ISSN 2304-0122 Ufa Mathematical Journal. Volume 4. 2 (2012). Pp. 102-114.

UDC 517.9

ASYMPTOTIC ANALYSIS OF THE SURFING ACCELERATION MODEL

L.A. KALYAKIN

Abstract. A mathematical model of acceleration of the charge particles by the electromagnetic waves is under investigation. Averaging equations, describing the resonance interaction of the particle with the electromagnetic wave are obtained. We show that each particle leaves the resonance zone under growth of time. The time length of the stay in the resonance depending on the initial data is calculated.
Keywords: Non-linear oscillations, small parameter, perturbation, averaging, adiabatic.

1. Introduction

The aim of the present paper is development of asymptotic approach to the analysis of the surfing acceleration model (or simply surfing) of relative particles in the electromagnetic wave.

From the point of view of physics we consider here acceleration of charged particles by means of localized in the space package of electromagnetic waves. When the particle crosses the package in the domain, where the wave field is higher than some threshold value, there is an acceleration effect. With the period of time the particle escape from the domain of the package, making cyclotron revolution, acceleration stops with this. Synchronization of the particle with the wave occurs on the finite, but rather big time interval. In the space conditions an effective width of the package can be sufficiently big, and therefore the energy of the particle captured can grow 3–5 orders of magnitude of its initial (before the capture) value.

The physics of this phenomenon, discovered in [1, 2, 3, 4], was in detail studied in paper [5]. We consider mathematical model in which base there is a non-linear non-autonomous equation of the second order [6, 7, 8]:

$$\beta\frac{d^2\psi}{d\tau^2}=\sigma\frac{\Big[1-\beta^2\Big(1-d\psi/d\tau\Big)^2\Big]^{3/2}}{\Big[1+h^2+\Big(J+(\tau-\Psi)\beta u\Big)^2\Big]^{1/2}}\cos\psi\cdot\exp\big[-((\psi-\tau)/\rho)^2\big]+ \\ +\frac{u\Big(J+(\tau-\psi)\beta u\Big)}{1+h^2+\Big(J+(\tau-\psi)\beta u\Big)^2}\Big[1-\beta^2\Big(1-d\psi/d\tau\Big)^2\Big]. \tag{1.1}$$

The equation contains six parameters $\beta, u, \sigma, J, h, \rho$, which are considered constant [1]. The required function $\psi(\tau)$ corresponds to the phase of the wave package of the electromagnetic wave on the carrier frequency. The surfing regime corresponds to the change of the function $\psi(\tau)$ within the finite bound. By its setting the problem of search of bounded solutions for the equation (1.1) resembles problems considered in the theory of synchronization [9]. However, we

L.A. Kalyakin, Asymptotic analysis of the surfing acceleration model.

The work was supported by RFFR (projects 10-01-00186, 11-02-97003).
Submitted on January 24, 2012.

[1]Dependence on slow time in these parameters is admitted and does not introduce principle changes into the described below method.

should note, that with physically significant coefficients all the solutions of the equation (1.1) unboundedly grow when $\tau \to \infty$. In the physical interpretation this corresponds to the escape with the period of time from the regime of surfing for any particle. Therefore synchronization here exists only during the final time interval.

It seems at first sight, that the equation for the function $\tilde{\psi}(\tau) = \psi(\tau) - \tau$ is more simple, non-autonomy there exists only in the argument of the cosine. However, to analyse the problem at the initial stage, which is connected with the notion of surfing, it is more useful to apply the equation in the form (1.1). Under the values of parameters considered below changes of coefficients by the time prove to be slow. This enables to apply ideas of adiabatic approximations and significantly simplify the initial model.

The general problem during the analysis of surfing consists in discovering conditions, when the beam of phase trajectories (or the family of solutions) is in the finite, previously fixed bounds during *a big* lime interval. The term *big* implies presence in the problem of big (or small) parameter which is being compared with. An exact mathematical formulation of the problem is possible within the frames of the theory of perturbations with application of the notion of asymptotic decomposition.

An opportunity of separating a small parameter in the problem and application of the theory of perturbation can be revealed from the analysis of characteristic values of the initial parameters applied in the model (1.1), see [8]:

$$0 < \beta < 1, \quad u \approx 10^{-1}, \ \sigma \approx 10^{-1}, \ \ J \approx 10^{1} \div 10^{2}, \ \ h \approx 10^{2}, \ \ \rho \approx 10^{4}.$$

In particular, numerical experiments with the equation (1.1) with such coefficients enabled to reveal the domain of the resonance capture, i.e. a number of initial points on the phase plane, which provides solutions, oscillating within the finite bounds (with a bounded average value) up to the remote time intervals of the order $\tau \approx 10^3 \div 10^4$, [7, 8]. For the initial data outside the capture domain average values of the phase immediately start to monotonously grow. The examples of equations of different types are given on Fig. 1.

FIGURE 1. Phase trajectories of different types for the initial equation (1.1) at the initial stage. Parameters $\sigma = 0.1; \beta = 0.4; u = 0.1; h = 100; J = 10; \rho = 10^4$. Initial points $\psi(0) = \pi/2$ and $\psi(0) = -\pi/4$ when $\dot{\psi}(0) = 0$. Evolution of trajectories: on the left up to the moment $4 \cdot 10^3$, on the right up to the moment $4 \cdot 10^2$

Such division of solution into two classes resembles situation with a non-linear pendulum. By its structure the equation (1.1) resembles the equation of a pendulum with the torque moment[1]: $\psi'' = A\cos\psi + B$. If the coefficients A, B are constant, then such an equation is integrated,

[1]The limiting process to the equation of the pendulum can be considered when $u \approx \sigma \approx \beta \to 0$.

FIGURE 2. Phase portrait of pendulum with a torque moment. On the left when $0 < \nu < 1$ there are closed trajectories inside a separatrix loop; on the right when $\nu > 1$ there no closed trajectories

and when $\nu = |B/A| < 1$ we can single out movements of two types: periodic oscillations and revolutions, see the left part of Fig. 2.

