)F\big(cu + s(u_x)\big) = \lambda(u). \tag{3.66}$$

Since the variables u, u_x, u_y are independent, by (3.66) we conclude that $\lambda(u) = -c_2$ and rewriting (3.66),

$$p'(u_y)\big(cu_y - c_1 - p(u_y)\big) = -c_2, \quad -s'(u_x)F\big(cu + s(u_x)\big) = c_2. \tag{3.67}$$

We differentiate the second identity in (3.67) w.r.t. the variable u,

$$s'(u_x)cF'\big(cu + s(u_x)\big) = 0.$$

Hence, $c = 0$ or $F' = 0$.

Let $c = 0$, then the second identity in (3.67) gives

$$s'(u_x)F\big(s(u_x)\big) = c_2.$$

And we arrive at the equations

$$u_{xy} = \frac{c_1 + p(u_y)}{s'(u_x)}, \tag{3.68}$$

$$v = s(u_x), \tag{3.69}$$

$$v_{xy} = F(v). \tag{3.70}$$

We substitute (3.69) into (3.70), bearing in mind (3.68),

$$\frac{p'(u_y)(c_1 + p(u_y))}{s'(u_x)} = \frac{c_2}{s'(u_x)}.$$

Thus, we have

$$\begin{gathered} u_{xy} = \frac{c_1 + p(u_y)}{s'(u_x)}, \quad v = s(u_x), \quad v_{xy} = F(v), \\ s'(u_x)F\big(s(u_x)\big) = c_2, \quad p'(u_y)\big(c_1 + p(u_y)\big) = c_2. \end{gathered} \tag{3.71}$$

The change $p(u_y) + c_1 \to p(u_y)$ leads us to the equation $p'(u_y)p(u_y) = c_2$, whose solution is

$$p(u_y) = \sqrt{2c_2u_y + c_3}.$$

The change $u \to u - c_3y/(2c_2)$ transforms system (3.71) to

$$u_{xy} = \frac{\sqrt{2c_2u_y}}{s'(u_x)}, \quad v = s(u_x), \quad v_{xy} = F(v), \quad s'(u_x)F\big(s(u_x)\big) = c_2.$$

Applying the scaling of the variable $y \to c_2y$, and then, having made the changes $s(u_x) \to c_2s(u_x)$ and $F(c_2S) \to F(S)$, we arrive at equations (3.22).

Let $c \neq 0$, then $F = c_3$, where c_3 is an arbitrary constant. The second relation in (3.67) yields

$$s'(u_x)c_3 = c_2.$$

If here $c_2 = 0$, then $c_3 = 0$ and it follows from the first relation in (3.67) that

$$p'(u_y)\big(cu_y - c_1 - p(u_y)\big) = 0.$$

It implies $p(u_y) = cu_y - c_1$. We get

$$u_{xy} = 0, \quad v = cu + s(u_x), \quad v_{xy} = 0.$$

The scaling of the variable $cu \to u$ and the change $s(u_x/c) \to s(u_x)$ leads us to the equations (3.23). If $c_2 \neq 0$, then $F = c_3 \neq 0$, and we obtain

$$\begin{gathered} u_{xy} = \frac{c_1 + p(u_y) - cu_y}{s'(u_x)}, \quad v = cu + s(u_x), \quad v_{xy} = c_3, \\ s'(u_x) = \tfrac{c_2}{c_3}, \quad p'(u_y)\big(cu_y - c_1 - p(u_y)\big) = -c_2. \end{gathered}$$

Or

$$u_{xy} = \frac{\big(c_1 + p(u_y) - cu_y\big)c_3}{c_2}, \quad v = cu + \frac{c_2}{c_3}u_x + c_4, \quad v_{xy} = c_3,$$
$$p'(u_y)\big(cu_y - c_1 - p(u_y)\big) = -c_2.$$

The shift $p + c_1 \to p$, $v - c_4 \to v$ lead us to equations (3.24). Lemma is proven.

Thus, case (3.7) is studied completely. Let us consider condition (3.8). The following statement holds true.

