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PERIODIC SOLUTIONS OF THE TELEGRAPH EQUATION WITH A DISCONTINUOUS NONLINEARITY

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Abstract. We consider telegraph equations with a variable inner energy, discontinuous by phase, and the homogeneous Dirichlet boundary condition. Question of existence of general periodic solutions in the resonant case, when the operator created by a linear part of the equation with the homogeneous Dirichlet boundary condition and the condition of periodicity has a non zero kernel, and nonlinearity appearing in the equation is limited. We obtained an existence theorem for the general periodic solution by means of the topological method. The proof is based on the Leray-Schauder principle for convex compact mappings. The main difference from similar results of other authors is an assumption that there are breaks in the phase variable of the inner energy of the telegraph equation.

Keywords: nonlinear telegraph equation, discontinuous nonlinearity ,periodic solutions, resonance problem

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^n$ with the boundary $\partial\Omega$ of class C^2,

$$Lu(x) = -\sum_{i,j=1}^{n} (a_{ij}(x)u_{x_i})_{x_j} + a(x)u(x)$$

be a uniformly elliptic differential operator in the domain Ω [1] with coefficients $a_{ij} \in C^{1,\alpha}(\overline{\Omega}),\ a_{ij}(x) = a_{ji}(x),\ a \in C^{\alpha}(\overline{\Omega}),\ 0 < \alpha < 1$.

We consider the problem of existence of the solution of telegraph equation with discontinuous nonlinearity

$$u_{tt} + Lu(x,t) + \mu u_t + g(x,t,u) = f(x,t),\ (x,t) \in Q, \tag{1}$$

satisfying homogeneous Dirichlet boundary condition

$$u(x,t) = 0 \tag{2}$$

on $S = \partial\Omega \times (0, 2\pi)$, and the condition of periodicity

$$u(x,0) = u(x,2\pi) \tag{3}$$

for $x \in \Omega$, where $Q = \Omega \times (0, 2\pi),\ \mu \neq 0$ (considering dissipation of energy), $f \in L^2(Q)$.

It is assumed, that nonlinearity $g(x,t,u)$ satisfies the i-condition:

$i1$ is the function $g : Q \times \mathbb{R} \to \mathbb{R}$ of Borel (mod 0) [2], that denotes existence of the set $l \subset Q \times \mathbb{R}$, which projection on Q has a zero measure, and Borel on $Q \times \mathbb{R}$ function, coinciding with $g(x,t,u)$ on $(Q \times \mathbb{R}) \setminus l$;

$i2$ — for almost all $(x,t) \in Q$ the section $g(x,t,\bullet)$ has on $\mathbb{R}$ discontinuities just of the first kind and for the arbitrary $u \in \mathbb{R}$ the following inclusion holds $g(x,t,u) \in [g_-(x,t,u), g_+(x,t,u)]$, where $g_-(x,t,u) = \liminf_{\eta\to u} g(x,t,\eta),\quad g_+(x,t,u) = \limsup_{\eta\to u} g(x,t,\eta)$;

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$i3$ — (boundedness of nonlinearity) there is the function $b(x,t)$ from $L^2(Q)$ such that for almost all $(x,t) \in Q$

$$|g(x,t,u)| \leqslant b(x,t) \quad \forall u \in \mathbb{R}. \tag{4}$$

Let us note, that the condition $i1$ ensures superposition measurability $g(x,t,u)$ on Q, that is measurability on Q of the composition $g(x,t,u(x,t))$ for any function $u(x,t)$ measurable on Q.

The differential operator L with homogeneous Dirichlet boundary condition generates in $L^2(\Omega)$ a self-adjoint linear operator B with the definitional domain $D(B) = H^2(\Omega) \cap H_0^1(\Omega) : Bu = Lu \ \forall u \in$ where all the derivatives of the function $u(x)$ are the Sobolev ones. By $H^m(\Omega)$ $(m \in \mathbb{N})$ we denote Sobolev space $W_2^m(\Omega)$[1], and by $H_0^m(\Omega)$ we denote closure of the set of continuously differentiated finite in Ω functions in the metric $H^m(\Omega)$. The spectre σ of the operator B consists of characteristic constants of the finite order

$$\lambda_0 \leqslant \lambda_1 \leqslant \lambda_2 \leqslant \ldots; \quad \lambda_j \to \infty.$$