A significant difference of the equation (1.1) from a simple integrated model of a pendulum contains in presence of variables of coefficients reducing with the lapse of time. Due to this reducing oscillating at the initial stage solutions start to grow rapidly with the lapse of time. Usually such effects are described approximately with the application of the method of adiabatic approximations or averaging [10, 11]. In such an approach change of the type of the solution corresponds to the passage of a slowly deforming trajectory of a quick oscillating movement through a slowly deforming separatrix loop. Similar problems have been long and thoroughly studied, see, for example, the review [11]. In particular, on this way there was solved a group of problem on surfing [12, 13, 14] and there were revealed conditions of the capture in the resonance.

The model considered here (1.1) with the precision to substitution of variables coincides with the one, analysed in the papers [12, 13]. The asymptotic analysis which is conducted here corresponds to the paper [12], from which it is possible to extract the results obtained below. Though, the analysis of the problem in the formulation (1.1) with the application of the phase $\psi(t)$ can be of an special interest due to its transparent physical interpretation.

The general result consists in the proof of impossibility of a constant capture in the resonance: any trajectory with the lapse of time leaves the resonance domain. Duration of the stage of the resonance capture depends on the initial data and parameters of the system. For instance, summands with the first derivative in the equation (1.1), which owe the record of relativistic effects, reduce to prolongation of the stage of the resonance capture. These summands play at the initial stage the same role as dissipation (resistance) in the system of a non-linear pendulum.

The analytical results obtained here give opportunity to calculate such characteristics as domain and duration of the resonance capture in their dependence on the initial parameters. Therewith there is no need in hard calculations conducted with the help of direct control of the beam of trajectories [8]. The approach under discussion corresponds one of the variants of the theory of perturbations and is based on presence of a small parameter in the equations. Therefore we should remember, that the formulae presented below are of some approximate character and their precision depends on the value of this parameter.

2. The scale of fast oscillations

The initial equation is transformed with the aim to minimise the number of independent parameters. The time scale is chosen so that in the simplest model of an oscillator (when $u = 0$) oscillations nearby the equilibrium possess frequency equivalent to the unit. This enables to identify the fast time scale and single out a small parameter.

To choose the fast time scale $t = T \cdot \tau$ we apply the scale multiplier T, determined by the correlation

$$T^2 = \frac{(1-\beta^2)^{3/2}\sigma}{\beta\sqrt{1+h^2+J^2}}.$$

New parameters are introduced by the expressions

$$\varepsilon = \frac{\beta^{3/2}\sqrt{\sigma}}{(1-\beta^2)^{1/4}(1+h^2+J^2)^{1/4}}, \quad \mu = \frac{u^2}{\sigma^2(1-\beta^2)},$$

$$\nu = \frac{uJ}{\sigma\sqrt{(1-\beta^2)(1+h^2+J^2)}}, \quad \lambda = \frac{\sqrt{1+h^2+J^2}}{\sigma\beta\rho\sqrt{1-\beta^2}}, \quad b = \frac{1-\beta^2}{\beta^2}.$$

Meanwhile we obtain the correlations: $T = b\,\varepsilon$, $\lambda = 1/\rho b\varepsilon^2$, $\nu^2 < \mu$. The remained coefficients are recalculated by the formulae

$$\nu\beta u/J = \frac{u^2\beta}{\sigma\sqrt{(1-\beta^2)(1+h^2+J^2)}} = \varepsilon^2\mu b, \quad \frac{(\beta u)^2}{\varepsilon^2 T^2(1+h^2+J^2)} = \mu.$$

Alongside with the fast time t we also introduce the slow time $\theta = \varepsilon t$. After this in the initial equation there remain five independent parameters $\varepsilon, b, \lambda, \mu, \nu$:

$$\begin{aligned}\frac{d^2\psi}{dt^2} = {} & \frac{\left[1+\varepsilon\frac{d\psi}{dt}\Big(2-\varepsilon b\frac{d\psi}{dt}\Big)\right]^{3/2}}{\left[1+(\theta-\varepsilon^2 b\psi)\Big(2\nu+(\theta-\varepsilon^2 b\psi)\mu\Big)\right]^{1/2}}\cdot\cos\psi\cdot\exp\left(-\lambda^2(\theta-b\varepsilon^2\psi)^2\right)+ \\ & +\frac{\nu+\mu\theta-\varepsilon^2\mu b\psi}{1+(\theta-\varepsilon^2 b\psi)\Big(2\nu+(\theta-\varepsilon^2 b\psi)\mu\Big)}\left[1+\varepsilon\frac{d\psi}{dt}\Big(2-\varepsilon b\frac{d\psi}{dt}\Big)\right].\end{aligned} \tag{2.1}$$

For this equation below we conduct an asymptotic analysis of the equations with the application of the small parameter $0 < \varepsilon \ll 1$. Meanwhile we consider, that other parameters are not big; their probable infinitesimality is not applied. In particular, characteristic values of new constants, recalculated from the old ones, which are usually taken to the considered models, correspond to the studied conditions:

$$\varepsilon = 10^{-1} \div 10^{-2},\ \nu = 1 \div 10^{-1},\ \mu \approx 1,\ \lambda = 10^{-1} \div 10^{-2},\ \ b = 1 \div 10,\ T = 1 \div 10^{-2}.$$

The infinitesimality of the parameter ε enables to significantly simplify the equation and obtain a number of analytical results. The simplifications are based on ideas of two-scale decompositions, which provide division of the fast and slow dependences on the tome of the solution.

3. Simplification of equations

In the initial equation (2.1) there are small parameter ε and the slow time $\theta = \varepsilon t$ identified. The most rough approximation is obtained within the limit when $\varepsilon \to 0$:

$$\frac{d^2\psi_0}{dt^2} = \cos\psi_0 + \nu.$$

This equation corresponds to the pendulum with a constant torque moment, we also relate with it the description of process capture in the resonance in different situations [15]. The solution of such an approximate equation reveals the dominant term pf the asymtotics for the solution of the complete equation (2.1): $\psi(t) = \psi_0(t) + \mathcal{O}(\varepsilon)$ when $\varepsilon \to 0$. This approximation holds at any finite time interval[1] $0 < t \leqslant T_0$, independent on the parameter ε. The law of conservation $(\dot{\psi}_0)^2/2 - \sin\psi_0 - \nu\psi_0 = E = \text{const}$ enables to write the solution by an integral. Though, for the analysis of the structure of all the possible solution a phase portrait is sufficient (see Fig. 2).

