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NEW SOLUTIONS OF YANG-BAXTER EQUATION WITH SQUARE

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Abstract. The paper is devoted to Yang-Baxter equation with the square, that is, to the equation

$$R([R(a),b]-[R(b),a])=R^2([a,b])+[R(a),R(b)],$$

where $a,b\in g$, g is a Lie algebra, and R is a linear operator on the vector space g. We construct two series of the operators R satisfying this equation. For constructing we employ the Lie subalgebras in the matrix algebra complement to the subspace of matrices with zero last row.

Keywords: Yang-Baxter equation with square, integrable differential equations, complementary subalgebras in the algebra of Laurent series.

1. Introduction

The main question studied in this paper is Yang-Baxter equation with square

$$R([R(a),b]-[R(b),a])=R^2([a,b])+[R(a),R(b)], \tag{1}$$

where $a,b\in g$, g is a Lie algebra, and R is a linear operator on the vector space g. Equation (1) plays an important role in the theory of integrable systems [1–4]. The main aim of the present paper is to construct new series of the solutions to Yang-Baxter equation with square (1).

In §3 we shall construct two examples of Lie subalgebras in the matrix algebra complement to the subspace of matrices with zero last row. Then in §4, employing subalgebras from §3, we construct two series of solutions to Yang-Baxter equation. The series 2 rests upon the method based on the proposition 3 in the work [1]. This series of the solutions to equation (1) is related to 3-graded Lie algebras. The series 1 is essentially new. The corresponding construction rests upon Theorem 1 in §2.

2. Homogenuous complement subalgebras in the algebra of polynomials over matrices

In the present work equation (1) is studied under the assumption that g is the Lie algebra of the matrices $g=C_m\oplus\ldots\oplus C_m$ being the direct sum of several copies of Lie alegbras C_m. The Lie algebra of matrices g is the direct sum of Lie algebras of $m\times m$ matrices over field C. We introduce the notations,

1) we call a subalgebra g_+ of the algebra g *diagonal* if it consists of the elements $\{(a,a,\ldots,a)|a\in C_m\}$;

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2) we call a subalgebra g_- of the algebra g *complement to* g_+ if the direct sum of the subspaces g_- and g_+ coincides with the Lie algebra g or, in other words, the following two conditions

$$g_+ \oplus g_- = g, \qquad g_+ \cap g_- = \{0\}$$

hold true;

3) we call subalgebra h of the algebra of polynomials $C_m[x]$ *homogenous* if the subalgebra satisfies the condition $xh \subset h$.

We define the operator $R : C_m \to C_m$ by the formula

$$(\alpha_1 p, \alpha_2 p, \ldots, \alpha_m p)_+ = -(R(p), \ldots, R(p)). \tag{2}$$

Here

$$q = (p, p, \ldots, p) \in g_+, \qquad \lambda = (\alpha_1, \ldots, \alpha_m),$$

where α_i are different and by $(\lambda q)_+$ we denote the projection of the element λq on g_+ parallel to g_-.

Theorem 1. *Let g_+ be a diagonal subalgebra of the algebra g, g_- be a homogenous subalgebra complement to g_+. Then the operator R defined by formula (2) satisfies equation* (1) *on g_+.*

Proof. We consider Lie algebra $C_m[x]$ of the polynomials of the form $\sum a_i x^i$, where the coefficients a_i belong to the ring of $m \times m$ complex matrices, x is a scalar variable.

Let φ be a linear operator acting from $C_m[x]$ into the direct sum of k copies of the algebra C_m by the formula

$$\varphi(\sum(a_i x^i) = \sum_i \lambda^i(a_i, \ldots, a_i) = \sum_i (\alpha_1^i a_i, \alpha_2^i a_i, \ldots, \alpha_m^i a_i). \tag{3}$$

It is easy to check that φ is a homomorphism of Lie algebras, i.e., it preserves the commutator. The full image $\varphi^{-1}(g_-) = G_-$ of subalgebra g_- under the homomorphism φ is a subalgebra of the algebra $C_m[x]$. By G_+ we denote the subalgebra in $C_m[x]$ formed by the polynomials independent of x.

Let us prove that the following three conditions similar to the assumptions for Lie algebra g in Theorem 1

$$a)\, xG_- \subseteq G_-; \quad b)\, G_+ + G_- = C_m[x]; \quad c)\, G_+ \cap G_- = \{0\}$$

hold true.

