ISSN 2304-0122 Ufa Mathematical Journal. Volume 4. 2 (2012). Pp. 80-86.

UDC 517.928

PHASE SHIFT FOR THE COMMON SOLUTION OF THE KDV AND THE FIFTH ORDER DIFFERENTIAL EQUATION

R.N. GARIFULLIN

Abstract. We investigate the special solution of Korteweg–de Vries equation. This solution describes the influence of small dispersion on the process of transformation from weak to strong discontinuities in inviscid fluid dynamics. This solution also satisfies the fifth order ordinary differential equation. We construct the asymptotic solution in the Witham zone up to a phase shift. We obtain an equation for phase shift and, using the numerical experiments, we choose the concrete solution of this equation. This solution is a constant function.

Keywords: phase shift, Korteweg–de Vries equation, nondissipative shock waves.

1. Introduction

In their papers [1–3] A.M. Ilyin and S.V. Zaharov started to research the problem of influence of small dissipation on processes of transformation of weak discontinuities to strong ones. It is shown in these papers, that this process is in general described by special solution of the Burgers' equation. It is shown in the paper [4], that in problems with small the analogous role is played by two special solutions of the Korteweg–de Vries equation (KdV)

$$u_t + uu_x + u_{xxx} = 0. \tag{1.1}$$

In this paper we are intended to research one of them with the present asymptotic forms:

$$u\big|_{x\to-\infty} = 0, \qquad u\big|_{x\to\infty} = (-t - \sqrt{t^2+4x})/2. \tag{1.2}$$

The solution $u(x,t)$ plays a universal role [4] in problems about generation of nondissipative shock waves [4–6]. In the paper [4] to solve the problem (1.1,1.2) there is an asymptotic solution when $x^2+t^2\to\infty$, that is given by quasiprime solutions of the Whitham equations in the domain of continuous oscillation. However, there still was an indefinite phase shift in this solution. In this paper this phase shift is defined by the method, described in [7].

It is shown in [4], that the solution $u(x,t)$ satisfies an fifth-order ordinary differential equation by the variable x :

$$\left(u_{xxxx} + \frac{5u_{xx}u}{3} + \frac{5u_x^2}{6} + \frac{5u^3}{18}\right)'_x - \frac{2u + xu_x - 3t(u_{xxx}+uu_x)}{6} = 0. \tag{1.3}$$

The equation (1.3) corresponds to a combination of stationary parts of symmetries of KdV equations. One of them is the highest (generalized) symmetry of the fifth order:

$$u_{\tau_5} = \left(u_{xxxx} + \frac{5u_{xx}u}{3} + \frac{5u_x^2}{6} + \frac{5u^3}{18}\right)'_x, \tag{1.4}$$

the second one is a classical symmetry of extension:

$$u_{\tau_r} = 2u + xu_x - 3t(u_{xxx}+uu_x). \tag{1.5}$$

R.N. Garifullin, Phase shift for the common solution of the KdV and the fifth order differential equation.

Submitted on May, 19, 2012.

The equations (1.3) can be called the first highest analogue of the Painleve equation I, see 6.2 [4].

The article consists of two parts. In part one it is shown how the problem (1.1,1.2) generates during description of nondissipative shock waves. For this purpose we consider the Cauchy problem for perturbed generalized Hopf equation with initial data, experiencing a weak discontinuity. It is shown, that in the neighbourhood of the point of the gradient catastrophe for the general term of asymptotic there occurs the problem under study. In part two we construct an asymptotic solution of the problem (1.1,1.2) by method, applying presence of two equations, for which the equation $u(x,t)$ holds. For the indefinite phase shift we obtain an ordinary linear equation fo the third order. The certain solution of this equation is chosen with the help of numerical experiments: modelling of the solution $u(x,t)$ and constructed asymptotic solution.

2. Origin of the problem (1.1,1.2)

In the paper [4] the origin of the problem (1.1,1.2) is shown on the example of perturbed generalized Hopf equation, equations of shallow water and dispersion non-linear Schrodinger equation. The present paper describes the origin of this problem in more details.