[3]. Here every characteristic constant is repeated the number of times its order demands. There is the orthonormalized basis (v_j) in $L^2(\Omega)$ from the characteristic constants of the operator B $(Bv_j = \lambda_j v_j)$. In the complex space $L^2(Q)$ the sequence $\{\psi_{jk}(x,t) = \frac{1}{\sqrt{2\pi}} v_j(x) e^{ikt}$, $j = 0, 1, 2, \ldots;\ k \in \mathbb{Z}\}$ is an orthonormalized basis. For any real-value function $u \in L^2(Q)$

$$u(x,t) = \sum_{k \in \mathbb{Z}} \sum_{j=0}^{\infty} a_{jk} \psi_{jk}(x,t), \quad a_{j,-k} = \overline{a_{j,k}}.$$

Assume $D(A_0) = \{u(x,t) = \sum_{k=-m}^{m} \sum_{j=0}^{n} a_{jk}\psi_{jk}(x,t) |\ a_{j,-k} = \overline{a_{j,k}},\ \ m,n \in \ \mathbb{N} \cup \{0\}\}$ and define the operator $A_0 : D(A_0) \subset L^2(Q) \to L^2(Q)$ in the real $L^2(Q)$ by the equality $A_0 u = u_{tt} + \mu u_t + Lu(x,t)$ for any $u \in D(A_0)$. Let us note, that the formula, which defines A_0, can set the extension A_0 for the linear shell of the sequence $(\psi_{jk}(x,t))$ in complex $L^2(Q)$, and for this expansion $\psi_{jk}(x,t)$ are eigenfunctions which satisfy the characteristic constants $\mu_{jk} = -k^2 + \lambda_j + i\mu k$. In particular, this implies, that the kernel of the operator A_0 $(Ker A_0)$ coincides with $Ker B$.

Definition 1. *The generalized solution of the problem* (1)-(3) *is the function* $u(x,t) \in L^2(Q)$ *with values in* $\mathbb{R}$ *such that there is the function* $z(x,t) \in [g_-(x,t,u(x,t)), g_+(x,t,u(x,t))]$, *measurable almost everywhere on* Q, *for which the following integral identity holds:*

$$\int_Q u(x,t)(\varphi_{tt} + L\varphi - \mu\varphi_t)dxdt = \int_Q \varphi(x,t)(f(x,t) - z(x,t))dxdt \ \forall \varphi \in D(A_0). \tag{5}$$

Remark 1. In case, when $g(x,t,u)$ is a caratheodory function, that is for almost all $(x,t) \in Q$ the section $g(x,t,\bullet)$ is continuous on $\mathbb{R}$ and for any $u \in \mathbb{R}$ the function $g(\bullet,\bullet,u)$ is measurable on Q, in the definition $z(x,t) = g(x,t,u(x,t))$, and we come to the accepted notion of the generalized solution of the problem (1)–(3). It is shown in [4], that if $u \in L^2(Q)$ satisfies (5) with $r(x,t) = f(x,t) - z(x,t) \in L^2(Q)$, then $u(x,t) \in H_0^1(\Omega)$ for $t \in [0, 2\pi]$ (regularity of the generalized solution) does hold (3). If we assume, that the generalized solution $u(x,t) \in H^2(Q)$, then with the help of part integrating in (5) we can obtain, that $u_{tt} + Lu(x,t) + \mu u_t + z(x,t) = f(x,t)$ almost everywhere on Q.

The general result of the paper is the following theorem (it considers a resonance problem when the equation $u_{tt} + Lu(x,t) + \mu u_t = 0$ possesses in Q nontrivial solution, satisfying the conditions (2) and (3), that is equivalent to null membership to the spectre σ of the operator B).

Theorem 1. *Let us assume, that $0 \in \sigma$, function $g(x,t,u)$ satisfies i - condition. Moreover, for any function $v(x)$ from the kernel of the operator B the Landesmann - Lazer condition holds*

$$\int\limits_{v>0} \underline{g}_+(x,t)v(x)dxdt + \int\limits_{v<0} \overline{g}_-(x,t)v(x)dxdt > \int\limits_{\Omega} f(x,t)v(x)dxdt,$$

where $\underline{g}_+(x,t) = \liminf\limits_{u\to+\infty} g(x,t,u)$, $\overline{g}_-(x,t) = \limsup\limits_{u\to-\infty} g(x,t,u)$.
Then the problem (1)–(3) has a generalized solution $u(x,t) \in L^2(Q)$.

The proof of Theorem 1 is reduced to the problem of existence of a fixed point in a convex value compact mapping. The existence of a fixed point is set with the help of Leray-Schauder principle for multivalued mappings [5].