The inclusion $xG_- \subseteq G_-$ is valid since by the assumption of Theorem 1 $\lambda g_- \subseteq g_-$ and $\varphi^{-1}(\lambda g_-) = xG_- \subseteq G_-$.

Let $a = b + c$, $b \in G_+$, $c \in G_-$. Then $\varphi(a) = \varphi(b) + \varphi(c)$, where $\varphi(b) \in g_+$, $\varphi(c) \in g_-$. Thus, $\varphi(G_+ + G_-) = g_+ + g_- = C_m[x]$. Therefore, $G_+ + G_- + Ker\varphi = C_m[x]$. Since $Ker\varphi \subseteq G_-$, then $G_+ + G_- = C_m[x]$. Thus, the condition b) holds true as well.

Next, let a belong to $G_+ \cap G_-$. Then $\varphi(a) \in g_+ \cap g_- = 0$, i.e., $Ker\varphi \in G_-$. Hence, $a \in Ker\varphi \cap G_+ = 0$. Therefore, $\varphi(a) = (a, \ldots, a) = 0$ and the condition c) is satisfied.

In order R to satisfy equation (1), it is sufficient to show that G_- can be represented as

$$G_- = \sum_i x^i(xa_i + R(a_i)). \tag{4}$$

Since by definition (3) of the function φ we have

$$\varphi(xp + R(p)) = \lambda(p, p, \ldots, p) + (R(p), R(p), \ldots, R(p)) \in g_-,$$

$xp + R(p) \in G_-$. Denote $G^- = \sum_i x^i(xa_i + R(a_i))$. The condition $\lambda g_- \subseteq g_-$ follows that $x^i(xa_i + R(a_i)) \subseteq G_-$. We get that $G^- \subseteq G_-$, $G^- + G_+ = C_m[x]$, and since $G_- \cap G_+ = \{0\}$, then $G_- \subseteq G^-$. Hence, $G^- = G_-$, and identity (4) is proven. Let us deduce the equation for the operator R. In order to do it, we consider the commutator

$$[xa + R(a), xb + R(b)] \in G_-.$$

Denote $d = [a, b]$, then $[xa + R(a), xb + R(b)] = x(xd + R(d)) + x(c) + R(c)$,

$$x^2[a, b] + x[a, R(b)] + x[R(a), b] + [R(a), R(b)] = x(xd + R(d)) + x(c) + R(c).$$

Equating the coefficients at like powers of x in the left and right hand sides of the last identity, we obtain the relations

$$[a, R(b)] + [R(a), b] = R(d) + c, \qquad R(c) = [R(a), R(b)],$$

$$c = [a, R(b)] + [R(a), b] - R(d), \qquad R(c) = R([a, R(b)] + [R(a), b] - R(d)).$$

It follows that $R([R(a), b] - [R(b), a]) = R^2([a, b]) + [R(a), R(b)]$. Theorem 1 is proven.

In the work [1] it was shown that the following theorem holds.

Theorem 2. *Let the operator $R : G \to G$ be diagonalizable, $\lambda_1, \dots, \lambda_k$ be its spectrum, and G_i be the associated eigensubspaces. Then R satisfies equation* (1) *if and only if the subspaces G_i and $G_i + G_j$ are Lie sublagebras in G for all different i and j from 1 to k.*

3. Frobenius subspace

Definition 1. We call a subspace in the space of matrices $C_{n\times n}$ a *Frobenius subspace* if all the space of matrices is the direct sum of its subspace and the space of the matrices with zero last row.

For constructing a series of the examples of the operators R satisfying Yang-Baxter equation with square in the work we consider Frobenius subspaces being Lie subalgebras.

Example 1. Consider the block matrices

$$h = \left\{ \begin{pmatrix} \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \lambda_{m_1} \end{pmatrix} & 0 & 0 \\ 0 & \sum \lambda_s D_s & 0 \\ 0 & \mu_1, \dots, \mu_{m_2} & \mu_m \end{pmatrix} | \lambda_s, \mu_s \in C \right\}. \tag{5}$$

These matrices consist of the blocks of size $m_i \times m_j$, where $i = \{1, 2, 3\}$, $j = \{1, 2, 3\}$, the index $s \in \{1, \dots, m_1\}$, m_3=1, $m = m_1 + m_2 + m_3$. The matrices D_s in formula (5) are fixed diagonal matrices of size $m_2 \times m_2$, λ_s, μ_t are arbitrary parameters. At that the parameters λ_s in the block (2,2) are the same as in the block (1, 1).