As we can see, phase trajectories separated by a separatrix loop, and also a stable branch of a separatrix correspond to the solutions with a bounded function $\psi(t)$. On the remaining trajectories the function $\psi(t)$ undoundedly grows when $t \to \infty$. With regard to the complete equation (2.1) these results can make an impression, that the domain bounded by a separatrix loop corresponds a required domain of the capture, which gives start to the solutions with a bounded by the time phase $\psi(t)$. Moreover, the solution of the equation (2.1) does not possess such a property of boundedness. Numerical experiments show, that all the solutions $\psi(t)$ unboundedly grow when $t \to \infty$; the examples are demonstrated on Fig. 1. Moreover, this fact is easily proved on the level of the exact statement, see Supplement.

Significant discrepancy between approximate and complete equations is discovered on remote times $t \geq \varepsilon^{-1/2}$; in particular, there occurs a spiral instead of closed trajectories. First of all these discrepancies owe to the change of coefficients, which depend on slow time $\theta = \varepsilon t$.

Counting of the slow deformation of coefficients in such problems is conducted by method of averaging and the corresponding asymptotic constructions are sometimes called adiabatic approximations. Besides, in the considered problem at the remote times $t > \varepsilon^{-1/2}$ the influence of perturbations-members of the order $O(\varepsilon)$ from the complete equation (2.1) prove to be significant. Their influence is displayed in the effect of the spiral compression on Fig. 1. Average equations which consider such perturbations do not already guarantee an adiabatic invariant. Obtaining such equations is sometimes related either to the method of two-scale decompositions or to the non-linear Wentzel-Kramers-Brillouin method. Application of one of variants of such an approach enables to construct for the equation (2.1) asymptotics of the solution by a small parameter $\varepsilon \to 0$, applicable for the remote times $t \approx \varepsilon^{-1}$. Following this way it becomes possible to describe both the capture domain, and the time spent by the phase trajectory in the capture domain.

For the part of coefficients which change into the value of the unit order at the remote times $t \approx \varepsilon^{-1}$, we introduce symbols

$$A(\theta) = \frac{\exp\big(-\lambda^2\theta^2\big)}{\sqrt{1 + 2\nu\theta + \mu\theta^2}}, \quad B(\theta) = \frac{\nu + \mu\theta}{1 + 2\nu\theta + \mu\theta^2}, \quad \theta = \varepsilon t. \tag{3.1}$$

Let us single out in the initial equation members up to the order $\mathcal{O}(\varepsilon)$ and write the corresponding system of two equations of the first order

$$\frac{d\psi}{dt} = \phi, \quad \frac{d\phi}{dt} = A(\theta)\cdot\cos\psi + B(\theta) + \varepsilon F(\psi, \phi; \theta). \tag{3.2}$$

This system with the function $F(\psi, \phi; \theta) = \big[3A(\theta)\cdot\cos\psi + 2B(\theta)\big]\phi$ is the general object of research. The summand εF with the small multiplier ε is called a perturbation. It is not Hamiltonian and is responsible for the dissipation, which leads to attenuation of oscillations at the stage of resonance capture. In formal constructions of the asymptotic solution dissipation is developed in slow change of the value[2], which in the absence of the perturbation εF is an adiabatic invariant.

We should note, that some part of effects from the perturbation εF can be obtained from the solution in the form of an exponential multiplier so, that the remaining perturbation in general is Hamiltonian and the problem possesses an adiabatic invariant. This fact the author derived from A.I.Neistadt. In the simplest case of a linear oscillator with low resistance $\ddot{x} + x + \varepsilon\dot{x} = 0$

[1] At remote times $t \approx \varepsilon^{-1/2}$ such an approximation becomes useless, that is discovered in occurrence of secular summands in corrections of the order ε.

[2] the area, covered by a non-perturbed closed trajectory on the phase plane

such approach corresponds to the separating from the solution a slowly attenuating multiplier $\exp(-\varepsilon t/2)$. Though, for the equation (3.2) the structure of a similar multiplier is not that obvious. We are not intended to make reductions to Hamiltonian perturbations not to make long explanations which are not understandable for the reader, who is not closely acquainted to the method of averaging.

The complete equation (2.1) is corresponded by a more complex expression for the perturbation $\varepsilon F = \varepsilon F(\psi, \phi; \theta, \varepsilon)$, which includes corrections of higher orders, starting with the summands of the order $\mathcal{O}(\varepsilon^2)$. Though, such corrections do not significantly contribute to the effects under study and therefore they are not counted.

Let us note, that in the coefficients of the equations (3.2) apart from ε there remain three independent parameters μ, ν, λ. The parameter b is not contained in the members of the order $\mathcal{O}(\varepsilon^2)$ and therefore it is not counted in the further constructions[1].

4. Formulation of the result

Construction of asymptotics applicable even in the remote times $t = \mathcal{O}(\varepsilon^{-1})$ is based on the autonomous equation with "frozen" coefficients

$$\frac{d\psi_0}{ds} = \phi_0, \quad \frac{d\phi_0}{ds} = A(\theta) \cdot \cos\psi_0 + B(\theta), \tag{4.1}$$

in which the parameter θ is considered as independent on "time" s. We also consider the family of periodic solutions $\psi_0 = \psi_0(s, E; \theta)$, $\phi_0 = \phi_0(s, E; \theta)$. For parametrisation (enumeration of solutions) we apply the value of the first integral (energy) $\phi_0^2/2 - A\sin\psi_0 - B\,\psi_0 = E = \text{const}$. Let us note, that in these solutions parameters E and θ are independent[2].

