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OPTIMAL BOUNDARY CONTROL IN A SMALL CONCAVE DOMAIN

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Abstract. The paper is devoted to investigation of an asymptotics of a solution of the problem of optimal boundary control [1] in a small concave domain. Construction of an asymptotics of a boundary value problem for an elliptic operator in a small concave domain is considered in [2], and an asymptotics of the distributed control in a small concave domain in [3]. The Asymptotics of boundary control for an operator with a small factor at the higher derivative was considered in [4], [5]. Other problems of control by solutions of boundary value problems of the optimal control containing a small parameter are considered in [6], [7].

Keywords: asymptotic, boundary control, matching method, boundary value problems, systems of equations in partial derivatives.

1. Formulation of the problem

In the biconnected bounded domain $\Omega_\varepsilon := \Omega \setminus \varepsilon\omega \subset \mathbb{R}^3$ $(O \in \overset{\circ}{\omega}, \overline{\omega} \subset \overset{\circ}{\Omega})$ with the smooth boundary $\Gamma_\varepsilon = \Gamma \cup \varepsilon\gamma := \partial\Omega \cup \varepsilon\partial\omega$ (Ω_ε is smooth variety with boundary) we consider the following problem of optimum control [1, chapter 2, correlations (2.41), (2.9)]

$$\begin{cases} Az_\varepsilon = f(x), & x \in \Omega_\varepsilon,\ z_\varepsilon \in H^1(\Omega_\varepsilon), \\ \dfrac{\partial z_\varepsilon}{\partial n_A} = g(x) + u_\varepsilon(x), & x \in \Gamma_\varepsilon, \end{cases} \tag{1.1}$$

$$u \in \mathcal{U}_\varepsilon \text{ is convex closed set in } L_2(\Omega_\varepsilon), \tag{1.2}$$

$$J(u) := ||z_\varepsilon - z_d||_\varepsilon^2 + \nu^{-1}|||u_\varepsilon|||_\varepsilon^2 \to \inf, \tag{1.3}$$

where $A = -\nabla \cdot \big(A_{3\times 3}(x) \cdot \nabla\big) + a_0(x)$, $A_{3\times 3}(x) = (a_{ij}(x))$, notably

$$Az := -\sum_{i,j=1}^{3} \frac{\partial}{\partial x_i}\Big(a_{ij}(x)\frac{\partial z}{\partial x_j}\Big) + a_0(x)z,$$

$$f, a_0, a_{ij} \in C^\infty(\overline{\Omega}), g \in C^\infty(\Gamma_\varepsilon), \tag{1.4}$$

$$\frac{\partial z}{\partial n_A} := \sum_{i,j=1}^{3} a_{ij}\frac{\partial z}{\partial x_i}\cos(n, x_i) = \nabla z \cdot \big(A^T_{3\times 3} n\big) \text{ is} \tag{1.5}$$

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conormal derivative, denoted by the operator A, $\cos(n, x_i)$ is i-direction cosine of the outer normal n to the boundary Γ_ε domain Ω_ε, $A^T_{3\times3}$ is transposed matrix $A_{3\times3}$, ν is a positive constant, and $||\cdot||_\varepsilon$ and $|||\cdot|||_\varepsilon$ are norms in the space $L_2(\Omega_\varepsilon)$ and $L_2(\Gamma_\varepsilon)$ correspondingly.

Relative to coefficients of the operator A we also assume the following:

$$\exists\, \alpha > 0\, \forall\, x \in \overline{\Omega}\, \forall \xi \in \mathbb{R}^3$$
$$\sum_{i,j=1}^{3} a(x)_{ij}\xi_i\xi_j \geqslant \alpha \sum_{i=1}^{3} \xi_i^2, \quad a_0(x) \geqslant \alpha > 0 \tag{1.6}$$
$$a_{ii}(0) = 1, a_{ij}(0) = 0\; (i \neq j).$$

In the sequel scalar products in $L_2(\Omega_\varepsilon)$ and $L_2(\Gamma_\varepsilon)$ we denote by $(\cdot,\cdot)_\varepsilon$ and $\langle\cdot,\cdot\rangle_\varepsilon$, the norm in $H^1(\Omega_\varepsilon)$ by $||\cdot||_{\varepsilon,1}$, and norms and scalar products in $L_2(\Omega)$ and $L_2(\Gamma)$ we denote by $||\cdot||_0$, $(\cdot,\cdot)_0$ and $|||\cdot|||_0$, $\langle\cdot,\cdot\rangle_0$ correspondingly.

