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ON SOLUTIONS OF THE FIRST-ORDER PDE WITH A MULTIDIMENSIONAL SYMMETRIC INTEGRAL AND THEIR MODELLING

F.S. NASYROV, E.V. YUREVA

Abstract. The deterministic analog of the multidimensional Stratonovich integral is constructed. Method of solution of a system of equations with a multidimensional symmetric integral is elaborated. The method of characteristics for solving the Cauchy problem for the first-order partial differential equations with a multidimensional symmetric integral is developed. This method reduces solving the initial-value problem of the above equations to solving a system of equations with a multidimensional symmetric integral.

Keywords: multidimensional symmetric integral, differential equations system with multidimensional symmetric integral, partial differential equations with multidimensional symmetric integral, the method of characteristics

1. Introduction

Let $(\Omega, \mathscr{F}, (F_t), P)$ is a probabilistic space with filtration (F_t), where a standard d-dimensional Wiener process is given $\overline{W}(t,\omega) = (W_1(t,\omega), \dots, W_d(t,\omega))$. Further, as a rule, the variable $\omega \in \Omega$ is omitted.

Let $\overline{\eta}(t,\bar{x}) = (\eta_1(t,\bar{x}), \dots, \eta_n(t,\bar{x}))$ be a diffusion process which is the solution of Ito's system of equations:

$$\begin{cases} \eta_i(t,\bar{x}) = x_i + \displaystyle\int_0^t \Bigg[B^i(s,\overline{\eta}(s,\bar{x})) + \\ \qquad + \dfrac{1}{2} \displaystyle\sum_{k=1}^{n} \sum_{j=1}^{d} \sigma^{kj}(s,\overline{\eta}(s,\bar{x})) \left(\sigma^{ij}\right)'_{x_k}(s,\overline{\eta}(s,\bar{x})) \Bigg] ds + \\ \qquad + \displaystyle\sum_{j=1}^{d} \int_0^t \sigma^{ij}(s,\overline{\eta}(s,\bar{x})) dW_j(s), \qquad i = 1,2,\dots,n, \end{cases} \tag{1}$$

where the latest d integrals in every equation of the system (1) are stochastic Ito's integrals by multi-dimensional Wiener process $\overline{W}(t)$, and the variable $\bar{x} \in R^n$ points out the dependence of the process $\overline{\eta}(t,\bar{x})$ on the initial conditions $\eta_i(0,\bar{x}) = x_i$, $i = 1,2,\dots,n$.

It is known, that there is a close relationship (see, for instance, [6, 12, 14]) between ordinary stochastic differential Ito's equations and differential Ito's equations in partial derivatives. Let the function $u = u(t,\bar{x}) : [0,T] \times R^n \to R$ with every t belong to the class of functions $C^m(R^n)$, continuous by totality of variables and having continuous (relative to $\bar{x}$) variables by $\bar{x}$ up to some order m. The function which is with every t reciprocal by $\bar{x}$ to the diffusion process $\overline{\eta}(t,\bar{x})$ is denoted by $\overline{\eta}^{\,-1}(t,\bar{x})$. Let us fix the value $K > 0$ and the whole $m \geq 3$.

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Theorem ([6, 12])**.** *Let with every $\bar{x}$ coefficients $B^i(s,\bar{x})$, $\sigma^{ij}(s,\bar{x})$, $i=1,2,\dots,n$, $j=1,2,\dots,d$ are measurable by (t,ω) and are compatible with the family of σ-algebras $\{\mathscr{F}_t\}$, $t\in[0,T]$, the functions themselves $B^i(s,\bar{x})$, $\sigma^{ij}(s,\bar{x})$, $i=1,2,\dots,n$, $j=1,2,\dots,d$ and their derivatives by $\bar{x}$ up to the order m by the absolute value do not accede K. Then with all ω from some subset of the probability 1 and every $t\in[0,T]$ the mapping $\overline{\eta}(t,\cdot):\bar{x}\in R^n\to\overline{\eta}(t,\bar{x})\in R^n$ is a diffeomorphism of class $C^{m-1}(R^n)$, moreover every coordinate of the inverse mapping $\overline{\eta}^{-1}(t,\bar{x})$, $i=1,2,\dots,n$ is the solution of the corresponding Cauchy problem*

$$d_tu(t,\bar{x})=-\sum_{i=1}^{n}\left[\frac{1}{2}\sum_{k=1}^{n}\sum_{j=1}^{d}\sigma^{ij}(t,\bar{x})\sigma^{kj}(t,\bar{x})u''_{x_ix_k}(t,\bar{x})+\right.$$
$$\left.+\left(\frac{1}{2}\sum_{k=1}^{n}\sum_{j=1}^{d}\sigma^{kj}(t,\bar{x})\left(\sigma^{ij}\right)'_{x_k}(t,\bar{x})+B^i(t,\bar{x})\right)u'_{x_i}(t,\bar{x})\right]dt- \qquad (2)$$
$$-\sum_{i=1}^{n}\sum_{j=1}^{d}\sigma^{ij}(t,\bar{x})u'_{x_i}(t,\bar{x})dW_j(t)$$

with the initial condition $u(0,\bar{x})=x_i$, where i is the number of the coordinate of the vector $\bar{x}\in R^n$.

Here and below the symbol $d_tu(t,\bar{x})$ denotes the differential by the variable t unlike the total differential $du(t,\bar{x})$.

Let us note, that stochastic differential Ito's equation (2) is an equation of the second order of the parabolic type, but if we write this equation with the integral of Stratonovich we obtain a first-order equation in partial derivatives.

Let us provide all the necessary symbols and definitions which are used in the paper. Let $X(s)$, $s\in[0,T]$ be an arbitrary continuous function, $f(s,v)$, $s\in[0,T]$, $v\in R$ be a determined function, measurable by s and v. Let us consider the segmentations T_n, $n\in N$, of the section $[0,T]$: $T_n=\{t_k^{(n)}\}$, $0=t_0^{(n)}\le t_1^{(n)}\le\ldots\le t_k^{(n)}\le\ldots\le t_{m_n}^{(n)}=T$, $n\in N$, such that $T_n\subset T_{n+1}$, $n\in N$ and $\lambda_n=\max\limits_k\left|t_k^{(n)}-t_{k-1}^{(n)}\right|\to 0$ with $n\to\infty$. Let us denote the broken line constructed by the function $X(s)$ and satisfying the segmentation T_n by $X^{(n)}(s), s\in[0,t]$. Assume $\Delta t_k^{(n)}=t_k^{(n)}-t_{k-1}^{(n)}$, $\left[\Delta t_k^{(n)}\right]=\left[t_{k-1}^{(n)},t_k^{(n)}\right]$, $\Delta X_k^{(n)}=X(t_k^{(n)})-X(t_{k-1}^{(n)})$.
A symmetrical integral by a continuous function $X(s)$ is

$$\int_0^t f(s,X(s))*dX(s)\overset{def}{=}\lim_{n\to\infty}\sum_k\frac{1}{\Delta t_k^{(n)}}\int_{[\Delta t_k^{(n)}]}f(s,X^{(n)}(s))ds\,\Delta X_k^{(n)},$$

if the limit in the first side of the equality does exist and does not depend on the choice of the succession of segmentations $T_n, n\in N$. A sufficient condition for the existence of a symmetrical integral is a so called condition (S) (see [7, 8]). In the case when $X(t)$ is a trajectory of the standard Wiener process, a symmetrical integral with the probability 1 coincides with the stochastic integral of Stratonovich.