The question of existence of periodic solutions of the telegraph equation with nonlinear internal energy has been studied by many authors. The problem (1)–(3) with caratheodory nonlinearity $g(x,t,u)$ of the linear growth, symmetric elliptic part L of the order 2m with independent of the time coefficients was considered in the combined work of Brezis and Nirenberg [4] (the condition (2) in this case is substituted by the membership $u(x,t)$ to $H_0^m(\Omega)$ for any $t \in (0,T)$). In the resonance case, when the problem $Lu = 0, u \in H_0^m(\Omega)$ has a non-nil solution, we obtain the theorem of existence of a generalized solution under the harder constraint for f, than the Landesmann - Lazer condition in Theorem 1. We research here regularity of the generalized case $m = 2$. In particular, it is shown, that if $f \in L^2(Q)$, then $u(x,t) \in H_0^1(\Omega)$ for any $t \in (0,T)$.

In the paper of I.A. Rudakov [6] the problem (1)–(3) with caratheodory nonlinearity $g(x,t,u)$ of the degree growth is considered for $n = 1$, $Lu = -u_{xx}$ with the supplementary term νu_x in the nonresonance case. We also prove the existence of the generalized solution and research its regularity. Let us also note the papers [7],[8], in which the problem of existence of periodic solutions of the nonlinear telegraph equation is studied in resonance case when $n = 1$, $Lu = -u_{xx}$. The general difference of the present research from the works of other authors is assumption of discontinuities in $g(x,t,u)$ by the phase variable u.

2. The operator formulation of the problem(1)–(3)

Let us denote $A : D(A) \subset L^2(Q) \to L^2(Q)$ the closure of the operator A_0. As it is shown in [4],

$$D(A) = \{u(x,t) = \sum_{k\in\mathbb{Z}} \sum_{j=0}^{\infty} a_{jk}\psi_{jk}(x,t)| \;\; a_{j,-k} = \overline{a_{j,k}},$$

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty} |a_{j,k}|^2((\lambda_j - k^2)^2 + \mu^2 k^2) < +\infty\},$$

and for any $u \in D(A)$ the value $Au = \sum_{k\in\mathbb{Z}}\sum_{j=0}^{\infty} \mu_{jk}a_{jk}\psi_{j,k}(x,t)$. The real spectre of the operator A coincides with σ(the spectre of the operator B), $D(A^*) = D(A)$ and $KerA^* = KerA$ (A^* - operator conjugated with A),

$$A^*u = u_{tt} + Lu_t - \mu u_t,$$

for any $u \in D(A_0)$.

For $\lambda \notin \sigma, \lambda \in \mathbb{R}$ the resolvent of the operator A

$$(A - \lambda I)^{-1}u = \sum_{k\in\mathbb{Z}}\sum_{j=0}^{\infty} \frac{a_{jk}}{\mu_{jk} - \lambda}\psi_{j,k}(x,t),$$

Since $\frac{1}{\mu_{jk}-\lambda} \to 0$, when $j + k \to +\infty$, then the operator $(A - \lambda I)^{-1}$ compact in $L^2(Q)$.

Let us denote the Nemytsky operator G by the equality

$$Gu = g(x, t, u(x, t)), \quad \forall u \in L^2(Q).$$

Whereas $g(x, t, u)$ satisfies $i1$ and $i3$ conditions, then the operator G functions from $L^2(Q)$ to $L^2(Q)$, and the following estimate holds for it:

$$\|Gu\| \leqslant \|b\|, \quad \forall u \in L^2(Q), \tag{6}$$

here and further $\|\|$ is the norm in $L^2(Q)$. Let us denote convexity of the operator G by $G^{\square}$:

$$G^{\square}u = \bigcap_{\varepsilon<0} clco\{y = Gz \mid \ \|z - u\| < \varepsilon\},$$

where $clco\Lambda$ is a closed convex hull of the set $\Lambda \subset L^2(Q)$.

Let us consider the inclusion

$$f - Au \in G^{\square}u. \tag{7}$$

Its validity denotes existence of $z \in G^{\square}u$ such that

$$f - Au = z. \tag{8}$$

As it is shown in [2], $z \in G^{\square}u$ is equivalent to the fact, that the function $z(x, t)$ measurable on Q and for almost all $(x, t) \in Q$ $\ z(x, t) \in [g_-(x, t, u), g_+(x, t, u)]$. It results from the equality (8), that u is a generalized solution of the problem (1)–(3). Let us prove it. Let us denote $(\cdot, \cdot)$ to be a scalar product in $L^2(Q)$. For any $\varphi \in D(A_0)$ we have the equality $(Au, \varphi) + (z, \varphi) = (f, \varphi)$, that is equivalent $(u, A^*\varphi) + (z, \varphi) = (f, \varphi)$ $\forall \varphi \in D(A_0)$ (by definition of the adjoint operator), and it is equivalent to the integral identity