It was proved in [1, chapter 2, s. 2.4], that the problem (1.1) — (1.3) has only one solution.

We are intended to consider this problem with the following suppositions:

$$\begin{gathered} g\Big|_{\varepsilon\gamma} \equiv 0, \\ \mathcal{U}_\varepsilon = \mathcal{U}_\varepsilon(1), \text{ where } \mathcal{U}_\varepsilon(r) := \{u \in L_2(\Gamma_\varepsilon) : u\Big|_{\varepsilon\gamma} \equiv 0,\ |||u|||_0 \leqslant r\}, \end{gathered} \tag{1.7}$$

i.e. the process control is carried out only by the exterior boundary.

We are mainly interested in asymptotic expansion z_ε and u_ε when $\varepsilon \to 0$.

2. Determinative correlations

As it is shown in [1, chapter 2, s. 2.4], the only solution of the problem (1.1) — (1.3) is the pair z_ε and u_ε, which is characterized by the following conditions: there is $p_\varepsilon \in H^1(\Omega_\varepsilon)$ such that

$$\begin{cases} Az_\varepsilon = f(x), & A^*p_\varepsilon = z_\varepsilon - z_d, \quad x \in \Omega_\varepsilon, \\ \dfrac{\partial z_\varepsilon}{\partial n_A} = g(x) + u_\varepsilon(x), & \dfrac{\partial p_\varepsilon}{\partial n_{A^*}} = 0, \qquad x \in \Gamma_\varepsilon \end{cases} \tag{2.1}$$

and

$$\forall\, v \in \mathcal{U} \qquad \langle p_\varepsilon + \nu^{-1}u_\varepsilon, v - u_\varepsilon\rangle \geqslant 0, \tag{2.2}$$

where the operator A^* is formally conjugated to A, i.e.

$$A^* := -\nabla \cdot \left(A^T_{3\times3}(x)\cdot\nabla\right) + a_0(x).$$

Lemma 1. *The condition* (2.2) *for* $\mathcal{U}_\varepsilon = \mathcal{U}_\varepsilon(r)$ *is equivalent to the following*

$$\begin{gathered} \exists\, \lambda \in (0;\nu] : \left(u_\varepsilon(\cdot) = -\lambda p_\varepsilon(\cdot)\Big|_\Gamma\right) \wedge \\ \wedge \Big(\lambda|||p_\varepsilon|||_0 \leqslant r\Big) \wedge \Big((\nu - \lambda)\cdot(r - \lambda_\varepsilon|||p_\varepsilon|||_0) = 0\Big). \end{gathered} \tag{2.3}$$

Proof is carried out by analogy to the proof of Lemma 1 from [4]. ∎

Subject to (2.3) the system (2.1) takes the form

$$\begin{cases} Az_\varepsilon = f(x), & A^*p_\varepsilon = z_\varepsilon - z_d, \quad x \in \Omega_\varepsilon, z_\varepsilon, p_\varepsilon \in H^1(\Omega_\varepsilon), \\ \dfrac{\partial z_\varepsilon}{\partial n_A} + \lambda_\varepsilon p_\varepsilon = g(x), & \dfrac{\partial p_\varepsilon}{\partial n_{A^*}} = 0, \qquad x \in \Gamma \\ \dfrac{\partial z_\varepsilon}{\partial n_A} = 0, & \dfrac{\partial p_\varepsilon}{\partial n_{A^*}} = 0, \qquad x \in \varepsilon\gamma \end{cases} \tag{2.4}$$

$$\big(\lambda_\varepsilon \in (0;\nu]\big) \wedge \Big(\lambda_\varepsilon|||p_\varepsilon|||_0 \leqslant 1\Big) \wedge \Big((\nu - \lambda_\varepsilon)\cdot\big(1 - \lambda_\varepsilon|||p_\varepsilon|||_0\big) = 0\Big). \tag{2.5}$$