In papers [7, 8] there were studied determined analogues of stochastic differential equations with symmetrical integral, and there was obtained the method of their solution by means of reducing to the solution of finite chains of ordinary differential equations. In her paper Zaharova O.V. [5] obtained a method of solution of a definite class of systems of stochastic differential equations with symmetrical integrals by means of reducing solutions of the latter to the solution of systems of equations in total differentials. In the paper [15] there are considered mathematical models, containing a transformation from ordinary Ito's equations to equations in partial derivatives, and in papers [3, 4] there was found a relationship between solutions of determined analogues of stochastic differential equations with one-dimensional symmetrical integral and equations in partial derivatives with one-dimensional symmetrical integral. in the monograph [9]there are considered the general notions of the theory of symmetrical integrals.

The present paper continues study of this approach. Firstly, there are constructed multi-dimensional symmetrical integrals by arbitrary continuous functions, which generalise stochastic integrals of Stratonovich by multi-dimensional Wiener process. Secondly, the method of solution of equations with one-dimensional symmetrical integral and corresponding stochastic differential equations, found in papers [7, 8], was developed for the solution of systems of equations with multi-dimensional symmetrical integrals. Finally, there was constructed the method of characteristics for the solution of differential equations in partial derivatives with multi-dimensional symmetrical integrals, which, in particular, generalises the presented above result of Krylov N.V., Rozovsky B. L. (see [6, 12]), therewith: (a) instead of multi-dimensional Wiener process $\overline{W}(t)$ we take an arbitrary continuous vector-function $\overline{X}(t)$, all the components of which possess an unbounded variation on any section; (b) the function $\overline{\eta}(t,\bar{x})$ where the solution of the system of ordinary differential equations with multi-dimensional symmetrical integrals is not already a diffusion process, and multi-dimensional symmetrical integrals themselves are generalisation of the stochastic integral of Stratonovich (see [7, 8]).

Therefore, it is shown in the paper, that some results which were earlier valid within the frames of stochastic analysis, possess a more general character and can be formulated for some classes of equations with symmetrical integrals.

2. General results

2.1. Let $\overline{X}(s)=(X_1(s),\ldots,X_d(s))$, $s\in[0,T]$, be an arbitrary continuous vector-function and there are also functions $\sigma^1(s,\overline{X}(s)),...,\sigma^d(s,\overline{X}(s))$. Let us consider segmentations T_n, $n\in N$ of the section $[0,T]$: $T_n=\{t_k^{(n)}\}$, $0=t_0^{(n)}\le t_1^{(n)}\le ...\le t_k^{(n)}\le ...\le t_{m_n}^{(n)}=T$, $n\in N$, such that $T_n\subset T_{n+1}$, $n\in N$ and $\lambda_n=\max_k\left|t_k^{(n)}-t_{k-1}^{(n)}\right|\to 0$ when $n\to\infty$. Let us denote broken lines constructed by the functions $X_k(s)$ according to the sequence of concentrating segmentations T_n by $X_k^{(n)}(s)$, $s\in[0,T]$, $k=1,2,\ldots,d$.

A symmetrical integral by the function $\overline{\sigma}(s,\overline{X}(s))=(\sigma^1(s,\overline{X}(s)),\ldots,\ \sigma^d(s,\overline{X}(s)))$ relative to the continuous function $\overline{X}(s)$ is

$$\int_0^t\overline{\sigma}(s,\overline{X}(s))*d\overline{X}(s)=\lim_{n\to\infty}\sum_{k=1}^d\int_0^t\sigma^k(s,X_1^{(n)}(s),\ldots,X_d^{(n)}(s))\left(X_k^{(n)}\right)'(s)ds,\tag{3}$$

if the limit in the right side does exist and does not depend on the choice of succession of segmentations $T_n, n\in N$.

Alongside with the denotation (3) we apply the following:

$$\int_0^t\overline{\sigma}(s,\overline{X}(s))*d\overline{X}(s)\equiv\sum_{k=1}^d\int_0^t\sigma^k(s,X_1(s),\ldots,X_d(s))*dX_k(s).\tag{4}$$

Let $f(s,\bar{v})=f(s,v_1,\ldots,v_d)$ be a continuous differentiated function, then its *differential with symmetrical integrals* is named

$$f(t,\overline{X}(t))-f(0,\overline{X}(0))=\int_0^t grad_{\bar{v}}f(s,\overline{X}(s))*d\overline{X}(s)+\int_0^t\frac{\partial}{\partial s}f(s,\overline{X}(s))ds,\tag{5}$$

where $grad_{\bar{v}}f(s,\bar{v})=(f'_{v_1},\ldots,f'_{v_d})$.

Remark 1. The formula (5) is a determined analogous of the stochastic differential Ito's equation with the integral of Stratonovich. Alongside with the notation of the differential in the form (5) we apply an abridged notation of the differential

$$df(t,\overline{X}(t))=grad_{\bar{v}}f(t,\overline{X}(t))*d\overline{X}(t)+\frac{\partial}{\partial t}f(t,\overline{X}(t))dt.$$

Let us show, that for the continuously differentiated function $f(s,\bar{v})$ the differential with symmetrical integrals does exist.