Let us show that the set H of such matrices h form a Lie algebra. Indeed, for the commutators of block matrices the identities

$$\left[\begin{pmatrix} \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \lambda_{m_1} \end{pmatrix} & 0 & 0 \\ 0 & \sum \lambda_s D_s & 0 \\ 0 & \mu_1, \dots, \mu_{m_2} & \mu_m \end{pmatrix}, \begin{pmatrix} \begin{pmatrix} \lambda'_1 & 0 & \dots & 0 \\ 0 & \lambda'_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \lambda'_{m_1} \end{pmatrix} & 0 & 0 \\ 0 & \sum \lambda'_t D'_t & 0 \\ 0 & \mu'_1, \dots, \mu'_{m_2} & \mu'_m \end{pmatrix} \right] =$$

$$= \begin{pmatrix} \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \lambda_{m_1} \end{pmatrix} & 0 & 0 \\ 0 & \sum \lambda_s D_s & 0 \\ 0 & \mu_1, \dots, \mu_{m_2} & \mu_m \end{pmatrix} \times \begin{pmatrix} \begin{pmatrix} \lambda'_1 & 0 & \dots & 0 \\ 0 & \lambda'_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \lambda'_{m_1} \end{pmatrix} & 0 & 0 \\ 0 & \sum \lambda'_t D'_t & 0 \\ 0 & \mu'_1, \dots, \mu'_{m_2} & \mu'_m \end{pmatrix} -$$

$$-\begin{pmatrix} \begin{pmatrix} \lambda_1' & 0 & \dots & 0 \\ 0 & \lambda_2' & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \lambda_{m_1}' \end{pmatrix} & 0 & 0 \\ 0 & \sum \lambda_t' D_t' & 0 \\ 0 & \mu_1', \dots, \mu_{m_2}' & \mu_m' \end{pmatrix} \times \begin{pmatrix} \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \lambda_{m_1} \end{pmatrix} & 0 & 0 \\ 0 & \sum \lambda_s D_s & 0 \\ 0 & \mu_1, \dots, \mu_{m_2} & \mu_m \end{pmatrix} =$$

$$= \begin{pmatrix} \begin{pmatrix} \lambda_1\lambda_1' & 0 & \dots & 0 \\ 0 & \lambda_2\lambda_2' & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \lambda_{m_1}\lambda_{m_1}' \end{pmatrix} & 0 & 0 \\ 0 & \sum \lambda_s\lambda_t' D_s D_t' & 0 \\ 0 & (\mu_1 \dots \mu_{m_2}) \sum \lambda_t' D_t' & \mu_m \mu_m' \end{pmatrix} -$$

$$- \begin{pmatrix} \begin{pmatrix} \lambda_1'\lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2'\lambda_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \lambda_{m_1}'\lambda_{m_1} \end{pmatrix} & 0 & 0 \\ 0 & \sum \lambda_t' D_t' D_s \lambda_s & 0 \\ 0 & (\mu_1' \dots \mu_{m_2}') \sum \lambda_s D_s & \mu_m' \mu_m \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & (\mu_1'' \dots \mu_{m_2}'') & 0 \end{pmatrix}$$

hold. This is why such commutator is the matric of the form (5) why $\lambda_i = 0$; we obtain that H is a Lie algebra.

Consider the matrix

$$T = \begin{pmatrix} E_{m_1} & 0 & 0 \\ 0 & E_{m_2} & 0 \\ 1 \dots 1 & 0 & 1 \end{pmatrix},$$

where E_{m_i} is the unit $m_i \times m_i$ matrix. It is easy its inverse is given by the formula

$$T^{-1} = \begin{pmatrix} E_{m_1} & 0 & 0 \\ 0 & E_{m_2} & 0 \\ -1 \dots -1 & 0 & 1 \end{pmatrix}.$$

Since H is a Lie sublagebra, the subspace THT^{-1} is also a Lie sublagebra.