Let us note, that subject to the conditions (1.6) the boundary operator $\partial/\partial n_A$ $(\partial/\partial n_{A^*})$ is normal, it covers the operator A (A^*) [8, Chapter 2. s. 1.4.], and the mapping of the trace

$$H^m(\Omega_\varepsilon) \ni w \mapsto \Big(w\Big|_{\Gamma_\varepsilon}, \frac{\partial w}{\partial n_A} \Big) \in H^{m-1/2}(\Gamma_\varepsilon) \times H^{m-3/2}(\Gamma_\varepsilon)$$

is surjective.

Indeed, if n is a unit vector of a normal to Γ_ε, subject to (1.5)

$$n \cdot \big(A^T_{3\times 3} \cdot n\big) = n \cdot \big(A_{3\times 3} \cdot n\big) \geqslant \alpha > 0,$$

that denotes normality of this boundary operator.

Let now $0 \neq \tau$ be a tangent vector to Γ_ε in the point $x \in \Gamma_\varepsilon$, n be a unit vector of the normal to Γ_ε in the point $x \in \Gamma_\varepsilon$, $\beta \neq 0$, $A^T_{3\times 3} \cdot n = \tau_1 + \beta_1 n$, where τ_1 is a tangent vector to Γ_ε in the point $x \in \Gamma_\varepsilon$. Then the polynomial

$$(\tau + \beta t \cdot n) \cdot \big(A^T_{3\times 3} n\big) = \tau \cdot \tau_1 + \beta \cdot \beta_1 t$$

from t possesses a different from zero coefficient with t, whereas $\beta_1 = n \cdot \big(A^T_{3\times 3} n\big)$. Therefore this root is real. Hence, this polynomial is not equal to zero by module of the polynomial $(t - t_1)$, where t_1 is a complex root of the second-order polynomial, generated by the symbol of the operator A and the vector $\tau + \beta t \cdot n$.

Finally, let us show the solution of the problem $H^m(\Omega_\varepsilon) \ni w$ $w = \varphi\Big|_{\Gamma_\varepsilon}$ and $\dfrac{\partial w}{\partial n_A} = \psi$ for the variables $\varphi \in H^{m-1/2}(\Gamma_\varepsilon)$ and $\psi \in H^{m-3/2}(\Gamma_\varepsilon)$.

Subject to the definition (1.5) and presentation $A^T_{3\times 3}(x) \cdot n(x) = \tau_1(x) + \beta_1(x) n(x)$ we obtain, that $\partial w/n_A = \nabla w \cdot \tau_1(x) + \beta_1(x) \partial w/n$. But $\nabla w \cdot \tau_1$ is a derivative by the tangent vector τ_1, therefore, it is expressed by φ: $\nabla w \cdot \tau_1 = B(\varphi)$. Hence, $\partial w/n = \beta_1^{-1}\big(\partial w/n_A - B(\varphi)\big) = \beta_1^{-1}\big(\psi - B(\varphi)\big)$. But subject to the theorem on traces [8, Chapter 1, theorem 8.3] the mapping

$$H^m(\Omega_\varepsilon) \ni w \mapsto \Big(w\Big|_{\Gamma_\varepsilon}, \frac{\partial w}{\partial n} \Big) \in H^{m-1/2}(\Gamma_\varepsilon) \times H^{m-3/2}(\Gamma_\varepsilon)$$

is surjection.