Let us consider the Cauchy problem for the function $U(X,T)$:

$$\begin{aligned} &U_T + g(U)U_X + \varepsilon^3 U_{XXX} = 0,\\ &U(X,0) = F(x) = \begin{cases} F_-(x), & x<0,\\ F_+(x), & x\ge 0. \end{cases}, \quad F_-(0) = F_+(0). \end{aligned} \tag{2.1}$$

By means of substitution of variables and transformations ε wa can obtain:

$$F_-(0) = F_+(0) = 0, \quad g(0) = 0, \quad g'(0) = 1. \tag{2.2}$$

We impose conditions

$$\begin{gathered} g'(U) > 0, \quad F'_-(0) > F'_+(0), \quad F'_+(0) < 0,\\ F'(X)g'(F(X)) \notin [F'_+(0), 0], \ \forall x \neq 0, \end{gathered} \tag{2.3}$$

on the initial data, which ensures existence of a weak discontinuity of the initial data and occurrence of the gradient catastrophe (strong discontinuity) for the characteristic $X = 0$ in some moment of time T^* in the unperturbed equation ($\varepsilon = 0$).

We are intended to construct an asymptotic solution of the Cauchy problem (2.1) in the form of the series:

$$U(X,T) = U_0(X,T) + \varepsilon^3 U_1(X,T) + \dots \tag{2.4}$$

The dominant term and the first correction satisfy the problems:

$$\partial_T U_0 + g(U_0)\partial_X U_0 = 0, \quad U_0(X,0) = F(X), \tag{2.5}$$

$$\partial_T U_1 + g(U_0)\partial_X U_1 + g'(U_0)\partial_X U_0 U_1 + \partial_X^3 U_0 = 0, \quad U_1(X,0) = 0. \tag{2.6}$$

The solution of the problem (2.5) is written in the implicit form by the method of characteristics:

$$U_0 = F(X - g(U_0)T), \quad X \neq 0. \tag{2.7}$$

The solution of the problem (2.6) can be also obtained by the method of characteristics and it can be written explicitly in terms of the function $U_0(X,T)$.

The point of the gradient of catastrophe X^*, T^*, U^* is defined by the correlations:

$$T^* = -\frac{1}{F'_+(0)}, \quad U^* = 0, \quad X^* = 0 \tag{2.8}$$

Let us define behaviour of the solution $U_0(X,T)$, $U_1(X,T)$ in the neighbourhood of the line $X=0$, subject to limits for the initial data we have:

$$U_0(X,T)=\begin{cases}\dfrac{F'_+(0)}{1+TF'_+(0)}X+\dfrac{F''_+(0)-TF'_+(0)g''(0)}{2(1+TF'_+(0))^3}X^2+O(X^3), & X>0,\\ \dfrac{F'_-(0)}{1+TF'_-(0)}X+\dfrac{F''_-(0)-TF'_-(0)g''(0)}{2(1+TF'_-(0))^3}X^2+O(X^3), & X<0\end{cases}$$

$$U_1(X,T)=\begin{cases}\dfrac{U^+_{10}(T)}{(1+TF'_+(0))^4}+U^+_{11}(T)X+O(X^2), & X>0,\\ \dfrac{U^-_{10}(T)}{(1+TF'_-(0))^4}+U^-_{11}(T)X+O(X^2), & X<0\end{cases}$$

Is is seen, that the first derivative is not a discontinuous function, moreover, the first correction $U_1(X,T)$ experiences a disconnection in the point $X=0$ when $T>0$. Therefore, in the neighbourhood of the line $X=0$ it is necessary to asymptotic another way. In the neighbourhood of this line it is needed to make an extension of the variables:

$$U(X,T,\varepsilon)=\varepsilon V(y,T,\varepsilon),\quad x=\varepsilon y. \tag{2.9}$$

After this the problem (2.1) in new variables takes the form:

$$\begin{gathered}V_T+VV_y+V_{yyy}+(U(\varepsilon V)/\varepsilon-V)V_y=0,\\ V(y,0)=F(\varepsilon y)/\varepsilon=\begin{cases}F'_-(0)y+\varepsilon F''_-(0)y^2+\dots, & y<0,\\ F'_+(0)y+\varepsilon F''_+(0)y^2+\dots, & y>0.\end{cases}\end{gathered} \tag{2.10}$$

Formal asymptotic of the function V can be constructed in the form of the series:

$$V(y,T,\varepsilon)=V_0(y,T)+\varepsilon V_1(y,t)+\dots. \tag{2.11}$$

The problem for the dominant term of the asymptotic is as follows:

$$\begin{gathered}V^0_T+V^0V^0_y+V^0_{yyy}=0,\\ V^0(y,0)=\begin{cases}F'_-(0)y, & y<0,\\ F'_+(0)y, & y>0.\end{cases}\end{gathered} \tag{2.12}$$

In case $F'_-(0)=0$ existence of solution of this problem on the section $T\in(0,T^*)$ was proved by Faminsky [8]. To solve this problem (2.12) the asymptotic, resulting from necessity of matching with the decomposition for the function $U(X,T)$

$$V(y,T)\to\frac{F'_+(0)}{1+TF'_+(0)}y, y\to+\infty,\quad V(y,T)\to 0, y\to-\infty$$

holds. Though, in the neighbourhood of the point of the gradient catastrophe X^*, T^* decompositions for $U(X,T)$ and for $V(y,T)$ do not hold. In the neighbourhood of this point there is another decomposition needed.