Proposition 1. *The subspace THT^{-1} is a Frobenius one (see Definition 1).*

Proof: The relations

$$ThT^{-1} = \begin{pmatrix} E_{m_1} & 0 & 0 \\ 0 & E_{m_2} & 0 \\ 1 \dots 1 & 0 & 1 \end{pmatrix} \times \begin{pmatrix} \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \lambda_{m_1} \end{pmatrix} & 0 & 0 \\ 0 & \sum \lambda_s D_s & 0 \\ 0 & \mu_1 \dots \mu_{m_2} & \mu_m \end{pmatrix} \times$$

$$\times \begin{pmatrix} E_{m_1} & 0 & 0 \\ 0 & E_{m_2} & 0 \\ -1 \dots -1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \lambda_{m_1} \end{pmatrix} & 0 & 0 \\ 0 & \sum \lambda_s D_s & 0 \\ \lambda_1 \dots \lambda_{m_1} & \mu_1 \dots \mu_{m_1} & \mu_m \end{pmatrix} \times$$

$$\times\begin{pmatrix} E_{m_1} & 0 & 0 \\ 0 & E_{m_2} & 0 \\ 1\dots 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \lambda_{m_1} \end{pmatrix} & 0 & 0 \\ 0 & \sum \lambda_s D_s & 0 \\ \lambda_1-\mu_m \dots \lambda_{m_1}-\mu_m & \mu_1\dots\mu_{m_2} & \mu_m \end{pmatrix} \tag{6}$$

hold. We denote by I the space of the matrices with zero leas row, i.e., the space of the matrices

$$I = \begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & 0 \end{pmatrix}.$$

Let $q \in I \cap h$. We need to show that $q = 0$. Relations (6) follow the identities $\mu_n = 0$, $\lambda_j - \mu_n = 0$ $(j = \overline{1,k})$. Since $THT^{-1} \cap I = 0$, the sum of the dimensions of the spaces ThT^{-1} and I equals n^2 because ThT^{-1} contains n parameters and the dimension of I equals $n^2 - n$. The dimension of this sum of spaces $dim(ThT^{-1} + I)$ coincides with the dimension of the space of complex $n \times n$ matrices. Hence, these spaces ThT^{-1} and I are complement subspaces each to the other. Thus, ThT^{-1} is a Frobenius subspace being Lie subalgebra. The lemma is proven.

Example 2. Consider the block matrices

$$h = \{ \begin{pmatrix} \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \lambda_{m_1} \end{pmatrix} & 0 & 0 & 0 \\ 0 & 0 & \sum \lambda_s A_s & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \mu_1\dots\mu_{m_2} & \mu_{m_2+1}\dots\mu_{m_3+m_2} & \mu_m \end{pmatrix} | \tag{7}$$

$\lambda_i, \mu_i \in C\}$. These matrices consist of the blocks of size $m_i \times m_j$ where $i = \{1,2,3,4\}$,$j = \{1,2,3,4\}$, the index $s \in \{1,\dots,m_1\}$, m_4=1, $m = m_1 + m_2 + m_3 + m_4$). The matrices A_i in formula (7) are constant matrices not necessary diagonal, λ_s, μ_t are arbitrary parameters. At that the parameters λ_s in the block (2,3) are the same as in the block (1, 1).

The calculations similar to those done in Example 1 show that

$$[\begin{pmatrix} \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \lambda_{m_1} \end{pmatrix} & 0 & 0 & 0 \\ 0 & 0 & \sum \lambda_s A_s & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \mu_1\dots\mu_{m_2} & \mu_{m_2+1}\dots\mu_{m_3+m_2} & \mu_m \end{pmatrix},$$

$$\begin{pmatrix} \begin{pmatrix} \lambda'_1 & 0 & \dots & 0 \\ 0 & \lambda'_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \lambda'_{m_1} \end{pmatrix} & 0 & 0 & 0 \\ 0 & 0 & \sum \lambda'_t A'_t & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \mu'_1\dots\mu'_{m_2} & \mu'_{m_2+1}\dots\mu'_{m_3+m_2} & \mu'_m \end{pmatrix}] =$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \mu''_1 \dots \mu''_{m_2} & \mu''_{m_2+1} \dots \mu''_{m_2+m_3}) & \mu''_m \end{pmatrix}.$$

The last matrix is the matrix of the form (7) with $\lambda_i = 0$, i.e., the set H of matrices (7) is a Lie algebra.