Subject to the properties of elliptical equations from the condition (1.6) it results, that

$$\forall\, m \in \mathbb{N} \qquad z_\varepsilon, p_\varepsilon \in H^m(\Omega_\varepsilon),$$

and, consequently, $z_\varepsilon, p_\varepsilon \in C^\infty(\overline{\Omega_\varepsilon})$.

Let us note, that the boundary value problem (2.4) with every fixed λ_ε is by definition equivalent to the correlations

$$\begin{cases} \forall\, \varphi, \psi & \in & H^1(\Omega_\varepsilon) \\ (f, \varphi) & = & \pi_\varepsilon(\nabla z_\varepsilon, \nabla \varphi) + (a_0 z_\varepsilon, \varphi)_\varepsilon - \langle g - \lambda_\varepsilon p_\varepsilon, \varphi\rangle_0, \\ (z_\varepsilon - z_d, \psi) & = & \pi_\varepsilon(\nabla \psi, \nabla p_\varepsilon) + (a_0 p_\varepsilon, \psi)_\varepsilon, \end{cases} \tag{2.6}$$

where

$$\pi_\varepsilon(\varphi, \psi) := \sum_{i,j=1}^{3} \Big(a_{ij} \frac{\partial \varphi}{\partial x_j}, \frac{\partial \psi}{\partial x_i} \Big)_\varepsilon.$$

In the sequel we are intended to use the fact that if $\Omega_1 \supset \Omega_2$, then the following continuous embedding is defined $H^m(\Omega_1) \hookrightarrow H^m(\Omega_2)$ — "narrowing on Ω_2". We do not distinguish between elements from $H^m(\Omega_1)$ and its narrowing on Ω_2. Let us also note, that the norm of this embedding operator is equal to 1.

Lemma 2. *Let z_ε, p_ε and λ_ε, be the solution of the problem* (2.4), (2.5). *Then*

$$||z_\varepsilon||_\varepsilon^2 + \lambda_\varepsilon|||p_\varepsilon|||_0^2 = (f, p_\varepsilon)_\varepsilon + (z_d, z_\varepsilon)_\varepsilon + \langle g, p_\varepsilon\rangle_0 \tag{2.7}$$

and

$$||z_\varepsilon||_{\varepsilon,1}, ||p_\varepsilon||_{\varepsilon,1} = \mathcal{O}\big(||f||_\varepsilon + ||z_d||_\varepsilon + |||g|||_0\big), \varepsilon \to 0. \tag{2.8}$$

Proof. Let $\overset{\circ}{z}_\varepsilon$ be the solution of the problem (1.1) when $u \equiv 0$. Then by definition $\overset{\circ}{z}_\varepsilon$ we obtain, that $||z_\varepsilon - z_d||_\varepsilon \leqslant ||\overset{\circ}{z}_\varepsilon - z_d||_\varepsilon$, which results in

$$||z_\varepsilon||_\varepsilon \leqslant ||\overset{\circ}{z}_\varepsilon||_\varepsilon + 2||z_d||_\varepsilon. \tag{2.9}$$

Whereas $\overset{\circ}{z}_\varepsilon$ satisfies (1.1) when $u \equiv 0$, then

$$(f, \overset{\circ}{z}_\varepsilon)_\varepsilon = (A\overset{\circ}{z}_\varepsilon, \overset{\circ}{z}_\varepsilon)_\varepsilon = \pi_\varepsilon(\nabla\overset{\circ}{z}_\varepsilon, \overset{\circ}{z}_\varepsilon) + (a_0\overset{\circ}{z}_\varepsilon, \overset{\circ}{z}_\varepsilon)_\varepsilon - \langle g, \overset{\circ}{z}_\varepsilon\rangle_0,$$

subject to (1.6) provides

$$\alpha||\overset{\circ}{z}_\varepsilon||_{\varepsilon,1}^2 \leqslant ||f||_\varepsilon \cdot ||\overset{\circ}{z}_\varepsilon||_\varepsilon + |||g|||_0 \cdot |||\overset{\circ}{z}_\varepsilon|||_0. \tag{2.10}$$