Lemma 1. *Let $\overline{X}(s) = (X_1(s),\ldots,X_d(s))$, $s \in [0,T]$ be an arbitrary continuous vector-function, and the function $f(s,\bar{v})$, $s \in [0,T]$, $\bar{v} \in R^d$ possess continuous partial derivatives of the first order for all their variables. Therefore with any $t \in [0,T]$ there is the symmetrical integral $\int_0^t grad_{\bar{v}} f(s,\overline{X}(s)) * d\overline{X}(s)$ and the formula* (5) *is valid.*

Proof. Let $X_k^{(n)}(s)$, $s \in [0,T]$, $k = 1,2,\ldots,d$ be broken lines, constructed by the functions $X_k(s)$, $k = 1,2,\ldots,d$ by sequence of concentrating segmentations T_n, $n \in N$ of the section $[0,T]$. Let us consider the expression

$$\begin{gathered} f(t,X_1^{(n)}(t),\ldots,X_d^{(n)}(t)) - f(0,X_1^{(n)}(0),\ldots,X_d^{(n)}(0)) = \\ = \sum_{k=1}^{d} \int_0^t \frac{\partial}{\partial v_k} f(s,X_1^{(n)}(s),\ldots,X_d^{(n)}(s)) \left(X_k^{(n)}\right)'(s)ds + \\ + \int_0^t \frac{\partial}{\partial s} f(s,X_1^{(n)}(s),\ldots,X_d^{(n)}(s))ds, \end{gathered} \tag{6}$$

obtained by means of differentiation and integration of the function $f(s,X_1^{(n)}(s),\ldots,X_d^{(n)}(s))$ by the variable $s \in [0,t]$. Let us note, that on the strength of the continuity of the function $f(s,v_1,\ldots,v_d)$ the limit of the left side of the expression (6) where $n \to \infty$ does exist and is equivalent to

$$\begin{gathered} \lim_{n\to\infty} [f(t,X_1^{(n)}(t),\ldots,X_d^{(n)}(t)) - f(0,X_1^{(n)}(0),\ldots,X_d^{(n)}(0))] = \\ = f(t,X_1(t),\ldots,X_d(t)) - f(0,X_1(0),\ldots,X_d(0)). \end{gathered}$$

Likewise, due to the continuity of the partial derivative of the function $f(s,v_1,\ldots,v_d)$ by s there is a limit of the latter summand in the right side of the equality (6):

$$\lim_{n\to\infty} \int_0^t \frac{\partial}{\partial s} f(s,X_1^{(n)}(s),\ldots,X_d^{(n)}(s))ds = \int_0^t \frac{\partial}{\partial s} f(s,X_1(s),\ldots,X_d(s))ds,$$

whence it appears, that there exists the limit (3).

Remark 2. Lemma 1 does not guarantee the existence of the limit of every summand in the expression (4), but the notation in the right side of the expression (4) corresponds to the accepted in the stochastic analysis system of symbols.

Remark 3. If $\overline{X}(s)$, $s \in [0,T]$ is a multi-dimensional Wiener process, then every summand in the formula (3) from the definition of the symmetrical integral by the function $\overline{X}(s)$ is valid and with the probability to 1 coincides with the corresponding stochastic integral of Stratonovich.

2.2. Let us introduce determined analogous of systems of stochastic differential equations in the Stratonovich form by multi-dimensional continuous functions.

Let us consider Cauchy problem for the system of differential equations with multi-dimensional symmetrical integrals:

$$\begin{cases} \eta_i(t) = \eta_i^0 + \displaystyle\int_0^t B^i(s,\overline{\eta}(s),\overline{X}(s))ds + \sum_{j=1}^{d} \int_0^t \sigma^{ij}(s,\overline{\eta}(s),\overline{X}(s)) * dX_j(s), \\ \qquad i = 1,2,\ldots,n, \end{cases} \tag{7}$$

where $\overline{X}(s) = (X_1(s),\ldots,X_d(s))$ is a continuous vector-function.

The solution of the system of equations (7) is a set of functions of the form $\eta_i(t) = \varphi_i(t,\overline{X}(t))$, $t \in [0,T]$, $i = 1,2,\ldots,n$, such that:

1. the functions $\varphi_i(t,\bar{v})$ possess continuous partial derivatives for all their arguments;

2. the right sides of the system (7) in the process of substitution of the functions $\varphi_i(t,\overline{X}(t))$, $i=1,2,\dots,n$ form differentials with symmetrical integrals of some functions $\psi_i(t,\overline{X}(t))$;
3. differentials with symmetrical integrals $d\varphi_i(t,\overline{X}(t))$ and $d\psi_i(t,\overline{X}(t))$, $t\in[0,T]$, $i=1,2,\dots,n$ of the right and the left sides of the system (7) coincide.

Let us denote $\overline{X}_{[k]}(t,v_k)=(X_1(t),\dots,X_{k-1}(t),v_k,X_{k+1}(t),\dots,X_d(t))$, where the index $[k]$ points out, that instead of k coordinate $X_k(t)$ of the vector $\overline{X}(t)$ there is the variable v_k. Let us show, that the solution of the system of equations with multi-dimensional symmetrical integrals is reduced to the solution of finite chains of systems of ordinary differential equations (further ODE).

Theorem 1. *Let the vector-function $\overline{X}(t)=(X_1(t),\dots,X_d(t))$ be fixed and its elements are continuous functions, and the functions $\sigma^{ik}(t,\overline{\eta},\bar{v})$, $k=1,2,\dots,d$, $i=1,2,\dots,n$ and $B^i(t,\overline{\eta},\bar{v})$, $i=1,2,\dots,n$ are continuously differentiated. Assume, that continuously differentiated by all their arguments functions $\overline{\varphi}(t,\bar{v})$, $t\in[0,T]$, $\bar{v}\in R^d$ satisfy the finite chain of the ODE:*

$$\left\{\ (\varphi_i)'_{v_1}(t,\overline{X}_{[1]}(t,v_1))=\sigma^{i1}(t,\overline{\varphi}(t,\overline{X}_{[1]}(t,v_1)),\overline{X}_{[1]}(t,v_1)),\quad i=1,2,\dots,n,\right.\tag{8}$$

$$\dots$$

$$\left\{\ (\varphi_i)'_{v_k}(t,\overline{X}_{[k]}(t,v_k))=\sigma^{ik}(t,\overline{\varphi}(t,\overline{X}_{[k]}(t,v_k)),\overline{X}_{[k]}(t,v_k)),\quad i=1,2,\dots,n,\right.\tag{9}$$

$$\dots$$

$$\left\{\ (\varphi_i)'_{v_d}(t,\overline{X}_{[d]}(t,v_d))=\sigma^{id}(t,\overline{\varphi}(t,\overline{X}_{[d]}(t,v_d)),\overline{X}_{[d]}(t,v_d)),\quad i=1,2,\dots,n,\right.\tag{10}$$

$$\begin{cases} (\varphi_i)'_t(t,\bar{v})\big|_{\{v_j=X_j(t),j=1,2,\dots,d\}}=B^i(t,\overline{\varphi}(t,\overline{X}(t)),\overline{X}(t)),\\ \qquad\qquad\qquad\qquad \varphi_i(0,\overline{X}(0))=\eta_i^0,\quad i=1,2,\dots,n.\end{cases}\tag{11}$$

Then the function $\overline{\varphi}(t,\overline{X}(t))$, $t\in[0,T]$, $\overline{X}(t)\in R^d$ is the solution of Cauchy problem (7).