We study the neighbourhood of the point $(0,T^*)$ with the help of the external decomposition (2.4). Let us write the behaviour of the solution $U_0(X,T)$ in the neighbourhood of the point of the gradient catastrophe in the left and in the right from the line $X=0$:

$$X-\left(g''(0)+F''_+(0)(T^*)^2\right)U^2/2-U(T-T^*)+\dots=0,\ X>0 \tag{2.13}$$

$$X-\left(\frac{1}{F'_-(0)}+T\right)U+\dots=0,\ X<0. \tag{2.14}$$

Here and below the constant

$$\delta=(g''(0)+F''_+(0)(T^*)^2)/2>0 \tag{2.15}$$

is considered to be positive, due to the condition (2.3) it is not negative, and it is positive in general position.

With respect to the method of matching of asymptotic decompositions [?] we are intended to carry out expansion of variables in the neighbourhood of the point of the gradient catastrophe:

$$U(X,T)=a\varepsilon^{\alpha}u,\quad T-T^{*}=b\varepsilon^{\beta}t,\quad X=c\varepsilon^{\gamma}x. \tag{2.16}$$

The equation (2.1) and formulae (2.14) with new variables take the form:

$$\begin{aligned}&\frac{a}{b}\varepsilon^{\alpha-\beta}u_t+\frac{a^2}{c}\varepsilon^{2\alpha-\gamma}uu_t+\frac{a}{c^3}\varepsilon^{3+\alpha-3\gamma}u_{xxx}+\frac{a^3}{c}\varepsilon^{3\alpha-\gamma}g''(0)u^2u_x+\ldots=0,\\&c\varepsilon^{\gamma}x-\delta a^2\varepsilon^{2\alpha}u^2-ab\varepsilon^{\alpha+\beta}ut+\ldots=0,\quad x\gg 1,\\&c\varepsilon^{\gamma}x-\left(\frac{1}{F'_{-}(0)}+T^{*}\right)a\varepsilon^{\alpha}u+\ldots=0,\quad x\ll -1.\end{aligned} \tag{2.17}$$

Having demanded the equality of coefficients relative to the first three summands in the first two equations, we obtain the system:

$$\alpha-\beta=2\alpha-\gamma=3+\alpha-3\gamma,\quad \gamma=2\alpha=\alpha+\beta,$$
$$a/b=a^2/c=a/c^3,\quad c=\delta a^2=ab.$$

This system has the following solution:

$$\alpha=\beta=\frac{3}{5},\quad \gamma=\frac{6}{5},\quad a=\delta^{-2/5},\quad b=\delta^{3/5},\quad c=\delta^{1/5}.$$

After the expansion (2.16) with the given parameters, the equations (2.17) take the form:

$$u_t+uu_x+u_{xxx}+O(\varepsilon^{3/5})=0,$$
$$x-u^2-ut+O(\varepsilon^{3/5})=0, x\gg 1,\quad u+O(\varepsilon^{3/5})=0, x\ll -1.$$

This form in general coincides with the problem (1.1,1.2).

3. Defining the phase shift

The asymptotic solution of the problem (1.1,1.2,1.3) when $t\to\infty$ consists of several parts [4]. The zone of continuous oscillations is of special interest. Let us make a natural substitution of the variables

$$u=t\,U(t,s),\quad s=\frac{x}{t^2}.$$

After it the equations (1.1,1.3) take the form:

$$t^{-5}U_{sss}+tU_t-2sU_s+UU_s+U=0, \tag{3.1}$$

$$t^{-10}U_{sssss}+\frac{1}{6}t^{-5}(20U_sU_{ss}+(10U+3)U_{sss})+\frac{1}{6}(5U^2-s+3U)U_s-\frac{1}{3}U=0. \tag{3.2}$$