Whereas $H^s(\Gamma)$ when $s > 0$ is embedded into $L_2(\Gamma)$ tightly and continuously, then subject to the theorem on traces (see [8, Chapter 1, theorem 8.3]) operator of sampling of the trace is continuous like an operator from $H^m(\Omega)$ in $L_2(\Gamma)$ when $m \geqslant 1$, i.e.

$$\exists\, K > 0\, \forall\, z \in H^1(\Omega) \qquad |||z|||_0 \leqslant K||z||_{H^1(\Omega)}, \tag{2.11}$$

therefore, subject to (2.10)

$$||\overset{\circ}{z}_\varepsilon||_{\varepsilon,1} = \mathcal{O}\big(||f||_\varepsilon + |||g|||_0\big). \tag{2.12}$$

Subject to (2.12) and (2.9) we obtain, that

$$||z_\varepsilon||_\varepsilon = \mathcal{O}\big(||f||_\varepsilon + ||z_d||_\varepsilon + |||g|||_0\big). \tag{2.13}$$

Now, after embedding into (2.6) $\varphi = z_\varepsilon$ and $\psi = p_\varepsilon$, we obtain

$$\begin{aligned} &\alpha|||z_\varepsilon|||_{\varepsilon,1}^2 \leqslant (f, z_\varepsilon)_\varepsilon + \langle g, z_\varepsilon\rangle_0 - \langle\lambda_\varepsilon p\varepsilon, z_\varepsilon\rangle_0, \\ &\alpha|||p_\varepsilon|||_{\varepsilon,1}^2 \leqslant (p_\varepsilon, z_\varepsilon)_\varepsilon - (z_d, p_\varepsilon)_\varepsilon. \end{aligned} \tag{2.14}$$

Taking into account (2.13) we obtain from the latest inequality, that

$$||p_\varepsilon||_{\varepsilon,1} = \mathcal{O}\big(||f||_\varepsilon + ||z_d||_\varepsilon + |||g|||_0\big). \tag{2.15}$$

Now, from the first correlation in (2.14) and the correlation (2.15) with the application of the inequality (2.11) and boundedness of λ_ε we obtain, that

$$||z_\varepsilon||_{\varepsilon,1} = \mathcal{O}\big(||f||_\varepsilon + ||z_d||_\varepsilon + |||g|||_0\big).$$

Finally, after taking in (2.6) $\varphi = p_\varepsilon$ and $\psi = z_\varepsilon$ and after subtracting of the second equality from the first obtained, we obtain the correlation (2.7). ∎

Now, applying a priori estimates (2.8), we obtain analogous estimates for the following boundary value problem of more general form in comparison with (2.4)

$$\begin{cases} Az = f_1(x), & A^*p - z = f_2(x), \quad x \in \Omega_\varepsilon, z, p \in H^1(\Omega_\varepsilon), \\ \dfrac{\partial z}{\partial n_A} + \lambda p = g_{1,\Gamma}(x), & \dfrac{\partial p}{\partial n_{A^*}} = g_{2,\Gamma}(x), \quad x \in \Gamma \\ \dfrac{\partial z}{\partial n_A} = g_{1,\gamma}(x), & \dfrac{\partial p}{\partial n_{A^*}} = g_{2,\gamma}(x), \quad x \in \varepsilon\gamma, \end{cases} \tag{2.16}$$

where λ is some positive constant, $f_i \in L_2(\Omega_\varepsilon)$, $g_{i,\Gamma} \in H^{1/2}(\Gamma)$ and $g_{i,\gamma}(x) \in H^{1/2}(\varepsilon\gamma)$.