We should not think that the function $\psi_0(t, E; \theta)$ immediately provides the dominant term of asymptotics for the initial equation[3]. Construction of the dominant term of the asymptotics applicable for remote times consists in obtaining a suitable slow deformation of the solution of frozen equations.

Theorem 1. *If the parameter $0 < \nu < 1$ is not big, then for the equation (2.1) there is a two-parameter family of asymptotic solutions*
$\psi(t; s_0, E_0)$, $(s_0, E_0 = \text{const})$, *which are presented in the dominant term of decomposition by the periodic solution of the "frozen" system*

$$\psi(t; s_0, E_0) = \psi_0(\varepsilon^{-1}\mathcal{S}(\theta), \mathcal{E}(\theta); \theta) + \mathcal{O}(\varepsilon), \quad \varepsilon \to 0, \ (\theta = \varepsilon t).$$

Here the arguments $\mathcal{S} = \mathcal{S}(\theta; E_0, s_0)$, $\mathcal{E} = \mathcal{E}(\theta; E_0)$ depend on θ and are determined from scalar differential equations. This asymptotics is uniform at the large time interval $0 < t \leqslant \theta^* \cdot \varepsilon^{-1}$ (the stage of the resonance capture), which length $\theta^* = \theta^*(E_0)$ depends on the initial parameter E_0.

The proof of the theorem is the basic contents of the following sections; the aim is to obtain and research equations for the functions $\mathcal{S}(\theta; E_0, s_0)$,
$\mathcal{E}(\theta; E_0)$.

The geometrical interpretation of the theorem. The trajectory of the frozen system on the phase plane are presented by closed curves inside a separatrix loop (see Fig. 2). The phase portrait depends on the parameter θ; in particular, with the growth of θ stationary points shift and the separatrix loop squeezes. With the fixed value of the parameter θ a closed trajectory is determined by the value E. When we include the relation $\theta = \varepsilon\, t$ and $E = \mathcal{E}(\theta; E_0)$ the trajectory slowly deforms with the lapse of time and reminds a squeezing spiral. The solution

---

[1] In the supposition, that the value b is small: $b \ll \varepsilon^{-1}$

[2] Sometimes instead of energy there applied either action or the area Π, covered by a closed phase trajectory. Transformation to such a parametrisation is described below.

[3] It is not true even in the trivial example of the equation with a variable coefficient $\dot{x} = 2\theta \cdot x$, $(\theta = \varepsilon t)$. Here the exact solution $x = C \cdot \exp(\varepsilon t^2)$, $\forall\, C = \text{const}$ differs from the approximate "frozen" one $x_0 = C \cdot \exp(2\theta t)$, $(\theta = \varepsilon t)$ on the value of the order of the unit on remote times $t = \mathcal{O}(\varepsilon^{-1/2})$.

of the equation at the stage of the resonance capture (see Fig. 1) possesses exactly the structure. An approximate description of the solution by means of periodic function stops at the remote times $t = \mathcal{O}(\varepsilon^{-1})$ when the slowly deforming trajectory meets the separatrix. Such a stop is inevitable for any trajectory due to the squeezing of the separatrix loop with the growth θ and its collapse in the moment Θ, when the coefficients become equivalent: $A(\Theta) = B(\Theta)$. In the problem considered there is no an adiabatic invariant: the area $\Pi(\theta)$ bounded by a fast trajectory depends on the slow time and reduces with the growth θ. This property is conditioned by presence of dissipation and leads to prolonging the stage of the resonance capture.

For the applications the general result is in the formula (7.1), which relates the moment of the resonance stop $\theta^* = \theta^*(\Pi_0)$ with the area Π_0 covered by the frozen trajectory in the initial moment $\theta = 0$.

5. Transformation of a perturbed system

The shortest and the most transparent for the construction of asymptotics by the small parameter is the approach based on the standard transformation to the variables of the type action - angle in the main part of the equations (3.2). For this purpose we apply a pair of periodical functions presenting the two-periodic solution of the frozen system (5.1) $\psi_0 = \psi_0(s, E; \theta)$, $\phi_0 = \phi_0(s, E; \theta)$. The solution considered is periodic if and only if $|B(\theta)/A(\theta)| < 1$. Let us note, that the period $T(E; \theta)$ and the frequency $\omega(E; \theta) = 2\pi/T > 0$ depend both on the parameter E, i.e. on the choice of the phase trajectory, and on θ.

In the further construction it is convenient to apply 2π-periodic functions

$$\Psi(S, E; \theta) \equiv \psi_0(S/\omega(E; \theta), E; \theta), \quad \Phi(S, E; \theta) \equiv \phi_0(S/\omega(E; \theta), E; \theta).$$

Functions introduced this way depend on three variables and on the strength of the equations (5.1) the satisfy the identities

$$\omega(E; \theta)\frac{\partial \Psi}{\partial S} = \Phi, \;\; \omega(E; \theta)\frac{\partial \Phi}{\partial S} = A(\theta) \cdot \cos\Psi + B(\theta). \tag{5.1}$$

Besides, there takes place an identity of the first integral:

$$\Phi^2/2 - A \sin\Psi - B\,\Psi = E.$$

Differentiating of the latter correlation by E and by θ provides two identities more

$$\begin{gathered} \omega(E; \theta)[\partial_S\Psi \cdot \partial_E\Phi - \partial_S\Phi \cdot \partial_E\Psi] = 1, \\ \omega(E; \theta)[\partial_S\Psi \cdot \partial_\theta\Phi - \partial_S\Phi \cdot \partial_\theta\Psi] = A'(\theta)\sin\Psi(S, E; \theta) + B'(\theta)\Psi(S, E; \theta). \end{gathered} \tag{5.2}$$

The determined above pair of functions is applied in the equations (3.2) for the change of variables by the formulae:

$$\psi = \Psi(\mathcal{S}, \mathcal{E}; \theta), \quad \phi = \Phi(\mathcal{S}, \mathcal{E}; \theta).$$