$$\int\limits_Q u(x, t)(\varphi_{tt} + Lu - \mu u_t)dxdt + \int\limits_Q z(x, t)\varphi(x, t)dxdt = \int\limits_Q f\varphi(x, t)dxdt,$$

for all $\varphi \in D(A_0)$. Assume $\varepsilon > 0$ and $[-\varepsilon, 0) \cap \sigma = \varnothing$ (such ε exists whereas the eigenvalues of the operator B are isolated). Let us transform the inclusion (7):

$$f - Au - \varepsilon u \in G^{\square}u - \varepsilon u$$

or

$$(A + \varepsilon I)u \in f - G^{\square}u + \varepsilon u,$$

the latter is equivalent to the inclusion

$$u \in (A + \varepsilon I)^{-1}(f - G^{\square}u + \varepsilon u) \equiv T.$$

Let us consider the properties of the mapping T. Let us prove, that the values T are convex compact sets in $L^2(Q)$. The values $G^{\square}$ are bounded and closed in $L^2(Q)$, and the operator $(A + \varepsilon I)^{-1} : L^2(Q) \to L^2(Q)$ is linear and compact. Therefore the values T are convex and precompact sets. To prove the compactness of Tu for $u \in L^2(Q)$, it is sufficient to set the closure Tu in $L^2(Q)$. Assume the sequence $(z_m) \subset Tu$ and $z_m \to z$ in $L^2(Q)$. Then there is $(y_m) \subset G^{\square}u$ such that $z_m = (A+\varepsilon I)^{-1}(f - y_m + \varepsilon u)$. This results in the equality $y_m = -(A+\varepsilon I)z_m + f + \varepsilon u$. From the boundedness of the set $G^{\square}u \subset L^2(Q)$ we derive existence of the subsequence (y_{m_k}), weakly converging to some y in $L^2(Q)$. Since $(y_{m_k}) \subset G^{\square}u$, and $G^{\square}u$ is a closed convex set, then $y \in G^{\square}u$. On the strength of the closure of the linear operator $(A + \varepsilon I)$ its graph in $L^2(Q) \times L^2(Q)$ is weakly closed, therefore $z \in D(A + \varepsilon I)$ and $y = -(A + \varepsilon I)z + f + \varepsilon u$, and, then, $z = (A + \varepsilon I)^{-1}(f - y + \varepsilon u) \in Tu$. The closure of the set Tu in $L^2(Q)$ has been set.

Let us demonstrate semi-continuity from the top of the mapping T on $L^2(Q)$. Assume the contrary and then we obtain $u \in L^2(Q)$ and the open set $D \supset Tu$ in $L^2(Q)$ such that for any $m \in \mathbb{N}$ there is $u_m \in L^2(Q)$ with $\|u_m - u\| < m^{-1}$ and $z_m \in Tu_m \backslash D$. Every element of (z_m) is presented in the form $z_m = (A + \varepsilon I)^{-1}(f - v_m + \varepsilon u_m)$, $\ v_m \in G^{\square}(u_m)$. Since the sequence (u_m) is limited in $L^2(Q)$, and the mapping $G^{\square}$ turns bounded sets into bounded (on the strength of the estimate (6)), then the sequence (v_m) is limited in $L^2(Q)$. This implies the existence of the weakly converging subsequence (v_{m_k}) to some v in $L^2(Q)$. Whereas $u_m \to u$ in $L^2(Q)$, then on the strength of the weak-strong closure $G^{\Box}$[9] we derive $v \in G^{\Box}(u)$. Since $(A+\varepsilon I)^{-1}$ is a linear compact operator, then $(A+\varepsilon I)^{-1}v_{m_k} \to (A+\varepsilon I)^{-1}v$. Therefore $z_{m_k} \to (A+\varepsilon I)^{-1}(f - v + \varepsilon u) \in Tu \subset D$. This implies, that since D is an open set in $L^2(Q)$, we conclude, that z_{m_k} belongs to D for sufficiently large k, that contradicts the choice z_m. The semi-continuity on the top of the mapping T on $L^2(Q)$ has been proved.