Lemma 3. *Let z and p be the solution of the problem* (2.16). *Then*

$$||z||_{\varepsilon,1}, ||p||_{\varepsilon,1} = \mathcal{O}\big(||f_1||_\varepsilon + ||f_2||_\varepsilon + |||g_{1,\Gamma}|||_0 + |||g_{1,\Gamma}|||_0 + + |||g_{1,\Gamma}|||_{\varepsilon\gamma} + |||g_{1,\Gamma}|||_{\varepsilon\gamma}\big), \varepsilon \to 0, \tag{2.17}$$

where $|||\cdot|||_{\varepsilon\gamma}$ is the norm in the space $L_2(\varepsilon\gamma)$.

Proof. Let $\widetilde{z}$ and $\widetilde{p}$ be the solutions of the boundary value problems

$$\begin{cases} A\widetilde{z} = 0, & A^*\widetilde{p} = 0, & x \in \Omega_\varepsilon, \\ \dfrac{\partial \widetilde{z}}{\partial n_A} = 0, & \dfrac{\partial \widetilde{p}}{\partial n_{A^*}} = g_{2,\Gamma}(x), & x \in \Gamma, \\ \dfrac{\partial \widetilde{z}}{\partial n_A} = g_{1,\gamma}(x), & \dfrac{\partial \widetilde{p}}{\partial n_{A^*}} = g_{2,\gamma}(x), & x \in \varepsilon\gamma. \end{cases}$$

Let us note, that they have only one solution and the following estimates hold for them: [9], [8]

$$||\widetilde{z}||_\varepsilon = \mathcal{O}\big(|||g_{1,\gamma}|||_{\varepsilon\gamma}\big), \quad |\widetilde{p}||_{\varepsilon,1} = \mathcal{O}\big(|||g_{2,\Gamma}|||_0 + |||g_{2,\gamma}|||_{\varepsilon\gamma}\big). \tag{2.18}$$

Therefore, the function $\widehat{z} := z - \widetilde{z}$ and $\widehat{p} := z - \widetilde{p}$ satisfy the following problem

$$\begin{cases} A\widehat{z} = f_1(x), & A^*\widehat{p} - \widehat{z} = f_2(x) + \widetilde{z}(x), & x \in \Omega_\varepsilon, \\ \dfrac{\partial \widehat{z}}{\partial n_A} + \lambda\widehat{p} = g_{1,\Gamma}(x) - \lambda\widetilde{p}(x), & \dfrac{\partial \widehat{p}}{\partial n_{A^*}} = 0, & x \in \Gamma \\ \dfrac{\partial \widehat{z}}{\partial n_A} = 0, & \dfrac{\partial \widehat{p}}{\partial n_{A^*}} = 0, & x \in \varepsilon\gamma, \end{cases}$$

Whereas this problem coincides with the problem (2.4), (2.5) when $z_d = f_2 + \widetilde{z}$, $g = g_{1,\Gamma} - \lambda\widetilde{p}$, $\nu > \lambda$ and $r = \lambda|||\widehat{p}|||_0$, then due to (2.8) we obtain

$$||\widehat{z}||_{\varepsilon,1}, ||\widehat{p}||_{\varepsilon,1} = \mathcal{O}\big(||f_1||_\varepsilon + ||f_2 + \widetilde{z}||_\varepsilon + |||g_{1,\gamma} - \lambda\widetilde{p}|||_0\big), \varepsilon \to 0.$$

Now we should apply inequality of the triangle for the norms and the correlation (2.18). ∎

Theorem 1. . *The problem* (2.16) *has only one solution for any $f_i \in L_2(\Omega_\varepsilon)$, $g_{i,\Gamma} \in H^{1/2}(\Gamma)$ and $g_{i,\gamma} \in H^{1/2}(\varepsilon\gamma)$ $(i = 1, 2)$ and its solution $z, p \in H^2(\Omega_\varepsilon)$.*

Hence, if $f_i \in C^\infty(\overline{\Omega}))$, $g_{i,\Gamma} \in C^\infty(\Gamma)$ and $g_{i,\gamma} \in C^\infty(\varepsilon\gamma)$ $(i = 1, 2)$, then for all $m \in \mathbb{N}$ $z, p \in H^m(\Omega_\varepsilon)$.