Lemma 2. *Suppose condition* (3.8) *and* $\varphi_{uu_x} \neq 0$. *Then equations* (1.1), (1.2), (1.9) *become*

$$u_{xy} = \frac{\alpha(u)F'\Big(F^{-1}\big(u_x\alpha'(u)\big)\Big) - \alpha''(u)u_xu_y}{\alpha'(u)}, \quad v_{xy} = F(v), \quad v = \big(F^{-1}(u_x\alpha'(u))\big). \tag{3.72}$$

$$u_{xy} = \frac{c - u_y\varphi_u(u, u_x)}{\varphi_{u_x}(u, u_x)}, \quad v_{xy} = 0, \quad v = \varphi(u, u_x). \tag{3.73}$$

Proof. Assume (3.8). Then it is easy to see that

$$\psi = \alpha(u) + c\ln u_y. \tag{3.74}$$

After the substitution of function (3.74) into relation (3.5), the latter can be represented as

$$u_x\varphi_{u_x}\alpha'(u) + \big(\alpha(u) + c\ln u_y\big)\frac{c}{u_y} - c\varphi_u = \varphi_{u_x}F(\varphi).$$

Since the functions appearing in the obtained identity are independent of the variable u_y, the coefficient at the expression $\ln u_y$ should vanish, i.e., $c = 0$, and therefore,

$$u_x\alpha'(u) = F\big(\varphi(u, u_x)\big).$$

From this relation we determine the function φ, defining the required differential substitution

$$\varphi = F^{-1}\big(u_x\alpha'(u)\big).$$

As $\alpha' \neq 0$, we arrive at equations (3.72). If $\alpha' = 0$, then $F = 0$, and we obtain equations (3.73). The lemma is proven.

And finally, to complete the classification, it is required to study case (3.9). The following statement is valid.

Lemma 3. *Suppose condition* (3.9) *and* $\psi_{uu_y} \neq 0$. *Then by the point change* $v \to \kappa(v)$ *equations* (1.1), (1.2), (1.9) *are reduced to the equations*

$$u_{xy} = u_x\big(\psi(u, u_y) - u_y\alpha'(u)\big), \quad v_{xy} = c_1\exp v, \quad v = \alpha(u) + \ln u_x, \tag{3.75}$$

respectively. Here c_1 *is an arbitrary constant, and the functions* ψ *are* α *are related by the identity* $\psi_u + \psi\psi_{u_y} - \alpha' u_y\psi_{u_y} = c_1\exp\alpha$.

Proof. Suppose condition (3.9) holds, then

$$\varphi = \alpha(u) + c\ln u_x. \tag{3.76}$$

Here $c \neq 0$, since we consider only the substitutions so that $\varphi_{u_x} \neq 0$. We substitute function (3.76) into relation (3.5), then the latter can be written as

$$c\psi_u(u, u_y) + \psi(u, u_y)\psi_{u_y}(u, u_y) - u_y\alpha'(u)\psi_{u_y}(u, u_y) = \frac{c}{u_x}F\big(\alpha(u) + c\ln u_x\big). \tag{3.77}$$

Due to the independence of the variables u_x and u_y, (3.77) is equivalent to the system

$$\begin{gathered} F\big(\alpha(u) + c\ln u_x\big) = \frac{1}{c}u_x\gamma(u), \\ c\psi_u(u, u_y) + \psi(u, u_y)\psi_{u_y}(u, u_y) - \alpha'(u)u_y\psi_{u_y}(u, u_y) = \gamma(u). \end{gathered} \tag{3.78}$$

We apply the operator $\frac{\partial}{\partial u_x}$ to both sides of relation (3.78),

$$F'\big(\alpha(u) + c\ln u_x\big) \cdot \frac{c}{u_x} = \frac{\gamma(u)}{c}. \tag{3.79}$$

Now bearing in mind the first equation in (3.78), we rewrite (3.79) as

$$F'\big(\alpha(u) + c\ln u_x\big) = \frac{1}{c} F\big(\alpha(u) + c\ln u_x\big). \tag{3.80}$$

Employing relation (3.80), we conclude that $cF'(v) - F(v) = 0$. Integrating the last equation, we determine the function F defining the right hand side of Klein-Gordon equation (1.2),

$$F(v) = c_2 \exp(v/c).$$

Substituting the function F into relation (3.78), we obtain that $\gamma(u) = cc_2 \exp\big(\alpha(u)/c\big)$.

Thus, we arrive at the equations

$$u_{xy} = \frac{u_x}{c}\big(\psi(u,u_y) - u_y\alpha'(u)\big), \quad v = \alpha(u) + c\ln u_x, \quad v_{xy} = c_2\exp(v/c),$$
$$c\psi_u + \psi\psi_{u_y} - \alpha' u_y \psi_{u_y} = cc_2 \exp\big(\alpha/c\big).$$

The change $\psi/c \to \psi$, $\alpha/c \to \alpha$, $v/c \to v$, and then $c_2/c \to c_1$ transform the obtained equations to (3.75). The lemma is proven.

The proof of Theorem 1 follows from Lemmas 1–3.

4. Differential substitutions $u = \psi(v, v_y)$

As it was said above, in this section we deal with a problem "inverse" in a sense to that in the first part of the work. Our aim to find all the equations (1.2) being reduced to equation (1.1) by differential substitutions (1.10). The following statement holds true.