These relations describe transformation from ψ, ϕ to new required functions - to the energy $\mathcal{E}(t; \varepsilon)$ and to the angle $\mathcal{S}(t; \varepsilon)$. Herewith the equations (3.2) for ψ, ϕ transform to the equations for $\mathcal{E}$, $\mathcal{S}$. After they are reduced to the norm due to (5.2) they take the form

$$\begin{gathered} \frac{d\mathcal{E}}{dt} = \varepsilon F\Phi - \varepsilon[A'(\theta)\sin\Psi(\mathcal{S}, \mathcal{E}; \theta) + B'(\theta)\Psi(\mathcal{S}, \mathcal{E}; \theta)]. \\ \frac{d\mathcal{S}}{dt} = \omega(\mathcal{E}; \theta) - \varepsilon\omega(\mathcal{E}; \theta)[F\partial_{\mathcal{E}}\Psi - \partial_{\mathcal{E}}\Psi\partial_\theta\Phi + \partial_{\mathcal{E}}\Phi\partial_\theta\Psi], \quad \theta = \varepsilon\, t. \end{gathered} \tag{5.3}$$

Here the function $F(\Psi, \Phi; \theta) = \big[3A(\theta) \cdot \cos\Psi + 2B(\theta)\big]\Phi$ determines perturbation of the integrated equation with the frozen coefficients (5.1).

6. An averaged equation

The equation (5.3) possess a standard form for the slow $\mathcal{E}$ and fast $\mathcal{S}$ variables. Asymptotics of the solution by the small parameter ε is constructed by a usual method of averaging. For the energy $\mathcal{E}(t;\varepsilon) = \mathcal{E}_0(\theta) + \mathcal{O}(\varepsilon)$, $\varepsilon \to 0$ the dominant term of the asymptotics $\mathcal{E}_0(\theta)$ corresponds a slowly changing function. An equation for it is obtained by means of averaging by the fast variable $\mathcal{S}$:

$$\frac{d\mathcal{E}_0}{d\theta} = \langle F(\Phi,\Psi;\theta)\Phi(\mathcal{S},\mathcal{E}_0;\theta)\rangle - \langle A'(\theta)\sin\Psi(\mathcal{S},\mathcal{E}_0;\theta) + B'(\theta)\Psi(\mathcal{S},\mathcal{E}_0;\theta)\rangle.$$

Here we denote by angle brackets an integral of the average value

$$\langle f(\mathcal{S})\rangle = \frac{1}{2\pi}\int_0^{2\pi} f(\mathcal{S})\,d\mathcal{S}.$$

Though, instead of $\mathcal{E}_0(\theta)$ it is efficient to apply another function $\Pi(\theta)$ possessing a simple geometric meaning. For this purpose, applying the identities (5.1), (5.2), the averaged equation for the energy is written in a different form[1]:

$$\langle \partial_S\Psi\partial_{\mathcal{E}}\Phi - \partial_S\Phi\partial_{\mathcal{E}}\Psi\rangle\frac{d\mathcal{E}_0}{d\theta} + \langle \partial_S\Psi\partial_\theta\Phi - \partial_S\Phi\partial_\theta\Psi\rangle = \frac{1}{\omega}\langle F\Phi\rangle.$$

It is easy to see, that the left side corresponds a complete derivative by θ from the integral of the average

$$\frac{1}{2}\frac{d}{d\theta}\langle \partial_S\Psi\Phi - \partial_S\Phi\Psi\rangle = \langle \partial_S\Psi\partial_{\mathcal{E}}\Phi - \partial_S\Phi\partial_{\mathcal{E}}\Psi\rangle\frac{d\mathcal{E}_0}{d\theta} + \langle \partial_S\Psi\partial_\theta\Phi - \partial_S\Phi\partial_\theta\Psi\rangle.$$

Let us count, that this integral is related to the area Π, which is covered on the phase plane (ψ,ϕ) by a closed trajectory $\psi = \Psi(S,\mathcal{E};\theta), \phi = \Phi(S,\mathcal{E};\theta)$ with fixed parameters $\mathcal{E},\theta$:

$$\Pi = 2\pi\langle \partial_S\Psi\Phi\rangle = \pi\langle \partial_S\Psi\Phi - \partial_S\Phi\Psi\rangle = \frac{2\pi}{\omega}\langle \Phi^2\rangle.$$

Therefore an averaged equation can be written in the form of the equation for the area

$$\frac{d\Pi}{d\theta} = \frac{2\pi}{\omega}\langle F(\Phi,\Psi;\theta)\Phi\rangle.$$

Whence it implies, that if there is no perturbation, when $F \equiv 0$, the area is an adiabatic invariant (preserves) in the scale of the slow time. Although, in the problem considered $F \not\equiv 0$. Moreover, the integral of the average differs from zero and is calculated by the area on the strength of the equations (5.1):

$$\langle F(\Phi,\Psi;\theta)\cdot\Phi\rangle = \langle [3A\cos\Psi + 2B]\Phi^2\rangle = -B(\theta)\langle\Phi^2\rangle = -B(\theta)\frac{\omega}{2\pi}\Pi.$$

As a result the averaged equation takes the form

$$\frac{d\Pi}{d\theta} = -B(\theta)\cdot\Pi. \tag{6.1}$$

It immediately results from the latter equation, that the area covered by the phase curve decreases with the deformation by time as $B(\theta) > 0$. For the considered problem with $B(\theta) = (\nu+\mu\theta)/(1+2\nu\theta+\mu\theta^2)$ the solution of this equation with the initial condition $\Pi|_{\theta=0} = \Pi_0$ is written in the form

$$\Pi(\theta;\Pi_0) = \Pi_0\Big(1 + 2\nu\theta + \mu\theta^2\Big)^{-1/2}. \tag{6.2}$$

This formula can be considered as the first stage of the asymptotics construction by the small parameter for the system of equations (5.3). The connection between the energy and the area $\omega(\mathcal{E};\theta)\cdot d\Pi = 2\pi\cdot d\mathcal{E}$ ensures to apply any of these variables in the construction.

[1]It could be done in the initial equation for the energy.