Proof. Let us consider mapping of the Hilbert space $E := H^2(\Omega_\varepsilon)^2$ in Hilbert space $G := L_2(\Omega_\varepsilon)^2 \times H^{1/2}(\Gamma_\varepsilon)^2$, denoted by the problem (2.16),

$$\mathcal{A}(z, p) := \Big(Az, A^*p - z, \Big(\frac{\partial z}{\partial n_A} + \lambda p\Big)\Big|_\Gamma, \frac{\partial p}{\partial n_{A^*}}\Big|_\Gamma, \frac{\partial z}{\partial n_A}\Big|_{\varepsilon\gamma}, \frac{\partial p}{\partial n_{A^*}}\Big|_{\varepsilon\gamma}\Big).$$

Assume $F := H^1(\Omega_\varepsilon)^2$. Hence E is compactly embedded into F. Let us show, that

$$\exists\, C > 0\, \forall\, z, p \in H^2(\Omega_\varepsilon) \qquad ||(z, p)||_E \leqslant C \cdot \Big(||\mathcal{A}(z, p)||_G + ||(z, p)||_F\Big). \tag{2.19}$$

On the strength of Theorem 5.1 from [8, Chapter 2, s. 5] $\exists\, C_1 > 0$:

$$||(z, p)||_E \leqslant ||z||_{H^2(\Omega_\varepsilon)} + ||p||_{H^2(\Omega_\varepsilon)} \leqslant C_1\Big(||Az||_\varepsilon + ||A^*p - z||_\varepsilon + ||z||_\varepsilon +$$

$$+\Big|\Big|\Big|\frac{\partial z}{\partial n_A} + \lambda p\Big|\Big|\Big|_{H^{1/2}(\Gamma_\varepsilon)} + \lambda|||p|||_{H^{1/2}(\Gamma_\varepsilon)} + \Big|\Big|\Big|\frac{\partial p}{\partial n_{A^*}}\Big|\Big|\Big|_{H^{1/2}(\Gamma_\varepsilon)} + ||z||_{1,\varepsilon} + ||p||_{1,\varepsilon}\Big).$$

But on the strength of the theorem on traces $\exists\, C_2 > 0$:

$$|||z|||_{H^{1/2}(\Gamma_\varepsilon)} \leqslant C_2|||z|||_{\varepsilon,1}, \quad |||p|||_{H^{1/2}(\Gamma_\varepsilon)} \leqslant C_2|||p|||_{\varepsilon,1},$$

that completes proof of the inequality (2.19).

Hence, with respect to Peetre lemma [10], [8, Chapter 2, lemma 5.1] the image of the operator $\mathcal{A}$ is closed, and its kernel is finite dimensional.

As for the kernel of the operator $\mathcal{A}$, on the strength of a priori estimates (2.17) it consists of one zero, that is the operator $\mathcal{A}$ is injective.

Let us show, that the operator is surjective.

Let $f_i^* \in L_2(\Omega_\varepsilon)$, $g_{i,\Gamma}^* \in H^{-1/2}(\Gamma)$ and $g_{i,\gamma}^* \in H^{-1/2}(\varepsilon\gamma)$ $(i = 1, 2)$ be such that $\forall\, u, v \in H^2(\Omega_\varepsilon)$

$$\begin{aligned} 0 \;=\;& (Au, f_1^*)_\varepsilon + (A^*v - u, f_2^*)_\varepsilon + \Big\langle \frac{\partial u}{\partial n_A} + \lambda v, g_{1,\Gamma}^* \Big\rangle_0 + \\ &+ \Big\langle \frac{\partial v}{\partial n_{A^*}}, g_{2,\Gamma}^* \Big\rangle_0 + \Big\langle \frac{\partial u}{\partial n_A}, g_{1,\gamma}^* \Big\rangle_{\varepsilon\gamma} + \Big\langle \frac{\partial v}{\partial n_{A^*}}, g_{2,\gamma}^* \Big\rangle_{\varepsilon\gamma}. \end{aligned} \tag{2.20}$$

To prove this theorem by $\langle\cdot\rangle_0$ and $\langle\cdot\rangle_{\varepsilon\gamma}$ we denote bilinear forms, setting duality between spaces $H^{1/2}(\Gamma)$ and $H^{-1/2}(\Gamma)$, $H^{1/2}(\varepsilon\gamma)$ and $H^{-1/2}(\varepsilon\gamma)$ correspondingly.