Next we consider the matrix

$$T = \begin{pmatrix} E_{m_1} & 0 & 0 & 0 \\ 0 & E_{m_2} & 0 & 0 \\ 0 & 0 & E_{m_3} & 0 \\ 1 \dots 1 & 0 & 0 & 1 \end{pmatrix}.$$

Its inverse is given by the formula

$$T^{-1} = \begin{pmatrix} E_{m_1} & 0 & 0 & 0 \\ 0 & E_{m_2} & 0 & 0 \\ 0 & 0 & E_{m_3} & 0 \\ -1 \dots -1 & 0 & 0 & 1 \end{pmatrix}.$$

Let us prove that the Lie subalgebra THT^{-1} is a Frobenius subspace. The identities

$$ThT^{-1} = \begin{pmatrix} E_{m_1} & 0 & 0 & 0 \\ 0 & E_{m_2} & 0 & 0 \\ 0 & 0 & E_{m_3} & 0 \\ 1 \dots 1 & 0 & 0 & 1 \end{pmatrix} \times$$

$$\times \begin{pmatrix} \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \lambda_{m_1} \end{pmatrix} & 0 & 0 & 0 \\ 0 & 0 & \sum \lambda_s A_s & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \mu_1 \dots \mu_{m_2} & \mu_{m_2+1} \dots \mu_{m_3+m_2} & \mu_m \end{pmatrix} \times \begin{pmatrix} E_{m_1} & 0 & 0 & 0 \\ 0 & E_{m_2} & 0 & 0 \\ 0 & 0 & E_{m_3} & 0 \\ -1 \dots -1 & 0 & 0 & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \lambda_{m_1} \end{pmatrix} & 0 & 0 & 0 \\ 0 & 0 & \sum \lambda_s A_s & 0 \\ 0 & 0 & 0 & 0 \\ \lambda_1 \dots \lambda_{m_1} & \mu_1 \dots \mu_{m_2} & \mu_{m_2+1} \dots \mu_{m_3+m_2} & \mu_m \end{pmatrix} \times \begin{pmatrix} E_{m_1} & 0 & 0 & 0 \\ 0 & E_{m_2} & 0 & 0 \\ 0 & 0 & E_{m_3} & 0 \\ -1 \dots -1 & 0 & 0 & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \lambda_{m_1} \end{pmatrix} & 0 & 0 & 0 \\ 0 & 0 & \sum \lambda_s A_s & 0 \\ 0 & 0 & 0 & 0 \\ \lambda_1 - \mu_m \dots \lambda_{m_1} - \mu_m & \mu_1 \dots \mu_{m_2} & \mu_{m_2+1} \dots \mu_{m_3+m_2} & \mu_m \end{pmatrix} \tag{8}$$

hold true.

Denote I the space of the matrices with zero last row,

$$I = \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Let $q \in I \cap h$. Then $q = 0$. Indeed, it follows from (8) that the identities $\mu_n = 0$, $\lambda_j - \mu_n = 0$ ($j = \overline{1,k}$) are valid. Since $THT^{-1} \cap I = 0$, the sum of the dimensions of THT^{-1} and I equals n^2, due to $dim(ThT^{-1}) = n$ and $dimI = n^2 - n$. The dimension of the sum of the spaces $dim(ThT^{-1} + I)$ coincides with the dimension of the space of complex $n \times n$ matrices. Hence, the spaces ThT^{-1} and I are subspaces complement each to the other. Thus, ThT^{-1} is a Frobenius subspace being a Lie subalgebra.

4. Series of equation to Yang-Baxter equation with square

On the basis of the examples in the previous section we construct two series of solutions to Yang-Baxter equation with square (1).

4.1. Series 1. Consider the ring of $m \times m$ matrices C_m over the field of complex numbers. The elements of this ring will be written as block matrices with the blocks formed by the matrices of size $m_i \times m_j$ ($i = \{1,2,3\}, j = \{1,2,3\}$), where the sum $m_1 + m_2 + m_3 = m$.

Let H_1, H_2, H_3 be Lie subalgebras in the algebras of matrices $C_{m_1}, C_{m_2}, C_{m_3}$, respectively, and H_i be Frobenius subspaces in the algebras of matrices (see Definition 1).

Denote by

$$L_1 = \begin{pmatrix} H_1 & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix}, \qquad L_2 = \begin{pmatrix} * & 0 & * \\ * & H_2 & * \\ * & 0 & * \end{pmatrix}, \qquad L_3 = \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ * & * & H_3 \end{pmatrix}$$

the sets of matrices; by stars we indicate arbitrary block matrices of the corresponding sizes. It is clear that L_i are Lie subalgebras in the matrices C_m and $L = L_1 + L_2 + L_3 = C_m$.