Proof. The fact, that the solution $\overline{\varphi}(t,\overline{X}(t))$ of the chain of the systems of the ODE (8)–(11) provides us the solution of the initial system of equations (7) is verified by substitution of the function $\overline{\varphi}(t,\overline{X}(t))$ into the system (7) and by applying the formula of differential with symmetrical integrals (5).

Remark 4. Let us show how by means of the chain of the system of the ODE (8)–(11) we can construct the solution of the system of equations (7). Herewith we suppose, that every from the considered below systems of ODE, constructed with the help of the chain (8)–(11), possesses some general solution.

Let ${}^r\overline{X}(t)=(X_r(t),X_{r+1}(t),\dots,X_d(t))$ be a vector-function, constructed from $\overline{X}(t)=(X_1(t),X_2(t),\dots,X_d(t))$ by omitting the first $r-1$ coordinates, $r=1,2,\dots,d$.

Solving the system of the ODE (8) relative to the variable v_1 and considering other variables as parameters we obtain the functions $\varphi_i(t,\overline{X}(t))$ in the form

$$\varphi_i(t,\overline{X}(t))=\varphi_i^{*1}(t,X_1(t),\overline{C}^1(t,{}^2\overline{X}(t)))\ ,\ i=1,2,\dots,n,\tag{12}$$

depending on the arbitrary vector-function $\overline{C}^1(t,{}^2\overline{X}(t))=(C_1^1(t,{}^2\overline{X}(t))\,,\dots,C_n^1(t,{}^2\overline{X}(t)))$. This vector-function in its turn occurs during substitution of the functions φ_i^{*1}, $i=1,2,\dots,n$ into the following system of ODE on the variable v_2 with the precision to the unknown vector-function $\overline{C}^2(t,{}^3\overline{X}(t))=(C_1^2(t,{}^3\overline{X}(t)),\dots,C_n^2(t,{}^3\overline{X}(t)))$. Proceeding this process at the $k-$step ($k<d$) we obtain the solution in the form

$$\varphi_i(t,\overline{X}(t))=\varphi_i^{*k}(t,X_1(t),\dots,X_k(t),\overline{C}^k(t,{}^{k+1}\overline{X}(t))),\tag{13}$$

$i=1,2,\dots,n$ with the precision to the arbitrary vector-function

$$\overline{C}^k(t,{}^{k+1}\overline{X}(t))=(C_1^k(t,{}^{k+1}\overline{X}(t)),\dots,C_n^k(t,{}^{k+1}\overline{X}(t))),$$

which in its turn occurs during substitution of the obtained at this step functions φ_i^{*k} into $(k+1)$ system of ODE. After we solve the first d systems of the given chain we obtain:

$$\varphi_i(t, \overline{X}(t)) = \varphi_i^{*d}(t, \overline{X}(t), \overline{C}^d(t)), \quad i = 1, 2, \dots, n,$$

where the unknown vector-function $\overline{C}^d(t) = (C_1^d(t), \dots, C_n^d(t))$ can be found by means of substitution of the obtained φ_i^{*d} into the system (11) with the initial conditions

$$\varphi_i(0, \overline{X}(0)) = \eta_i^0, \qquad i = 1, 2, \dots, n.$$

In the process of solving systems of ODE of the above chain in a succession different from the one presented here, it is possible to obtain a solution on other forms. In case when the system of equations with symmetrical integrals possesses an only solution, then all the solutions constructed should coincide. The method of solution of the systems of differential equations with a multi-dimensional symmetrical integrals remains valid even for the solving systems of stochastic differential equations with multi-dimensional Wiener process.

Remark 5. A sufficient condition of compatibility of the system of equations (7) in the suppositions of Theorem 1 is compatibility of every system from the from the system of ODE (8)–(11).

Let us assume, that continuous functions $X_1(s), \dots, X_d(s)$, $s \in [t_1, t_2]$ possessing unbounded variation on any finite interval, *are locally independent on the section* $[t_1, t_2]$, if there exists a continuously differentiated by all its variables function $\Phi(s, \bar{v}) = \Phi(s, v_1, ..., v_d)$, such that $grad_{\bar{v}}\, \Phi(s, \bar{v}) \neq 0$ for $\bar{v}$ from the "rectangle" $[\overline{X}(s_1), \overline{X}(s_2)]$ and $\Phi(s, \overline{X}(s)) \equiv 0$ on some section $[s_1, s_2] \subset [t_1, t_2]$, otherwise the functions $X_1(s), \dots, X_d(s)$ *are functionally independent of the section* $[t_1, t_2]$.

Remark 6. Let the continuous functions $X_1(s), \dots, X_d(s)$, $s \in [0, T]$ possess unbounded variation at any finite interval and be functionally independent on $[0, T]$. Then for any continuously differentiated function $\Phi(s, \overline{v})$ from the fact that $\Phi(s, \overline{X}(s)) = 0$, $s \in [0, T]$ results, that with every $s \in [0, T]$ for all $\overline{v} \in [\overline{X}(0), \overline{X}(s)]$ the following holds: $grad_{\overline{v}} \Phi(s, \overline{v}) = 0$.

The following statement reveals conditions allowing to transform Theorem 1.

Theorem 2. *Let us have a continuous vector-function $\overline{X}(t) = (X_1(t), \dots, X_d(t))$ which components possess unbounded variation on any section from $[0, T]$ and are functionally independent on the section $[0, T]$, and the functions $\sigma^{ik}(t, \overline{\eta}, \overline{v})$, $k = 1, 2, \dots, d$, $i = 1, 2, \dots, n$ and $B^i(t, \overline{\eta}, \overline{v})$, $i = 1, 2, \dots, n$ are continuously differentiated. If the vector-function $\overline{\varphi}(t, \overline{X}(t))$, $t \in [0, T]$ is the solution of Cauchy problem (7), then the function $\overline{\varphi}(t, \overline{v})$ satisfies the chain of systems of the ODE (8)–(11).*

Proof. Let us advance the proof of the theorem 2 in case $d = 2$ and $n = 1$, the general case is proved by analogy.

Let the function $\eta(t) = \varphi(t, X_1(t), X_2(t))$ be the solution of Cauchy problem (7). According to the definition of the solution of the equation with a symmetrical integral, there is the function $F(t, v_1, v_2)$ such that $F(t, X_1(t), X_2(t)) \equiv 0$ and

$$F'_t(t, X_1(t), X_2(t)) = B(t, \varphi(t, X_1(t), X_2(t)), X_1(t), X_2(t)) - \varphi'_t(t, X_1(t), X_2(t)),$$

$$F'_{v_1}(t, X_1(t), X_2(t)) = \sigma^1(t, \varphi(t, X_1(t), X_2(t)), X_1(t), X_2(t)) - \varphi'_{v_1}(t, X_1(t), X_2(t)),$$

$$F'_{v_2}(t, X_1(t), X_2(t)) = \sigma^2(t, \varphi(t, X_1(t), X_2(t)), X_1(t), X_2(t)) - \varphi'_{v_2}(t, X_1(t), X_2(t)).$$

Hence the functions $X_1(t), X_2(t)$ are functionally independent, then $grad_{\overline{v}} F(t, v_1, v_2) \equiv 0$ on $[\overline{X}(0), \overline{X}(t)]$, the the function $\varphi(t, v_1, v_2)$ satisfies the chain of equations (8)–(11).