The domimant term of the asymptotics for the angle is written by the integral

$$\mathcal{S}(t,\varepsilon) = \varepsilon^{-1}\int_0^\theta \omega(\mathcal{E}_0(\zeta))\,d\zeta + \mathcal{O}(1), \quad \varepsilon\to 0,\ (\theta = \varepsilon\, t).$$

Corrections of a higher order in the decomposition as $\mathcal{E}(t;\varepsilon)$, as $\mathcal{S}(t;\varepsilon)$ are made by a known method and can include periodic dependence on the fast variable [10, 11]. Though, for the analysis of the problem of the resonance capture higher corrections and even the formula for the angle are not needed.

7. The resonance capture

The resonance capture we consider as an opportunity to present asymptotics of the solution in the form of fast oscillating function with slowly changing parameters:

$$\Psi(t,\varepsilon) = \psi_0(s,\mathcal{E}_0(\theta);\theta) + \mathcal{O}(\varepsilon), \quad \varepsilon\to 0, \quad (s = \varepsilon^{-1}\mathcal{S}/\omega(\mathcal{E};\theta)).$$

Not all the solutions possess such a property. Moreover, description of the approximate solution by means of a periodic function is applicable only when a slowly deforming trajectory of the system (5.1) $\psi = \psi_0(s,\mathcal{E}_0(\theta);\theta),\ \phi = \phi_0(s,\mathcal{E}_0(\theta);\theta)$ is on the phase plane in the domain bounded by a separatrix loop. This so-called domain of the capture is described with the help of the first integral by the inequality

$$-B\psi_c - A\sin\psi_c \leqslant \frac{1}{2}\phi^2 - B\psi - A\sin\psi \leqslant -B(-\psi_c + 2\pi) + A\sin\psi_c. \tag{7.1}$$

Here we apply coordinates of stationary points $(\pm\psi_c(\theta),0)$ with the positive value $0<\psi_c = \arccos(-B(\theta)/A(\theta)) < \pi$. The loop of the separatrix escaping the saddle $2\pi - \psi_c$ covers the centre with the coordinate ψ_c. The condition of the existence of the capture domain is the inequality: $|B(\theta)/A(\theta)| < 1$, which due to (3.1) takes the form:

$$\frac{\nu+\mu\theta}{\sqrt{1+2\nu\theta+\mu\theta^2}} < \exp(-\lambda\theta).$$

It is obvious, that the demand $\nu<1$ is necessary and sufficient for the existence of the capture domain in the initial moment $\theta = 0$. With the lapse of time the area of this domain decreases, see Supplement.

7.1. A global break of the resonance capture. The upper boundary Θ of the time of the break of the resonance capture for all the solutions is determined by the moment of collapse of the separatrix loop, i.e. from the equation $A(\Theta) = B(\Theta)$. Dy this moment not a single solution is presented by periodic functions. In the considered problem the moment of the break is determined from the equation

$$\exp(-\lambda^2\Theta^2) = \frac{\nu+\mu\Theta}{\sqrt{1+2\nu\Theta+\mu\Theta^2}}.$$

In case, when $0<\lambda\ll 1$, and other parameters possess the order of the unit, the root of this equation Θ can be calculated approximately, i.e. we cam make the asymptotics of the solution when $\lambda\to 0$. The structure of the dominant term of such an asymptotics depends on the value μ.

If $\mu<1$, then the right side can be substituted by the asymptotics

$$\exp(-\lambda^2\Theta^2) = \sqrt{\mu} + \mathcal{O}(\Theta^{-1}),\ \Theta\to\infty.$$

Whence we obtain an approximate value for the moment of collapse of the separatrix loop

$$\Theta = \lambda^{-1}\sqrt{-\ln\mu} + \mathcal{O}(1),\ \lambda\to 0,\ (\mu<1).$$

It is obvious, that when $\lambda = 0,\ \mu<1$ there in no collapse of the separatrix loop and for one trajectory there exists a constant capture.

In the case $\mu > 1$ the approximate value of the moment of the brake is determined by the equation

$$1 + O(\lambda^2) = \frac{\nu + \mu\Theta}{\sqrt{1 + 2\nu\Theta + \mu\Theta^2}} \,.$$

Whence we derive the asymptotics

$$\Theta = \frac{\sqrt{\mu - \nu^2} - \nu}{\mu\sqrt{\mu - 1}} + \mathcal{O}(\lambda^2), \;\; \lambda \to 0, \;\;\; (\mu > 1).$$

As it is seen, in this case the value Θ does not grow when $\lambda \to 0$. Let us note, that $\mu > \nu^2$ due to the on the strength of the relations, which determine parameters of the problem.

Conclusion. *Conditions of the long-term capture:* $\mu < 1, \quad \lambda \ll 1.$

Remark. When applying the latter result we should take into account, that the formulae of two-scale decompositions hold only up to not too big times, at the best $\theta \ll \varepsilon^{-1/2}$.

7.2. Local brake of the resonance capture. The moment of the brake of the resonance capture on the definite trajectory is less than the upper boundary Θ and depends on the trajectory. The reason is that any trajectory in the considered weakly dissipative system slowly approaches to the slowly moving equilibrium position. The area covered by such a trajectory in any frozen moment of the slow time θ cannot turn to zero, though it tents to zero when $\theta \to \infty$. Therefore by the moment of collapse of the separatrix loop such a trajectory is already beyond the limits of its description by periodic functions. The only exception is the only trajectory corresponding to the slowly changing equilibrium position - to the centre $\psi = \psi_c(\theta)$ with zero area $\Pi(\theta) \equiv 0$. For such a trajectory the moment of the escape the resonance capture coincides with the upper boundary Θ.