Note that

$$L_1 \cap L_2 \cap L_3 = \begin{pmatrix} H_1 & 0 & 0 \\ 0 & H_2 & 0 \\ 0 & 0 & H_3 \end{pmatrix}.$$

Denote by L_4' the space of matrices in G with zero last row,

$$L_4 = T^{-1} L_4' T, \quad T = \begin{pmatrix} E_{m_1} & 0 & 0 \\ 0 & E_{m_2} & 0 \\ \begin{pmatrix} 0\dots 0 \\ 0 \dots 1 \end{pmatrix} & \begin{pmatrix} 0\dots 0 \\ 0 \dots 1 \end{pmatrix} & E_{m_3} \end{pmatrix}.$$

Then L_4 is Lie subalgebra.

Proposition 2. *The intersection of the spaces L_i is zero,*

$$L_1 \cap L_2 \cap L_3 \cap L_4 = \{0\}. \tag{9}$$

Proof. We have

$$T \begin{pmatrix} H_1 & 0 & 0 \\ 0 & H_2 & 0 \\ 0 & 0 & H_3 \end{pmatrix} T^{-1} \cap L_4' = \{0\},$$

$$T^{-1} = \begin{pmatrix} E_{m_1} & 0 & 0 \\ 0 & E_{m_2} & 0 \\ \begin{pmatrix} 0\dots 0 \\ 0\dots -1 \end{pmatrix} & \begin{pmatrix} 0\dots 0 \\ 0\dots -1 \end{pmatrix} & E_{m_3} \end{pmatrix}.$$

For $q_i \in H_i$ the identity

$$\begin{pmatrix} E_{m_1} & 0 & 0 \\ 0 & E_{m_2} & 0 \\ \begin{pmatrix} 0\dots 0 \\ 0\dots 1 \end{pmatrix} & \begin{pmatrix} 0\dots 0 \\ 0\dots 1 \end{pmatrix} & E_{m_3} \end{pmatrix} \begin{pmatrix} q_1 & 0 & 0 \\ 0 & q_2 & 0 \\ 0 & 0 & q_3 \end{pmatrix} =$$

$$= \begin{pmatrix} q_1 & 0 & 0 \\ 0 & q_2 & 0 \\ \begin{pmatrix} 0\dots 0 \\ 0\dots 1 \end{pmatrix} q_1 & \begin{pmatrix} 0\dots 0 \\ 0\dots 1 \end{pmatrix} q_2 & q_3 \end{pmatrix};$$

$$\begin{pmatrix} q_1 & 0 & 0 \\ 0 & q_2 & 0 \\ \begin{pmatrix} 0\dots 0 \\ 0\dots 1 \end{pmatrix} q_1 & \begin{pmatrix} 0\dots 0 \\ 0\dots 1 \end{pmatrix} q_2 & q_3 \end{pmatrix} \cdot \begin{pmatrix} E_{m_1} & 0 & 0 \\ 0 & E_{m_2} & 0 \\ \begin{pmatrix} 0\dots 0 \\ 0\dots -1 \end{pmatrix} & \begin{pmatrix} 0\dots 0 \\ 0\dots -1 \end{pmatrix} & E_{m_3} \end{pmatrix} =$$

$$= \begin{pmatrix} q_1 & 0 & 0 \\ 0 & q_2 & 0 \\ \begin{pmatrix} 0\dots 0 \\ 0\dots 1 \end{pmatrix} q_1 + \begin{pmatrix} 0\dots 0 \\ 0\dots -1 \end{pmatrix} q_3 & \begin{pmatrix} 0\dots 0 \\ 0\dots 1 \end{pmatrix} q_2 + \begin{pmatrix} 0\dots 0 \\ 0\dots -1 \end{pmatrix} q_3 & q_3 \end{pmatrix} \qquad (10)$$

holds. Therefore, if

$$q \in T \begin{pmatrix} H_1 & 0 & 0 \\ 0 & H_2 & 0 \\ 0 & 0 & H_3 \end{pmatrix} T^{-1} \cap L_4',$$

then the last row of the matrix q is zero.

It follows from identity (10) that the last rows from the elements q_1, q_2, q_3 lying in the algebras H_1, H_2, H_3, are zero. Since the subalgebras H_i are Frobenius, then the elements q_i are zero. Hence, the desired intersection is also zero.

In what follows we employ the results of Theorem 1.