2.3. Let us consider Cauchy problem for the equation in partial derivatives of the first order with multi-dimensional symmetrical integrals:

$$\begin{aligned} d_t u(t,\bar{x},\overline{X}(t)) = -\sum_{i=1}^{n} B^i(t,\bar{x},\overline{X}(t))u'_{x_i}(t,\bar{x},\overline{X}(t))dt- \\ -\sum_{i=1}^{n}\sum_{j=1}^{d} \sigma^{ij}(t,\bar{x},\overline{X}(t))u'_{x_i}(t,\bar{x},\overline{X}(t)) * dX_j(t), \end{aligned} \tag{14}$$

$$u(0,\bar{x},\overline{X}(0)) = x_k, \tag{15}$$

where x_k in the initial condition (15) is k coordinate of the variable $\bar{x} \in R^n$.

The solution of the equation (14) is the function $u(t,\bar{x},\overline{X}(t))$, such that during substitution of the function $u(t,\bar{x},\overline{X}(t))$ into the equation (14) all the integrals in the right side are valid, and the equation itself transforms into an identity. For any vector of the initial conditions $\bar{x} = (x_1,...,x_n)$ we denote the solution of Cauchy problem (14)-(15) by $\overline{U}(t,\bar{x}) = (u_1(t,\bar{x}),\ldots,u_n(t,\bar{x}))$ assuming $u_k(t,\bar{x}) = u_k(t,\bar{x},\overline{X}(t))$.

Alongside with the problem (14)–(15) we consider the corresponding system of equations with multi-dimensional symmetrical integrals:

$$\begin{cases} d\eta_i(t,\bar{x}) = B^i(t,\overline{\eta}(t,\bar{x}),\overline{X}(t))dt + \displaystyle\sum_{j=1}^{d} \sigma^{ij}(t,\overline{\eta}(t,\bar{x}),\overline{X}(t)) * dX_j(t), \\ \qquad\qquad\qquad\qquad \eta_i(0,\bar{x}) = x_i, \quad i = 1,2,\ldots,n. \end{cases} \tag{16}$$

Let us consider the following conditions:

(A) The functions $B^i(t,\overline{\eta},\overline{X}(t))$, $\sigma^{ij}(t,\overline{\eta},\overline{X}(t))$, $i = 1,2,\ldots,n$, $j = 1,2,\ldots,d$ are continuous on $(t,\overline{\eta})$ in some closed domain Q, which is the neighbouring of initial values of the systems of the equations (16).

(B) The functions $B^i(t,\overline{\eta},\overline{X}(t))$, $\sigma^{ij}(t,\overline{\eta},\overline{X}(t))$, $i = 1,2,\ldots,n$, $j = 1,2,\ldots,d$ satisfy in Q the Lipschitz condition relative to the variable $\overline{\eta}$: there exists such $N > 0$, that for any value t and any values $\overline{\eta}'$, $\overline{\eta}''$ of the variable $\overline{\eta}$ from the domain Q for all $i = 1,2,\ldots,n$, $j = 1,2,\ldots,d$ the following inequalities hold:

$$\left|B^i(t,\overline{\eta}',\overline{X}(t)) - B^i(t,\overline{\eta}'',\overline{X}(t))\right| \leqslant N\left|\overline{\eta}' - \overline{\eta}''\right|,$$

$$\left|\sigma^{ij}(t,\overline{\eta}',\overline{X}(t)) - \sigma^{ij}(t,\overline{\eta}'',\overline{X}(t))\right| \leqslant N\left|\overline{\eta}' - \overline{\eta}''\right|.$$

Theorem 3. *Let all the suppositions of Theorem 2 hold and for the coefficients $B^i(t,\overline{\eta}(t,\bar{x}),\overline{X}(t))$, $\sigma^{ij}(t,\overline{\eta}(t,\bar{x}),\overline{X}(t))$, $i = 1,2,\ldots,n$, $j = 1,2,\ldots,d$ the following conditions hold: (A)and (B). Then with every $t \in [0,T]$ the mapping $\overline{U}(t,\cdot) : \bar{x} \in R^n \to \overline{U}(t,\bar{x}) \in R^n$ is a diffeomorphism of the class $C^1(R^n)$, though the reverse mapping $\overline{U}^{-1}(t,\bar{x})$ is the solution of the system of equations with multi-dimensional symmetrical integrals* (16).

Proof. According to the theorems 1 and 2 the solution of the system of equations (16) is simultaneously the solution of the chains of the systems of ODE, therewith the variable $\bar{x}$ is a parameter. When imposing conditions (A) and (B) on the coefficients $B^i(t,\bar{x},\overline{X}(t))$, $\sigma^{ij}(t,\bar{x},\overline{X}(t))$, $i = 1,2,\ldots,n$, $j = 1,2,\ldots,d$ the solutions of the systems of ODE are continuously differentiated by the parameter $\bar{x}$ (see, for example, [1, Theorem 5.2.1]). Consequently, the solution of the system of equations (16) is continuously differentiated by the parameter $\bar{x}$.

Let $u(t,\bar{x},\overline{X}(t))$ be the solution of the problem (14)-(15). Applying the method of proving of the theorems 1,2 and applying the corresponding considerations to the equation (14), we come to the chain of correlations:

$$\begin{cases} u'_{v_j}(t,\bar{x},\overline{X}_{[j]}(t,v_j)) = -\sum_{i=1}^n \sigma^{ij}(t,\bar{x},\overline{X}_{[j]}(t,v_j))u'_{x_i}(t,\bar{x},\overline{X}_{[j]}(t,v_j)), \\ u'_t(t,\bar{x},\overline{X}(t)) = -\sum\limits_{i=1}^n B^i(t,\bar{x},\overline{X}(t))u'_{x_i}(t,\bar{x},\overline{X}(t)), \\ j=1,...,d. \end{cases} \tag{17}$$