In the general case the moment θ^* of the escape from the resonance capture for the definite trajectory can be related to the initial value of the area Π_0 on this trajectory. This condition can be formulated in the form of the demand of the equality of two areas. Whereas the separatrix is described by the equation

$$\frac{1}{2}\phi^2 - A(\theta)\sin\psi - B(\theta)\psi = A(\theta)\sin\psi_c + B(\theta)[\psi_c - 2\pi],$$

then the demand of the equality of the area $\Pi(\theta;\Pi_0)$ under the trajectory and the area $\Pi_c(\theta)$ under the separatrix results in the relationship:

$$\Pi_0\Big(1 + 2\nu\theta + \mu\theta^2\Big)^{-1/2} = 2\sqrt{2}\int_{\psi_-(\theta)}^{\psi_+(\theta)} \sqrt{A(\theta)[\sin\psi + \sin\psi_c] + B(\theta)[\psi + \psi_c - 2\pi]}\, d\psi. \quad (7.1)$$

Here $\psi_- < \psi_+$ are zeros of the radicand expression, in particular, $\psi_+ = -\psi_c + 2\pi$ is the coordinate of the saddle point, $0 < \psi_c(\theta) = \arccos(-B(\theta)/A(\theta)) < \pi$. The root of this equation $\theta^* = \theta^*(\Pi_0)$ determines the moment of escape from the resonance for the trajectory which starts with the initial value of the parameter $\Pi|_{\theta=0} = \Pi_0$.

On the other hand, the relationship (7.1) can be considered as a formula for the initial area of the set of those points which serve as the start for the trajectories which do not leave the resonance domain up to the moment θ. This set on the initial plane is bounded by a closed trajectory of a frozen system which area coincides with Π_0.

It is implicitly supposed in these considerations, that the trajectory can escape the domain of the capture, but cannot enter it. Such a property is not obvious and needs to be proved, what is done in Supplement.

8. Supplement

1. Unboundedness of solutions by the time.

In this section we discuss the problem of existence and behaviour of solutions of the equation (1.1) at the infinite interval.

The direction of the absence of solutions, bounded when $\tau \to \infty$, can be obtained from the following considerations. If the function $\Psi(\tau)$ is bounded, then the first summand in the right side of the equation (1.1) exponentially decreases when $\tau \to \infty$. After we remove this summand we obtain the equation, which can be integrated. It is seen from the explicit formula for the solution that the function $\Psi(\tau)$ cannot be bounded. Though, strictly speaking, the conducted considerations are not applicable to the complete equation (1.1) which is not integrated.

Theorem 2. *If* $\beta^2 \neq 1$, $u > 0$, *then any solution of the equation* (1.1) *continues to the infinite interval and linearly grows when* $\tau \to \infty$.

To prove the theorem it is convenient to apply the form of the equation different from (1.1) making a substitution of variables:

$$J + [\tau - \Psi(\tau)]\beta u = -\Phi(t),\ t = u \cdot \tau,\ \Rightarrow\ (1 - \dot{\Psi})\beta = -\dot{\Phi},\ \beta\ddot{\Psi} = u\ddot{\Phi}.$$

As the result the non-autonomous property occurs under the argument of the cosine:

$$\frac{d^2\Phi}{dt^2} = \frac{\sigma}{u} \cdot \frac{\Big[1 - \dot{\Phi}^2\Big]^{3/2}}{\Big[1 + h^2 + \Phi^2\Big]^{1/2}} \cos\Psi \cdot \exp\big[-((J+\Phi)/\beta u\rho)^2\big] + \frac{\Phi}{1 + h^2 + \Phi^2}\big[1 - \dot{\Phi}^2\big], \quad (\Psi = (\beta t + J + \Phi)/\beta u). \tag{8.1}$$

A part of the equation can be integrated and reduced to the integral relationship. For this purpose we should multiply the left and the right sides by $-2\dot{\Phi}/(1 - \dot{\Phi}^2)$ and take the integral at the interval (t_0, t). As a result we obtain the equation in the form

$$\frac{1 - \dot{\Phi}^2(t)}{1 + h^2 + \Phi^2(t)} - \frac{1 - \dot{\Phi}^2(t_0)}{1 + h^2 + \Phi^2(t_0)} =$$

$$= \exp\Big(-2\frac{\sigma}{u}\int_{t_0}^{t} \frac{\dot{\Phi}\sqrt{1 - \dot{\Phi}^2(\eta)}}{\sqrt{1 + h^2 + \Phi^2(\eta)}} \cos\Psi(\eta) \cdot \exp\big[-((J + \Phi(\eta))/\beta u\rho)^2\big]\, d\eta\Big).$$

Whence it is obviously provides a strict inequality

$$\frac{1 - \dot{\Phi}^2(t)}{1 + h^2 + \Phi^2(t)} > \frac{1 - \dot{\Phi}^2(t_0)}{1 + h^2 + \Phi^2(t_0)} = m_0. \tag{8.2}$$

Let us note, that the equation is determined in the domain where $\dot{\Phi}^2 \leqslant 1$. The boundary value $\dot{\Phi}^2 = 1$ corresponds to exact (growing) solutions of the initial equation $\Psi(\tau) = (1 \pm 1/\beta)\tau + \text{const}$. On the strength of the uniqueness of the Cauchy problem the value $\dot{\Phi}^2 = 1$ is not obtained in other solutions. Consequently at any other solution the right side in (8.2) should be a positive value: $m_0 > 0$. Then from the inequality (8.2) we obtain the estimate $\Phi^2(t) < 1/m_0 < \infty$, $\forall\, t > t_0$. Therefore we prove, that for any solution of the equation (8.1) the phase trajectory $(\Phi(t), \dot{\Phi}(t))$ remains in the bounded domain where conditions of existence and uniqueness are satisfied. In this case for the equation (8.1) we can apply known theorems on existence of the global solution [16] on the semiaxis $t_0 < t < \infty$.

Subject to the formulae of substitution the property of boundedness $\Phi(t)$ results in the asymptotics $\Psi(\tau) = \tau + \mathcal{O}(1)$, $\tau \to \infty$, which shows a linear growth of any solution of the initial equation (1.1). The theorem has been proved.