Theorem 2. *Assume Klein-Gordon equation* (1.2) *is reduced to equation* (1.1) *by differential substitutions* (1.10). *Then equations* (1.2), (1.1) *and substitution* (1.10) *up to point transformation $v \to \kappa(v)$, $u \to \theta(u)$, $x \to \xi x$, $y \to \eta y$, where ξ and η are constant, are of the form*

$$v_{xy} = F(v), \quad u_{xy} = F'\big(F^{-1}(u_x)\big)u, \quad u = v_y; \tag{4.1}$$

$$v_{xy} = 1, \quad u_{xy} = \frac{\psi''\big(\psi^{-1}(u)\big)u_y}{\psi'\big(\psi^{-1}(u)\big)}, \quad u = \psi(v_y); \tag{4.2}$$

$$v_{xy} = 0, \quad u_{xy} = 0, \quad u = cv + \mu(v_y); \tag{4.3}$$

$$v_{xy} = 0, \quad u_{xy} = -u_x \exp u, \quad u = \ln v_y - \ln v; \tag{4.4}$$

$$v_{xy} = v, \quad u_{xy} = u, \quad u = c_1 v + c_2 v_y; \tag{4.5}$$

$$v_{xy} = 1, \quad u_{xy} = 1, \quad u = v + cv_y. \tag{4.6}$$

Here c is an arbitrary constant, c_1 and c_2 are so that $(c_1, c_2) \neq (0,0)$.

The scheme of the proof. We substitute function (1.10) into relation (1.1), bearing in mind (1.2),

$$\begin{aligned}\big(\psi_{vv}v_y + \psi_{vv_y}v_{yy}\big)v_x + \psi_v F + \big(\psi_{v_yv}v_y + \psi_{v_yv_y}v_{yy}\big)F + \psi_{v_y}F'v_y = \\ = f\big(\psi, \psi_v v_x + \psi_{v_y}F, \psi_v v_y + \psi_{v_y}v_{yy}\big).\end{aligned} \tag{4.7}$$

We denote the first, second, and third argument of the function f by a, b, and c, respectively. We apply the operator $\frac{\partial}{\partial v_{yy}}$ to both sides of identity (4.7),

$$\psi_{vv_y}v_x + \psi_{v_yv_y}F = f_c \psi_{v_y}. \tag{4.8}$$

We apply the operator $\frac{\partial}{\partial v_{yy}}$ to both sides of identity (4.8), $f_{cc}\psi_{v_y}^2 = 0$. If $\psi_{v_y} = 0$, then instead of a differential substitution we obtain a point change $u = \psi(v)$. Thus,

$$f(a,b,c) = \alpha(a,b)c + \beta(a,b). \tag{4.9}$$

We substitute function (4.9) into relation (4.8),

$$\psi_{vv_y}v_x + \psi_{v_yv_y}F = \alpha\big(\psi, \psi_v v_x + \psi_{v_y}F(v)\big)\psi_{v_y}. \tag{4.10}$$

By (4.9) identity (4.7) becomes

$$\begin{gathered}\big(\psi_{vv}v_x + \psi_{vv_y}F\big)v_y + \psi_v F + \psi_{v_y}F'v_y \\ = \alpha\big(\psi, \psi_v v_x + \psi_{v_y}F\big)\psi_v v_y + \beta\big(\psi, \psi_v v_x + \psi_{v_y}F\big).\end{gathered} \tag{4.11}$$

Therefore, problem (1.2), (1.1), (1.10) is reduced to studying relations (4.10), (4.11). We apply the operator $\frac{\partial^2}{\partial v_x^2}$ to identities (4.10), (4.11),

$$\alpha_{bb}\psi_v^2\psi_{v_y} = 0, \quad \alpha_{bb}\psi_v^3 v_y + \beta_{bb}\psi_v^2 = 0. \tag{4.12}$$

Identities (4.12) are satisfied if one of the conditions

$$\psi_v = 0, \tag{4.13}$$

$$\psi_v \neq 0, \quad \alpha_{bb} = 0, \quad \beta_{bb} = 0. \tag{4.14}$$

holds true. The study of conditions (4.13), (4.14) leads us to equations (4.1) – (4.6).