Proposition 3. *Let*

$$g = C_m \oplus \ldots \oplus C_m; \qquad g_+ = \{(a, a, \ldots, a) | a \in C_m\}; \qquad g_- = (L_1, L_2, L_3, L_4).$$

Then the operator defined by formula (2), satisfies Yang-Baxter equation with square (1) *on* g_+.

Proof. Let us check that g_- is a homogenous subalgebra complement to g_+. L_i are Lie subalgebras. The validity of the condition $g_+ \cap g_- = \{0\}$ is implied by the fact that according to Proposition 2, $L_1 \cap L_2 \cap L_3 \cap L_4 = \{0\}$.

If $(a, a, \ldots, a) \in (L_1, L_2, L_3, L_4)$, then $a \in L_1 \cap L_2 \cap L_3 \cap L_4 = \{0\}$. This is the homogeneity condition $xh \subset h$ ($h \in C_m[x]$) for g_- is followed by $\alpha_i L_i \subseteq L_i$ (L_i is a subspace). It remains to check the condition $g_+ \oplus g_- = g$. It is sufficient to show that the dimensions of the space $g_- + g_+$ and g coincide. The identities $dimg = 4m^2$, $dimg_+ = m^2$,

$$dimg_- = dimL_1 + dimL_2 + dimL_3 + dimL_4$$
$$(m_2 + m_3)m + m_1 = dimL_1,$$
$$dimL_2 = (m_1 + m_3)m + m_2,$$

$$dimL_3 = (m_1 + m_2)m + m_3,$$
$$dimL_4 = m^2 - m,$$
$$dimg_- = m(m_2+m_3+m_1+m_3+m_1+m_2)+m_1+m_2+m_3+m^2-m = |m_1+m_2+m_3 = m| =$$
$$= 2m^2 + m + m^2 - m = 3m^2$$

hold. The identity
$$dimg_+ + dimg_- = dim(g_+ + g_-)$$
is valid since the intersection $g_+ \cap g_- = \{0\}$. This is why
$$dimg = dim(g_+ + g_-) = 4m^2.$$

Thus, the condition $g_+ \oplus g_- = g$ holds. By Theorem 1 the operator $R(q)$ defined by formula (2) satisfies Yang-Baxter equation with square (1).

Remark 1. Series 1 follows from Propositions 2 and 3 in the case if H_1, H_2, H_3 are block matrices of the form (5) and (7), respectively.

Remakr 2. All aforementioned in Series 1 remains true if the number of blocks is k, and Lie subalgebra $H_1, \ldots, H_k$ lying in the algebras of matrices $C_{m_1}, \ldots, C_{m_k}$ are Frobenius subspaces in these algebras of matrices.

4.2. Series 2. The work [1] contains the following propositions.

Proposition 4. *Let G be an arbitrary 3-graded Lie algebra, $p_1$be Lie subalgebra in g_0 and e be an element in g_1 such that $dimp_1 = dimg_1$ and $[p_1, e] = g_1$. Then $p_2 = exp(ad_e)(p_1 \oplus g_{-1})$ is a complement subalgebra to g_0.*

Proposition 5. *Suppose $R : G \longrightarrow G$ is diagonalizable, $\lambda_1, \ldots, \lambda_k$ is its spectrum, and G_i are the associated eigensubspaces. Then R satisfies Yang-Baxter equation with square (1) if and only if the subspaces G_i and $G_i + G_j$ are Lie subalgebras G for all different i and j from 1 to k.*

We shall also make use of the following remark made in the work [1].

Remark 3. Proposition 4 allows one to construct k-parametric family of the solutions $R = \sum_{i=1}^{k} \lambda_i \prod_i$ (where $\prod_i$ is the projector on G_i) to equation (1) if one knows the expansion of the Lie algebra G in a direct sum of the subspaces G_i such that G_i and $G_i + G_j$ are Lie subalgebras in G. The numbers λ_i serving as the parameters can be chosen arbitrarily.

Let us construct the series of the solutions to Yang-Baxter equation with square for certain 3-graded Lie algebras. Let G be the algebra of $(2m + n) \times (2m + n)$ matrices over the field of complex number. We shall write the elements of G as block matrices. The block are formed by the matrices of size $m_i \times m_j$ $(i = \{1, 2, 3\}, j = \{1, 2, 3\}$, $m_1 = n$, $m_2 = m_3 = m$.