The multivalued operator $G^{\Box}$ turns bounded sets in $L^2(Q)$ into bounded, and the operator $(A+\varepsilon I)^{-1}$ is rather continuous, therefore for the arbitrary sphere U from $L^2(Q)$ its image TU is the precompact set in $L^2(Q)$. Therefore, the values of the multi-mapping T in $L^2(Q)$ are convex compacts, T is semi-continuous on the top, and any sphere U from $L^2(Q)$ is turned into precompact set by the mapping T.

3. Proof of Theorem 1

Since the mapping T is convex-valued and compact, then to prove the existence of its fixed point it is sufficient to state an equivalent boundedness of the set of the solutions of the family of inclusions $u \in \tau Tu$, $0 \leqslant \tau < 1$ ([5], p.107). Let us assume the contrary. Then there are the sequences $(t_n) \subset [0,1)$ and $(u_n) \subset L^2(Q)$, $\|u_n\| > n$ such that $u_n \in t_n T u_n$ for any natural n. Assume $v_n = \dfrac{u_n}{\|u_n\|}$. There are $z_n \in Tu_n$ such that

$$Au_n + \varepsilon u_n = -t_n z_n + t_n \varepsilon u_n + t_n f, \tag{9}$$

let us divide both part into $\|u_n\|$, and we obtain:

$$Av_n + \varepsilon v_n = -t_n \frac{z_n}{\|u_n\|} + t_n \varepsilon v_n + t_n \frac{f}{\|u_n\|},$$

There is the growing sequence (n_k) of natural numbers such that $v_{n_k} \rightharpoonup v$, and $t_{n_k} \to t$, ($y_n \rightharpoonup y$ denote the weak convergence (y_n) to y in $L^2(Q)$). But

$$v_{n_k} = (A+\varepsilon I)^{-1}\left(\frac{t_{n_k} z_{n_k}}{\|u_{n_k}\|} + \frac{t_{n_k} f}{\|u_{n_k}\|} + t_{n_k}\varepsilon v_{n_k}\right),$$

$$\frac{t_n z_n}{\|u_n\|} \to 0, \ \frac{t_n f}{\|u_n\|} \to 0, \ t_{n_k}\varepsilon v_{n_k} \rightharpoonup t\varepsilon v.$$

Therefore $v_{n_k} \to (A+\varepsilon I)^{-1} t\varepsilon v$ and $v \neq 0$. Then $Av = (t-1)\varepsilon v$. Since v is a non-nil function, $t - 1 \leqslant 0$ and $[-\varepsilon, 0) \cap \sigma = \varnothing$ then this implies, that $t = 1$ and $Av = 0$. Therefore, v belongs to the kernel of the operator A, then $KerB$. Since $v_{n_k} \to v$ in $L^2(Q)$, we can suppose, that $v_{n_k} \to v$ almost everywhere on Q, turning on the contrary into subsequences. Let us multiply both parts (9) scalar by $v(x)$. We have for the arbitrary natural n

$$(Au_n, v) + \varepsilon(u_n, v) + (t_n z_n, v) - (t_n f, v) - t_n(\varepsilon u_n, v) = 0. \tag{10}$$

Since $(Au_n, v) = (u_n, A^* v) = 0$, then, if we divide both parts (10) into t_n, we obtain,

$$\left(\frac{1-t_n}{t_n}\right)\varepsilon(u_n, v) + (z_n, v) = (f, v).$$

This results in validity of the inequality $(f, v) > (z_{n_k}, v)$ for sufficiently large k, since $(u_{n_k}, v) = \|u_{n_k}\|(v_{n_k}, v)$, $(v_{n_k}, v) \to \|v\|^2 = 1$ and $\|u_{n_k}\| > n_k$. This implies, that

$$(f, v) \geq \liminf_{k\to\infty}(z_{n_k}, v) \geq \liminf_{k\to\infty}\Big(\int\limits_{v>0} g_-(x,t,u_{n_k}(x,t))v(x)dxdt +$$

$$+ \int\limits_{v<0} g_+(x,t,u_{n_k}(x,t))v(x)dxdt\Big) \geq \int\limits_{v>0} \underline{g}_+(x,t)v(x)dxdt + \int\limits_{v<0} \overline{g}_-(x,t)v(x)dxdt.$$

During the transition to the bound to the sign of the integral we applied the Fatou-Lebesgue lemma [10] subject to the estimate (4) for $g(x,t,u)$ and that for almost all $(x,t) \in Q$ $u_{n_k} \to +\infty$, if $v(x) > 0$, and $u_{n_k} \to -\infty$, if $v(x) < 0$. The obtained inequality contradicts the Landesmann - Lazer condition in Theorem 1. Theorem 1 has been proved.

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