Employing the Theorem 1 and 2 for certain pair of equations, it is possible to construct Bäcklund transformations. For instance, the equations $u_{xy} = -u_x \exp u$, $v_{xy} = 0$ are related by the Bäcklund transformation $v = \ln u_x - u$, $u = \ln(v_y/v)$. Next, the equations

$$u_{xy} = F'\big(F^{-1}(u_x)\big)u, \quad v_{xy} = F(v) \tag{4.15}$$

are related by the Bäcklund transformation

$$v = F^{-1}(u_x), \quad u = v_y.$$

In accordance with the work [10], the linearization of equations (4.15) are related by the first order Laplace transformation. As an example, we adduce the equations

$$u_{xy} = \big(\lambda - \beta n b^{n-1}(u_x)\big)u, \quad v_{xy} = \lambda v - \beta v^n, \quad n > 0, \tag{4.16}$$

where λ and β are arbitrary constants, and the function b satisfies the equation $\lambda b(u_x) - \beta b^n(u_x) = u_x$. The Bäcklund transformation relating the solutions to equations (4.16) reads as

$$u = v_y, \quad v = b(u_x).$$

It should be noticed that the second of equations (4.16) is a version [12] of so-called φ^4 equation in the elementary particles physics. The φ^4 equation and the corresponding Bäcklund transformation are obtained for $n = 3$. This model is important in the solid state physics and in the high energy particles physics [13].

Next, we obtain the equations

$$u_{xy} = \pm\left(\cos b(u_x) + \frac{1}{4}\cos\frac{b(u_x)}{2}\right)u, \quad v_{xy} = \pm\left(\sin v + \frac{1}{2}\sin\frac{v}{2}\right), \tag{4.17}$$

where the function b satisfies the relation $\pm\left(\sin b(u_x) + \frac{1}{2}\sin\frac{b(u_x)}{2}\right) = u_x$. The Bäcklund transformation is given by the formulas $u = v_y$, $v = b(u_x)$. The second of equations (4.17) is the double Sine-Gordon equation, with the plus sign having application in nonlinear optics, and with the minus sign being used in nonlinear optics and for studying B-phase of liquid helium [13].

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BIBLIOGRAPHY

1. Zhiber A.V. Shabat A.B. *The Klein-Gordon equation with nontrivial group* // Dokl. Akad. Nauk SSSR. V. 247, No. 5. 1979. P. 1102–1107. (in Russian)
2. I.M. Anderson, N. Kamran *The variational bicomplex for second order scalar partial differential equations in the plane* // Preprint. Montreal: Centre de Recherches Mathematiques, Universite de Montreal. 1994.
3. I.M. Anderson, N. Kamran *The variational bicomplex for hiperbolic second-order scalar partial differential equations in the plane* // Duke Math. J. V. 87, No. 2. 1997. Pp. 265–319.
4. Zhiber A.V., Sokolov V.V., Startsev S.Ya. *Darboux integrable nonlinear hyperbolic equations* // Dokl. Akad. Nauk, Ross. Akad. Nauk. V. 343, No. 6. 1995. P. 746–748. [Dokl. Math. V. 52, No. 1. 1995. P. 128-130]
5. Startsev S.Ya. *Laplace invariants of hyperbolic equations linearizable by a differential substitution* // Teor. matem. fiz. V. 120, No 2. 1999. P. 237–247. [Theor. math. phys. V. 120, No. 2. 1999. P. 1009-1018]
6. Startsev S.Ya. *Hyperbolic equations admitting differential substitutions* // Teor. matem. fiz. V. 127, No. 1. 2001. P. 63–74. [Theor. math. phys. V. 127, No. 1. 2001. P. 460-470]
7. Zhiber A.V., Sokolov V.V. *Exactly integrable hyperbolic equations of Liouville type* // Uspekhi mat. nauk. V. 56, No. 1. 2001. P. 63–106. [Russ. Math. Surv. V. 56, No. 1. 2001. P. 61-101]
8. Meshkov A.G., Sokolov V.V., *Hyperbolic equations with third-order symmetries* // Teor. matem. fiz. V. 166, No. 1. 2011. P. 51–67 [Theor. math. phys. V. 166, No. 1. 2011. P. 43-57]
9. Khabirov S.V. *Infinite-parameter families of solutions of nonlinear differential equations* // Mat. sb., V. 183, No. 11. 1992. P. 45–54. [Russ. AC SC Sb. Math. , V. 183, No. 11. 1992. P. 303-311]
10. Kuznetsova M.N.. *Laplace transformation and nonlinear hyperbolic equations* // Ufimskii matem. zhurn. V. 1, No. 3. 2009. P. 87–96. (in Russian)
11. Iskandarova M.N. (Kuznetsova M.N.) *Nonlinear hyperbolic equations and Tzitzeica equation* // International school-conference for students, post-graduated students and young scientists. Ufa. BSU. V. 1. 2009. P. 183–193. (in Russian).
12. Soliman A.A., Abdo H.A. *New exact solutions of nonlinear variants of the RLN, the PHI-four and Boussinesq equations based on modified extended direct algebraic method* arXiv: 1207.5127v1 [math.NA]
13. Dodd R.K., Eilbeck J.C., Gibbon J.D., Morris H.C. *Solitons and Nonlinear Wave Equations.* — London, Academic Press, 1982. — 630 pp.

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