In the equation (3.2) all derivatives by the variable x of the third and higher order can be substituted due to the equation (3.1):

$$2t^{-5}((U_s+9)U_{ss}+6sU_{sss})-6t^{-4}U_{sst}+(5s+U^2+8sU)U_s-(4U+3)tU_t-(5+4U)U=0. \tag{3.3}$$

The asymptotic solution U of this system (3.1,3.3) is constructed in the form of the series by inverse power series t

$$U=U_0(\varphi,s)+t^{-5/4}U_1(\varphi,s)+t^{-5/2}U_2(\varphi,s)+\ldots,\quad t\to\infty. \tag{3.4}$$

Here U_0, U_1 and U_2 are 2π periodic functions of the fast variable φ. This variable has the form

$$\varphi=t^{5/2}f(s)+n(s),$$

where $n(s)$ is the required phase shift.

We obtain the following non-linear system of equations by the fast variable φ for the function U_0:

$$\begin{aligned} (f')^2\partial_\varphi^3 U_0 + (a(s)+U_0)\,\partial_\varphi U_0 = 0,\\ 6a(s)(f')^2\partial_\varphi^3 U_0 - 2(f')^2\partial_\varphi^2 U_0\partial_\varphi U_0 + \partial_\varphi U_0\left(s - U_0^2 + 4a(s)U_0 + 3a(s)\right) = 0. \end{aligned} \tag{3.5}$$

For the function U_1, U_2 there are linear nonuniform systems of equations. The first from the equations on U_1 has the form:

$$\begin{aligned} (f')^2\partial_\varphi^3 U_1 + (a(s)+U_0)\,\partial_\varphi U_1 + \partial_\varphi U_0 U_1 = -3f'n'\partial_\varphi^3 U_0\\ +(2s-U_0)\frac{\partial_\varphi U_0 n' + \partial_s U_0}{f'} - 3\partial_s(f'\partial_\varphi^2 U_0) - \frac{U_0}{f'}. \end{aligned} \tag{3.6}$$

It is denoted here:

$$a(s) = \frac{5f}{2f'} - 2s.$$

If we exclude the expression $\partial_\varphi^3 U_0$ from (3.5), we obtain an equation of the second order for the function U_0:

$$(f')^2\partial_\varphi^2 U_0 + \frac{1}{2}U_0^2 + a(s)U_0 + 3a(s)^2 - \frac{s+3a(s)}{2} = 0. \tag{3.7}$$

The equation (3.7) can be once integrated:

$$(f'\partial_\varphi U_0)^2 + \frac{1}{3}U_0^3 + a(s)U_0^2 + (6a^2 - 3a - s)U_0 + b(s) = 0. \tag{3.8}$$

Here $b(s)$ is an arbitrary function (constant of integration).

Further it is suggested not to explicitly write the solution U_0, but simply consider, that it is some 2π periodic function, satisfying the equation (3.8). Subject to this equation, we can write all derivatives from U_0 as rational-fractional expressions in terms:

$$U_0, \partial_\varphi U_0, \partial_s U_0, \partial_s^2 U_0, \dots.$$

The equations on U_1 take the form:

$$\begin{aligned} (f')^2\partial_\varphi^3 U_1 + (a(s)+U_0)\,\partial_\varphi U_1 + \partial_\varphi U_0 U_1 = \frac{F_1(U_0, \partial_\varphi U_0, \partial_s U_0, a, a', n', s)}{f}\\ 6a(s)(f')^2\partial_\varphi^3 U_1 - 2(f')^2(\partial_\varphi^2 U_1\partial_\varphi U_0 + \partial_\varphi^2 U_0 \partial_\varphi U_1) + \partial_\varphi U_1(s - U_0^2 + 4aU_0 + 3a)\\ +2\partial_\varphi U_0(2a - U_0)U_1 = \frac{F_2(U_0, \partial_\varphi U_0, \partial_s U_0, a, b, a', b', n', s)}{f\partial_\varphi U_0}. \end{aligned} \tag{3.9}$$

Here F_1, F_2 are polynomial functions of their arguments. Excluding from the system (3.9) sequentially higher derivatives U_1 by the variable φ, we obtain the correlation, which does not contain the function U_1 which is the condition of compatibility of this system:

$$\begin{aligned} \big(3(2s+a)(-2s-24a+3+36a^2)a' + (4s+2a)b' + 6sa - 4s - 27a + 108a^2 - 6b\\ -108a^3\big)\,U_0 + 3(2s+a)(-72a^3 + 54a^2 - 9a + 12sa + 4b - 3s)a' + 3(4a-1)(2s+a)b'\\ +45a^2 - 36sa^2 + 216a^4 + 15b - 198a^3 + 15sa - 48ab = 0. \end{aligned} \tag{3.10}$$