Let us denote by $(\eta_i)'_{v_j}(t,\bar{x},\overline{X}(t)) = \frac{\partial}{\partial v_j}\eta_i(t,\bar{x},\overline{X}_{[j]}(t,v_j)) \mid_{v_j=X_j(t)}$ and find the differential with the symmetrical integral of the function $u(t,\overline{\eta}(t,\bar{x},\overline{X}(t)),\overline{X}(t))$:

$$\begin{gathered} d_t u(t,\overline{\eta}(t,\bar{x},\overline{X}(t)),\overline{X}(t)) = \Big[u'_t(t,\overline{\eta}(t,\bar{x},\overline{X}(t)),\overline{X}(t)) + \\ + \sum_{i=1}^n u'_{x_i}(t,\overline{\eta}(t,\bar{x},\overline{X}(t)),\overline{X}(t))\,(\eta_i)'_t(t,\bar{x},\overline{X}(t))\Big]dt + \\ + \sum_{j=1}^d \Big[u'_{v_j}(t,\overline{\eta}(t,\bar{x},\overline{X}(t)),\overline{X}(t)) + \\ + \sum_{i=1}^n u'_{x_i}(t,\overline{\eta}(t,\bar{x},\overline{X}(t)),\overline{X}(t))\,(\eta_i)'_{v_j}(t,\bar{x},\overline{X}(t))\Big] * dX_j(t). \end{gathered} \tag{18}$$

Let us write, according to the theorems 1,2, the chain of equations of the type (8)-(11) for the equation (16):

$$\begin{cases} (\eta_i)'_{v_j}(t,\bar{x},\overline{X}_{[j]}(t,v_j)) = \sigma^{ij}(t,\bar{x},\overline{X}_{[j]}(t,v_j)), \\ (\eta_i)'_t(t,\bar{x},\overline{X}(t)) = B^i(t,\bar{x},\overline{X}(t)), \\ i=1,2,\dots,n, \quad j=1,2,\dots,d. \end{cases}$$

Let us substitute these correlations into (18) and, omitting arguments of the functions in the right side, on the strength of (17) we obtain:

$$\begin{gathered} d_t u(t,\bar{\eta}(t,\bar{x},\overline{X}(t)),\overline{X}(t)) = \\ = \left[u'_t + \sum_{i=1}^n B^i u'_{x_i}\right]dt + \sum_{j=1}^d \left[u'_{v_j} + \sum_{i=1}^n \sigma^{ij}u'_{x_i}\right] * dX_j(t) \equiv 0. \end{gathered}$$

Therefore, we have made sure, that $d_t u(t,\bar{\eta}(t,\bar{x},\overline{X}(t)),\overline{X}(t)) \equiv 0$, therefore, $u(t,\bar{\eta}(t,\bar{x},\overline{X}(t)),\overline{X}(t)) = x_i$ with all t for every $i=1,2,\dots,n$. Theorem 3 has been proved.

Let us consider all the suppositions from Theorem 3 valid. Let us return to the equation in partial derivatives of the first order with a multi-dimensional symmetrical integral (14) and consider the system of equations with a multi-dimensional symmetrical integral and coefficients from the equation (14):

$$\begin{cases} x_i(t,\bar{z}) = z_i + \int_0^t B^i(t,\bar{x}(s,\bar{z}),\overline{X}(s))ds + \sum\limits_{j=1}^d \int_0^t \sigma^{ij}(t,\bar{x}(s,\bar{z}),\overline{X}(s)) * dX_j(s), \\ \qquad\qquad i=1,2,\dots,n. \end{cases} \tag{19}$$

Under the imposed conditions this system possesses n integrals

$$\begin{cases} \xi_1(t,\overline{X}(t),\bar{x}) = C_1, \\ \qquad \dots \\ \xi_n(t,\overline{X}(t),\bar{x}) = C_n. \end{cases} \tag{20}$$

The general solution of the equation (14) is the function $u = \Phi(\xi_1, \xi_2, \dots, \xi_n)$, where Φ is an arbitrary function and $\xi_1, \xi_2, \dots, \xi_n$ are left sides of the expressions (20).

Let $u = \theta(t, \bar{x}) = \theta(t, \overline{X}(t), \bar{x})$ be the solution of the equation (14). this equation is a hypersurface in the space $u, t, x_1, \dots, x_n$. The equations (20) together with $u = \theta(t, \bar{x})$ denote the family (of one-parameter) lines in this space. The lines of intersection of the cylinders (20) with the surface $\xi_{n+1} \equiv z = C$, where C is an arbitrary parameter, are named as *characteristic lines* of the equation (14), and the equations (19) are *equations of characteristics.*

Remark 7. The system of equations (19) can be formally written in the classical form, accepted in the theory of differential equations in partial derivatives of the first order for the equations of characteristics:

$$\begin{aligned} \frac{dt}{1} &= \frac{dx_1}{B^1(t, \bar{x}, \overline{X}(t)) + \sum_{j=1}^{d} \sigma^{1j}(t, \bar{x}, \overline{X}(t)) * (X_j(t))'_t} = \dots = \\ &= \frac{dx_n}{B^n(t, \bar{x}, \overline{X}(t)) + \sum_{j=1}^{d} \sigma^{nj}(t, \bar{x}, \overline{X}(t)) * (X_j(t))'_t}. \end{aligned} \tag{21}$$

Assuming $\sigma = 0$ in the equation (14), we proceed to the classical definition of characteristics and ordinary differential equations of the form (21) without symmetrical integrals.

Due to the fact that the coefficients B^i, σ^{ij}, $i = 1, 2, \dots, n$, $j = 1, 2, \dots, d$ of the system of equations (20) directly depend only on t, $\bar{x}$, $X(t)$, and do not depend on $\bar{z}$, the solution of the system (20) read:

$$\bar{x}(t, \bar{z}) = \overline{\Psi}(t) + \bar{z}, \quad \overline{\Psi}(0) = 0, \tag{22}$$

where $\bar{z}$ is the initial condition for the system of equations (20), $\overline{\Psi}(t) = \overline{\Psi}(t, \overline{X}(t))$ is the solution of the system of equations

$$\begin{cases} \Psi_i(t) = \displaystyle\int_0^t B^i(s, \overline{\Psi}(s), \overline{X}(s))ds + \sum_{j=1}^{d} \int_0^t \sigma^{ij}(s, \overline{\Psi}(s), \overline{X}(s)) * dX_j(s), \\ \hfill i = 1, 2, \dots, n. \end{cases} \tag{23}$$

Let us prove the formula (22). For all $i = 1, 2, \dots, n$ we write out differentials with symmetrical integrals for the functions $x_i(t, \overline{X}(t))$, $\Psi_i(t, \overline{X}(t))$ by the formula (5), and then we apply the systems of equations (19) and (23):

$$\begin{gathered} x_i(t, \overline{X}(t)) - x_i(0, \overline{X}(0)) = \\ = \int_0^t B^i(s, \overline{x}(s), \overline{X}(s))ds + \sum_{j=1}^{d} \int_0^t \sigma^{ij}(s, \overline{x}(s), \overline{X}(s)) * dX_j(s) = \\ = \Psi_i(t, \overline{X}(t)) - \Psi_i(0, \overline{X}(0)), \end{gathered} \tag{24}$$

whence we obtain the equality (22).