2. The monotony of the capture domain.

Theorem 3. *At any moment θ the area under the separatrix $\Pi_c(\theta)$ decreases faster, than the area under the trajectory $\Pi(\theta)$ nearby the separatrix*

$$\frac{d\Pi_c}{d\theta} < \frac{d\Pi}{d\theta}\Big|_{\Pi=\Pi_c} < 0. \tag{8.3}$$

Proof. Differentiating the formula for the area under the separatrix provides the expression

$$\frac{d\Pi_c}{d\theta} = \sqrt{2}\int_{\psi_c^-}^{\psi_c} \frac{A'(\theta)[\sin\psi + \sin\psi_c] + B'(\theta)[\psi+\psi_c-2\pi]}{\sqrt{A(\theta)[\sin\psi+\sin\psi_c]+B(\theta)[\psi+\psi_c-2\pi]}}\,d\psi.$$

For the functions $A(\theta), B(\theta)$, determined in (3.1), the derivatives in the expression under the integral are calculated and provide the form

$$A'(\theta) = -(2\lambda\theta A + BA), \quad B'(\theta) = \frac{\mu-\nu^2}{(1+2\nu\theta+\mu\theta^2)^2} - B^2.$$

The considered interval of integration corresponds to the interior of the separatrix loop where the following inequalities hold $\sin\psi \geq \sin\psi_c,\ \psi \leqslant -\psi_c + 2\pi$. Therefore, subject to the property $\mu > \nu^2$ the derivatives from the coefficients at such an interval are estimated from above

$$A'(\theta)[\sin\psi - \sin\psi_c] \leqslant -BA[\sin\psi - \sin\psi_c], \quad B'(\theta)[\psi+\psi_c+2\pi] \leqslant -B^2[\psi+\psi_c+2\pi].$$

Therefore, for the integral we obtain the estimate

$$\frac{d\Pi_c}{d\theta} < -\sqrt{2}\,B(\theta)\int_{\psi_c^-}^{\psi_c}\sqrt{A(\theta)[\sin\psi-\sin\psi_c]+B(\theta)[\psi+\psi_c-2\pi]}\,d\psi =$$

$$= -\frac{1}{2}B(\theta)\cdot\Pi_c(\theta).$$

Whence the required inequality (8.3) is obtained by means of substitution of the right side on the strength of the equation (6.1). The theorem has been proved.

Corollary. *Slowly deforming trajectories cannot enter from without into the capture domain.*

This paper should be considered as a response of a mathematician to the papers of physicists, dialogue with whom is always profitable for mathematics. We express our gratitude to N.S. Erokhin and N.N. Zolnikova for getting acquainted with the problem of surfing and to A.I. Neistadt for discussion.

BIBLIOGRAPHY

1. Sagdeev R.Z., Shapiro V.D. *Influence of the crosscut magnetic field on the attenuation of Landau* // Letters to JETP 1973. V.17, issue. 7. P. 389–394.
2. N. Katsouleas, J. Dawson *Unlimited electron acceleration in laser-driven plasma wave* // Physical Review Letters, 51:5 (1983). P. 392–395.
3. B.E. Gribov, R.Z. Sagdeev, V.D. Shapiro, V.I. Shevchenko *Damping of plasma waves and acceleration of resonant electrons in a transverse magnetic field* // JETP Letters, 15:2 (1985). P. 63–68.
4. N.S. Erokhin, S.S. Moiseev, R.Z. Sagdeev *Relativistic surfatron in an inhomogeneous plasma and cosmic ray generation* // Soviet Astr. Lett.(TR:PISMA) 15:1 (1989) P.1.
5. Kichigin G.N. *Surfatron machanism of acceleration of cosmic rays in the Galactic plasma* // JETP 2001. V.119, issue. 6. Pp. 1038–1049.
6. N. Erokhin, N. Zolnikova, L. Mikhailovskaya, P. Trenchev *On the charged particles surfatron acceleration in space plasmas* // Comptes rendus de l'Acade'mie bulgare des Sciences, Vol 60, No9, pp.967-972
7. N.S. Erokhin, N.N. Zolnikova, P.P. Grinevich, L.A. Mikhailovskaya *Generation of fast particle fluxes by finite amplitude electromagnetic waves in space plasma Problems of Atomic Science and Technology* // series "Plasma electronics". 2006. V 60, 5. P. 152–156.

8. N.S. Erokhin, N.N. Zolnikova, E.A. Kuznetsov, L.A. Mikhailovskaya *Surfing of relative charges on the electromagnetic wave with smooth rounding amplitude. Transformation of waves, coherent structures and turbulence* // Collected articles of the conference MSS-09. Moscow. 2009. P. 92–99.
9. Pikovsky A., Rozenblum M., Kurts Yu *Synchronisation. Fundamental non-linear phenomenon.* M.: Technosphera. 2003.
10. Bogolubov N.N., Mitropolsky Yu.A. *Asymptotic methods in theory of non-linear oscillations.* M.: Nauka, 1974.
11. Arnold V.I., Kozlov V.V., Neistadt A.I. *Mathematical aspects of classical and celestial.* M.: VINITI. 1985.
12. Physics of plasma, 15, (1989), P. 593–594.
13. A.P. Itin, A.I. Neishtadt, A.A. Vasiliev *Capture into resonance in dynamics of a charged partice in magnetic field and electrostatic wave* // Physica D, **141:**4 (2000). P. 281–296.
14. Neistadt A.I., Artemyev A.V., Zeleny L.M., Weinstein D.L. *Surfatron acceleration in the electromagnetic waves with low phase velocity* // JETP Letters. 2009. V.89, issue.9. P. 528–534
15. Chirikov B.V. *Passing of the non-linear oscillatory system through resonance* // Reports of AS of USSR. 1959. V.125, 5. P. 1015–1018.
16. Nemytsky V.V., Stepanov V.V. *Quality theory of differential equations.* Izd. 3, corrected. M.:Editorial URSS, 2004.

Leonid Anatolyevich Kalyakin,

Institute of Mathematics with Computer Center of the Ufa Science Center of the Russian Academy of Sciences,

112, Chernyshevsky str.,

Ufa, Russia, 450008

E-mail: `klenru@mail.ru`