We indicate by G_0, G_1, G_{-1} the following subspaces determining grade
$$g_0 = \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & * \end{pmatrix} \in G_0, \qquad g_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ * & * & 0 \end{pmatrix} \in G_1, \qquad g_{-1} = \begin{pmatrix} 0 & 0 & * \\ 0 & 0 & * \\ 0 & 0 & 0 \end{pmatrix} \in G_{-1}.$$
It is easy to check that $G = G_0 \oplus G_1 \oplus G_{-1}$ is 3-graded Lie algebra.

Denote by P_1 the subalgebra in G_0 formed by the matrices
$$P_1 = \begin{pmatrix} H_1 & 0 & 0 \\ I_1 & I_2 & 0 \\ 0 & 0 & H_2 \end{pmatrix},$$
where H_1, H_2 are Lie subalgebras in the algebras of matrices C_n and C_m being Frobenius subspaces (the examples in § 3). I_1, I_2 consist of block matrices having zero last row. It is clear that P_1 is a Lie subalgebra.

We note that $dimp_1 = dimH_1 + dimH_2 + dimI_1 + dimI_2 = n + m + (nm - n) + (m^2 - m) =$ $= n + m + nm - n + m^2 - m = m^2 + mn = m(m + n) = dimg_1$.

Define the element e in G_1 by the formula

$$e = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \begin{pmatrix} 0\dots0 \\ 0\dots1 \end{pmatrix} & E_m & 0 \end{pmatrix}.$$

Let us check the validity of the condition $[P_1, e] = G_1$ from Proposition 1. As $q_1 \in H_1, q_2 \in H_2$, $i_1 \in I_1, i_2 \in I_2$ the identities

$$[P_1, e] = \begin{pmatrix} q_1 & 0 & 0 \\ i_1 & i_2 & 0 \\ 0 & 0 & q_2 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \begin{pmatrix} 0\dots0 \\ 0\dots1 \end{pmatrix} & E_m & 0 \end{pmatrix} -$$

$$- \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \begin{pmatrix} 0\dots0 \\ 0\dots1 \end{pmatrix} & E_m & 0 \end{pmatrix} \begin{pmatrix} q_1 & 0 & 0 \\ i_1 & i_2 & 0 \\ & 0 & q_2 \end{pmatrix} =$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ q_2 \begin{pmatrix} 0\dots0 \\ 0\dots1 \end{pmatrix} & q_2 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \begin{pmatrix} 0\dots0 \\ 0\dots1 \end{pmatrix} q_1 + i_1 & i_2 & 0 \end{pmatrix} =$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ q_2 \begin{pmatrix} 0\dots0 \\ 0\dots1 \end{pmatrix} - \begin{pmatrix} 0\dots0 \\ 0\dots1 \end{pmatrix} q_1 - i_1 & q_2 - i_2 & 0 \end{pmatrix} \tag{11}$$

hold true. In order to check the condition $[P_1, e] = G_1$, we need to show that there are arbitrary elements on the positions of the block (3,1) and (3,2) of matrix (11). The subalgebras H_2 and I_2 are complement each to the other in the space of the $m \times m$ matrices, and moreover the subspaces $\begin{pmatrix} 0\dots0 \\ 0\dots1 \end{pmatrix} H_1$ and I_1 are complement each to the other in the space of the matrices of size $m \times n$. On the positions of the block (3,1) and (3,2) of matrix (11) there are arbitrary elements, since $q_2 - i_2$ is an arbitrary element of size $m \times m$ and $\begin{pmatrix} 0\dots0 \\ 0\dots1 \end{pmatrix} q_1 + i_1$ — is an arbitrary element of size $m \times n$, $q_1 \in H_1, q_2 \in H_2, i_1 \in I_1, i_2 \in I_2$.

We let

$$P_2 = exp(ad_e)(P_1 \oplus G_{-1})$$

and

$$G^1 = G_0, \qquad G^2 = P_2 \cap (G_0 \oplus G_1), \qquad G^3 = P_2 \cap (G_0 \oplus G_{-1}).$$

It is easy to see that P_i are Lie subalgebras in G and

$$G^1 + G^2 = G_0 + G_1, \qquad G^1 + G^3 = G_0 + G_{-1}, \qquad G^2 + G^3 = P_2$$

are also Lie subalgebras. According to the remark to Proposition 5 in [1], we have obtained the operators satisfying Yang-Baxter equation with square.

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