Following the classical theory of differential equations in partial derivatives of the first order, the family of characteristics of the equation (14) is written in the form:

$$x_i - \Psi_i(t) = \xi_i(t, x_1, \dots, x_n) = C_i, \qquad i = 1, 2, \dots, n, \tag{25}$$

where C_i, $i = 1, 2, \dots, n$ are some constants.

Let us show, that one of the general properties of characteristics holds (see, for instance, [11]), namely, that an arbitrary sufficiently smooth function from the characteristics of the differential equation in partial derivatives of the first order is its solution.

Theorem 4. *In the suppositions of Theorem 3 the general solution of the equation* (14) *can be written in the form $u(t, \bar{x}, \overline{X}(t)) = \Phi(\bar{x} - \overline{\Psi}(t))$ with the arbitrary continuous function $\Phi \in C^1(R^n)$, where $x_i - \Psi_i(t) = C_i$, C_i are the right sides of the integrals* (20), $i = 1, 2, \dots, n$.

Proof. Assume $u = \Phi(\xi_1, \xi_2, \ldots, \xi_n)$, where Φ is an arbitrary function, and $\xi_1, \xi_2, \ldots, \xi_n$ are the left sides of the expressions (20). Let us show, that when $\xi_i(t, x_1, \ldots, x_n) = x_i - \Psi_i(t)$, $i = 1, 2, \ldots, n$, where $\Psi_i(t) = \Psi_i(t, \overline{X}(t))$ is the solution of the system (23), the function $\Phi(\xi_1, \xi_2, \ldots, \xi_n)$ provides general solution of the equation (14).

Let us find differential by t of the function $\Phi(\overline{\xi}(t, \bar{x}))$:

$$\begin{gathered} d_t\Phi\left(\overline{\xi}(t,\bar{x})\right) = \sum_{i=1}^{n} \Phi'_{\xi_i} * d_t\xi_i(t,\bar{x}) = \\ = \sum_{i=1}^{n} \Phi'_{x_i} * d_t(x_i - \Psi_i\left(t, \overline{X}(t)\right)) = -\sum_{i=1}^{n} \Phi'_{x_i} * d\Psi_i\left(t, \overline{X}(t)\right). \end{gathered} \tag{26}$$

On the strength of (23) we have:

$$\begin{gathered} d\Psi_i(t, \overline{X}(t)) = grad_{\bar{v}}\Psi_i(t, \overline{X}(t)) * d\overline{X}(t) + (\Psi_i)'_t\,(t, \overline{X}(t))dt = \\ = \sum_{j=1}^{d} (\Psi_i)'_{v_j}\,(t, \overline{X}(t)) * dX_j(t) + (\Psi_i)'_t\,(t, \overline{X}(t))dt = \\ = \sum_{j=1}^{d} \sigma^{ij}(s, \overline{\Psi}(s), \overline{X}(s)) * dX_j(s) + B^i(s, \overline{\Psi}(s), \overline{X}(s))ds. \end{gathered}$$

Consequently, the right side (26) is equal to:

$$\begin{gathered} -\sum_{i=1}^{n} \Phi'_{x_i} \left(\sum_{j=1}^{d} \sigma^{ij}(t, \overline{\Psi}(t), \overline{X}(t)) * dX_j(t) + B^i(t, \overline{\Psi}(t), \overline{X}(t))dt \right) = \\ = -\sum_{i=1}^{n}\sum_{j=1}^{d} \sigma^{ij}(t, \overline{\Psi}(t), \overline{X}(t))\Phi'_{x_i} * dX_j(t) - \sum_{i=1}^{n} B^i(t, \overline{\Psi}(t), \overline{X}(t))\Phi'_{x_i}dt. \end{gathered}$$

Therefore,

$$\begin{gathered} d_t\Phi\left(\bar{x} - \overline{\Psi}(t)\right) = -\sum_{i=1}^{n} B^i(t, \overline{\Psi}(t), \overline{X}(t))\Phi'_{x_i}dt - \\ -\sum_{i=1}^{n}\sum_{j=1}^{d} \sigma^{ij}(t, \overline{\Psi}(t), \overline{X}(t))\Phi'_{x_i} * dX_j(t), \end{gathered}$$

hence, the function $\Phi\left(\bar{x} - \overline{\Psi}(t)\right)$ satisfies the equation (14). Consequently, general solution of the equation (14) can be presented in the form $\Phi\left(\bar{x} - \overline{\Psi}(t)\right)$, where Φ is an arbitrary smooth function.

Corollary. *Let $\overline{\eta} = \overline{\varphi}(t, \bar{x})$ be the solution of the system of equations* (16)*, $\overline{\varphi}^{\,-1}(t, \overline{\eta})$ be the function, with every t reciprocal by the variable $\bar{x}$ to the process $\overline{\varphi}(t, \bar{x})$. Then the structure of the process $\overline{\varphi}^{\,-1}(t, \overline{\eta})$ takes the form:*

$$\overline{\varphi}^{\,-1}(t, \overline{\eta}) = \overline{\eta} - \overline{\Psi}(t),$$

where the function $\overline{\Psi}(t)$ is the solution of the system of equations (23)*.*

Proof. From the supposition (22) we obtain, that $\overline{\varphi}(t, \bar{x}) = \overline{\eta} = \overline{\Psi}(t) + \bar{x}$, where $\bar{x}$ is the initial condition for the system of equations (16), $\overline{\Psi}(t)$ is the solution of the system of equations (23). Then the reciprocal function to $\overline{\varphi}(t, \bar{x})$ can be obtained by the formula: $\overline{\varphi}^{\,-1}(t, \overline{\eta}) = \bar{x} = \overline{\eta} - \overline{\Psi}(t)$.

2.4. Example 1. Let $X(t)$, $t \in [0, T]$ be an arbitrary continuous function of the unbounded variation, $\bar{x} = (x_1, x_2)$. Let us consider a differential equation in partial derivatives with the symmetrical integral:

$$\begin{aligned} d_t u(t, \bar{x}, X(t)) = \left(-tu'_{x_1} + (1 - X(t))u'_{x_2}\right) dt + \\ + \left(X(t)u'_{x_1} - tu'_{x_2}\right) * dX(t). \end{aligned} \tag{27}$$

Let us make the corresponding equations of characteristics:

$$\begin{cases} dx_1(t) = tdt - X(t) * dX(t), \\ dx_2(t) = (X(t) - 1)dt + t * dX(t). \end{cases} \tag{28}$$

After we solve the system (28) we obtain

$$\begin{cases} x_1(t) = \dfrac{1}{2}(t^2 - (X(t))^2) + C_1, \\ x_2(t) = (X(t) - 1)t + C_2, \end{cases} \tag{29}$$

whence we obtain, that the general solution of the equation (27) takes the form:

$$u(t, x) = \Phi\left(x_1 - \frac{1}{2}(t^2 - (X(t))^2), x_2 - (X(t) - 1)t\right), \tag{30}$$

where Φ is an arbitrary continuous differential function.