Whereas the equality (3.10) should hold identically, the coefficients with different degrees of U_0 are equal to 0, consequently, we obtain an equation for $a(s), b(s)$:

$$\begin{aligned} a' &= \frac{(2a-1)(-288a^3 + 192a^2 + 24sa - 27a - 4s + 4b)}{(2s+a)(-576a^3 + 504a^2 - 126a + 48sa + 8b - 12s + 9)},\\ b' &= (3s - 54a^2 + 36a - 9/2)a' + \frac{-6sa + 4s - 108a^2 + 27a + 6b + 108a^3}{4s+2a}. \end{aligned} \tag{3.11}$$

The system (3.9) is compatible if and only if $a(s)$ and $b(s)$ are defined from the equations (3.11). If this condition holds, then all derivatives by φ from U_1 which are higher than the second order, can be expressed by lower derivatives, for example:

$$(f')^2\partial_\varphi^2 U_1 = -(U_0+a)U_1 + (n' + \partial_s U_0/\partial_\varphi U_0)G_1(U_0,a,s)/s + G_2(U_0,a,b,s)/f/\partial_\varphi U_0,$$

where G_1, G_2 are some functions.

The equations on U_2 take the form :

$$\begin{aligned}
&(f')^2\partial_\varphi^3 U_2 + (a(s)+U_0)\,\partial_\varphi U_2 + \partial_\varphi U_0 U_2 = \frac{F_3}{f}\\
&6a(s)(f')^2\partial_\varphi^3 U_2 - 2(f')^2(\partial_\varphi^2 U_2\partial_\varphi U_0 + \partial_\varphi^2 U_0\partial_\varphi U_2) + \partial_\varphi U_2(s - U_0^2 + 4aU_0 + 3a)\\
&\quad +2\partial_\varphi U_0(2a-U_0)U_2 = \frac{F_4}{f\partial_\varphi U_0}.
\end{aligned} \tag{3.12}$$

Here F_3, F_4 are functions, depending on previous corrections.

Excluding from these equations consequently derivatives of the function U_2, we obtain correlation of the form:

$$\begin{aligned}
&\partial_{\varphi s}U_1 - \frac{\partial_\varphi^2 U_0}{\partial_\varphi U_0}\partial_s U_1 + \left(\frac{\partial_\varphi^2 U_0\partial_s U_0}{(\partial_\varphi U_0)^2} + \frac{G_3(U_0,a,b)}{(f\partial_\varphi U_0)^2(12a+2U_0-3)}\right)\partial_\varphi U_1\\
&-\left(\frac{\partial_\varphi^3 U_0\partial_s U_0}{(\partial_\varphi U_0)^2} - \frac{G_4(U_0,a,b)}{(f\partial_\varphi U_0)^2(12a+2U_0-3)}\right)U_1 = G_5(U_0,a,b,n',n'').
\end{aligned} \tag{3.13}$$

Differentiating this equation by φ, we obtain correlation of the same form, excluding from these two equations $\partial_{\varphi s}U_1$, we obtain:

$$\partial_\varphi U_1 = \frac{\partial_\varphi^2 U_0}{\partial_\varphi U_0}U_1 + \frac{n''G_6(s,a,b,f) + n'G_7(s,a,b,f)}{\partial_\varphi U_0} + G_8(\partial_s^3 U_0, \partial_s^2 U_0, \partial_s U_0, \partial_\varphi U_0, U_0, a, b, f, s). \tag{3.14}$$

Substituting it into the equation (3.13), we obtain the correlation of the form:

$$\begin{aligned}
\partial_\varphi U_0(n''' + A_1 n'' + A_2 n') + \partial_s^3 U_0 + B_1\partial_s^2 U_0\partial_s U_0 + B_2\partial_s^2 U_0 &\ +\\
B_3(\partial_s U_0)^3 + B_4(\partial_s U_0)^2 + B_5\partial_s U_0 + B_6 &= 0,
\end{aligned} \tag{3.15}$$

where

$$A_i = A_i(s,f,a,b), \quad B_i = B_i(U_0,s,f,a,b)$$

are some functions.