Let us verify, whether the function obtained is actually the solution of the equation (27). We have:

$$\xi_1(t, x_1, x_2) = x_1 - \frac{1}{2}(t^2 - (X(t))^2),$$
$$\xi_2(t, x_1, x_2) = x_2 - (X(t) - 1)t.$$

First we find derivatives by x_1, x_2 of the function (30):

$$\begin{aligned} u'_{x_1} = \Phi'_{\xi_1} \cdot {\xi_1}'_{x_1} + \Phi'_{\xi_2} \cdot {\xi_2}'_{x_1} = \Phi'_{\xi_1}, \\ u'_{x_2} = \Phi'_{\xi_1} \cdot {\xi_1}'_{x_2} + \Phi'_{\xi_2} \cdot {\xi_2}'_{x_2} = \Phi'_{\xi_2}, \end{aligned} \tag{31}$$

then we find differential by t:

$$\begin{aligned} d_t u = \Phi'_{\xi_1} \cdot d_t\xi_1 + \Phi'_{\xi_2} \cdot d_t\xi_2 = \\ = \Phi'_{\xi_1}\left(-tdt + X(t) * dX(t)\right) + \Phi'_{\xi_2}\left((1 - X(t))dt - t * dX(t)\right). \end{aligned} \tag{32}$$

Substituting expressions (31) into the right side of the equation (27), and (32) into the left side, we obtain an identity. Consequently, the function (30) is the solution of the equation (27). Though the function Φ is arbitrary, we have obtained general solution of the equation (27).

Example 2. Let $W(t)$ be a standard Wiener process. Let us consider Cauchy problem for the differential equation in partial derivatives with the symmetrical integral:

$$d_t u(t, x, W(t)) = -\left(t - \frac{\sin X^2(t)}{t^2}\right) u'_x dt - \left(W(t) + \frac{\sin 2W(t)}{t}\right) u'_x * dW(t), \tag{33}$$

with the initial conditions

$$u|_{\Gamma:\ x=\sqrt{t}} = x + \frac{\cos 2X(t)}{2t} - \frac{X^2(t) + t^2}{2} - \frac{1}{t}. \tag{34}$$

Let us make an equation of characteristics:

$$dx(t, X(t)) = \left(t - \frac{\sin X^2(t)}{t^2}\right) dt + \left(W(t) + \frac{\sin 2W(t)}{t}\right) * dW(t), \tag{35}$$

with which solution we obtain

$$x(t, X(t)) = -\frac{\cos 2X(t)}{2t} + \frac{X^2(t) + t^2}{2} + \frac{1}{t} + C. \tag{36}$$

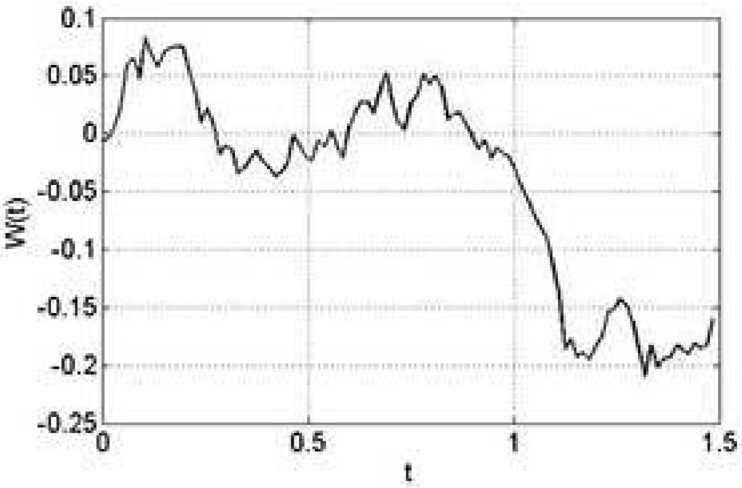

FIGURE 1. Graph of the process $W(t)$

Consequently, the general solution of the equation (33) takes the form:

$$u(t,x) = \Phi\left(x + \frac{\cos 2X(t)}{2t} - \frac{X^2(t)+t^2}{2} - \frac{1}{t}\right), \tag{37}$$

where Φ is an arbitrary continuously differentiated function. From the form of the initial condition we obtain, that the solution of the problem (33)-(34) is set by the expression:

$$u(t,x) = x + \frac{\cos 2X(t)}{2t} - \frac{X^2(t)+t^2}{2} - \frac{1}{t}. \tag{38}$$

Let us model trajectory of Wiener process $W(t)$ (Fig. 1),
and consider graphs of curves and the integral surface itself (38) (Fig. 2):

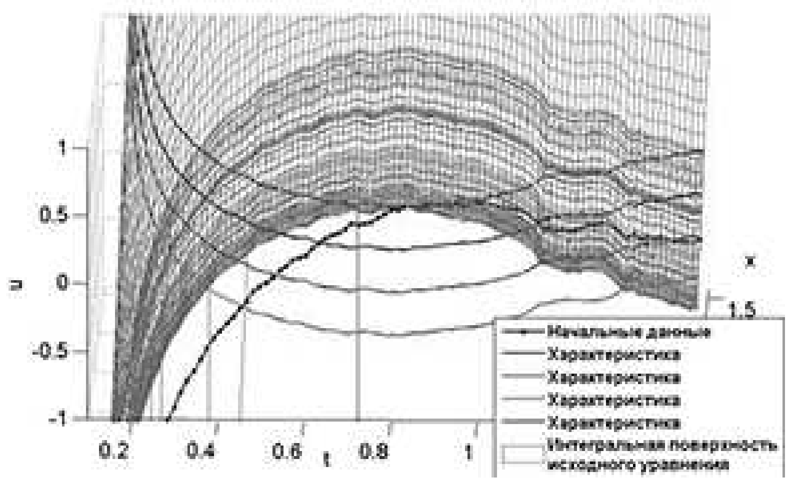

FIGURE 2. Integral surface of the equation (33)

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Ekaterina Viktorovna Yureva,
Ufa State Aviation Technical University,
12, K.Marx str.,
Ufa, Russia, 450000
E-mail: gap_kate@mail.ru

Farit Sagitovich Nasyrov,
Ufa State Aviation Technical University,
12, K.Marx str.,
Ufa, Russia, 450000
E-mail: farsagit@yandex.ru