If we do not limit generality, we can consider the function U_0 even by φ. Then, in (3.15) the first part is odd, the second in even by φ. Consequently, we immediately obtain from (3.15) two equations:

$$n''' + A_1 n'' + A_2 n' = 0. \tag{3.16}$$

$$\partial_s^3 U_0 + B_1\partial_s^2 U_0\partial_s U_0 + B_2\partial_s^2 U_0 + B_3(\partial_s U_0)^3 + B_4(\partial_s U_0)^2 + B_5\partial_s U_0 + B_6 = 0. \tag{3.17}$$

The general solution (3.16) takes the form:

$$n(s) = C_1 + C_2 n_1(s) + C_2 n_2(s). \tag{3.18}$$

Here n_1, n_2 are different from constant linearly independent solutions (3.16). If we apply numerical methods, we obtain:

$$n(s) = \pi.$$

It is numerically demonstrated, that the difference between numerical and asymptotic solution reduces as $t^{-5/2}$ for this solution $n(s)$. The Figure 1 presents a numerical modelling of the solution $U(t,z)$ where $t = 19$ and the dominant term of the asymptotic $U_0(\varphi, s)$.

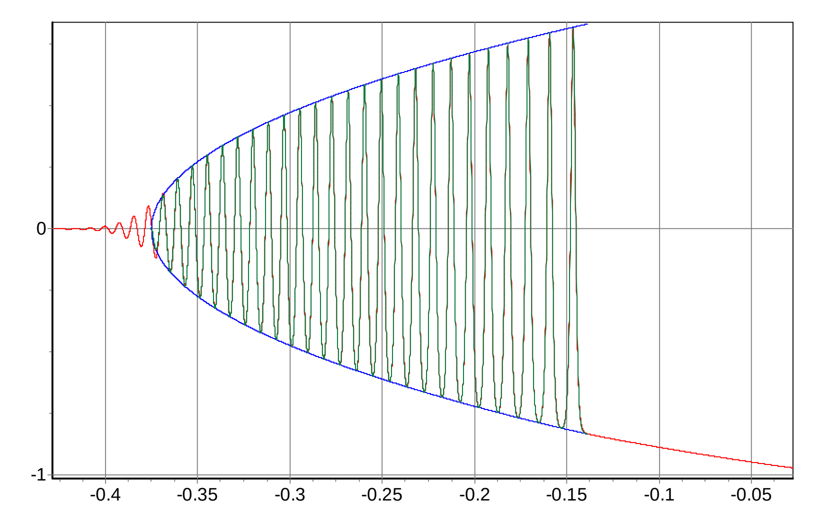

FIGURE 1. Numerical modelling of the function $U(t, z)$ where $t = 19$

BIBLIOGRAPHY

1. A.M. Il'in, S.V. Zakharov *On the influence of small dissipation on the evolution of weak discontinuities* // International Conference on Differential and Functional Differential Equations (Moscow, 1999). Funct. Differ. Equ. 8 (2001), no. 3–4. P. 257–271.
2. Zaharov S.V., Ilyin A.M. *From weak disconnection to gradient catastrophe* // Math. issue, 192:10 (2001). P. 3–18 .
3. Zaharov S.V. *Generating of the shock wave in one Couchy problem and Burgers' equation* // Journal of counting math. and math. physics., 44:3 (2004). P. 536–542.
4. Garifullin R.N., Suleimanov B.I. *From weal disconnections to nondissipative shock waves* // JETP. 2010. 137, issue. 1. P. 149–164.
5. Kamchatnov A.M., Korneev S.V. *Flow of Bose–Einstein condensate in quazi-one-dimentional canal under plunger influence* // JETP. 2010. 137, issue. 1. P. 191–204.
6. G.A. El, V.V. Khodorovskii, A.M. Leszczyszyn *Refraction of dispersive shock waves* arXiv:1105.1920v1 [nlin.PS]
7. R. Garifullin, B. Suleimanov, N. Tarkhanov *Phase Shift in the Whitham Zone for the Gurevich-Pitaevskii Special Solution of the Korteweg-de Vries Equation* Ph. Let. A 374 (2010). P. 1420–1424, DOI:10.1016/j.physleta.2010.01.057.
8. Faminsky A.V. *Couchy problem for the Korteweg–de Vries equation for an uneven unbounded initial function* // Math. notes, 83:1 (2008). P. 119–128 .

Rustem Nailevich Garifullin,
Institute of Mathematics with Computer Center of the Ufa Science Center of the Russian Academy of Sciences,
112, Chernyshevsky str.,
Ufa, Russia, 450008
E-mail: rustem@matem.anrb.ru