Let us note, that if $g_{i,\Gamma}^* \in L_2(\Gamma)$ and $g_{i,\gamma}^* \in L_2(\varepsilon\gamma)$, then these bilinear forms coincide with scalar product in $L_2(\Gamma)$ and $L_2(\varepsilon\gamma)$ correspondingly, and, by this they do not contradict the previous application of the symbols.

Our target is to prove the equalities $f_i^* = 0$, $g_{i,\Gamma}^* = 0$ and $g_{i,\gamma}^* = 0$ $(i = 1, 2)$, which on the strength of the closure of the image of the operator $\mathcal{A}$ give surjectiveness of this operator.

On the strength of independence of u and v the correlation (2.20) is divided into two

$$\forall\, u \in H^2(\Omega_\varepsilon) \quad 0 = (Au, f_1^*)_\varepsilon - (u, f_2^*)_\varepsilon + \Big\langle \frac{\partial u}{\partial n_A}, g_{1,\Gamma}^* \Big\rangle_0 + \Big\langle \frac{\partial u}{\partial n_A}, g_{1,\gamma}^* \Big\rangle_{\varepsilon\gamma}, \tag{2.21}$$

$$\begin{aligned} \forall\, v \in H^2(\Omega_\varepsilon) \quad 0 \;=\;& (A^*v, f_2^*)_\varepsilon + \langle \lambda v, g_{1,\Gamma}^* \rangle_0 + \Big\langle \frac{\partial v}{\partial n_{A^*}}, g_{2,\Gamma}^* \Big\rangle_0 + \\ &+ \Big\langle \frac{\partial v}{\partial n_{A^*}}, g_{2,\gamma}^* \Big\rangle_{\varepsilon\gamma}. \end{aligned} \tag{2.22}$$

The correlation (2.21) shows, that

$$(Au, f_1^*)_\varepsilon + \Big\langle \frac{\partial u}{\partial n_A}, g_{1,\Gamma}^* \Big\rangle_0 + \Big\langle \frac{\partial u}{\partial n_A}, g_{1,\gamma}^* \Big\rangle_{\varepsilon\gamma} = (u, f_2^*)_\varepsilon$$

Thereby, according to s. 2 of Theorem 5.1 from [8, Chapter 2, s. 5], applied to the operator $u \mapsto \left(Au, \frac{\partial u}{\partial n_A}\right)$, we obtain, that

$$f_1^* \in H^2(\Omega_\varepsilon),\ g_{1,\Gamma}^* \in H^{3/2}(\Gamma),\ g_{1,\gamma}^* \in H^{1/2}(\varepsilon\gamma).$$

Now, let us apply the fact, that $g_{1,\Gamma}^* \in H^{3/2}(\Gamma)$. Whereas mapping of the trace

$$H^m(\Omega_\varepsilon) \ni w \mapsto \left(w\Big|_{\Gamma_\varepsilon}, \frac{\partial w}{\partial n_A} \right) \in H^{m-1/2}(\Gamma_\varepsilon) \times H^{m-3/2}(\Gamma_\varepsilon)$$

is continuous and surjective, there is $g_1^* \in H^3(\Omega_\varepsilon)$: $\dfrac{\partial g_1^*}{\partial n_A} = g_{1,\Gamma}^*$ on Γ and $\dfrac{\partial g_1^*}{\partial n_A} = 0$ on $\varepsilon\gamma$. Then, on the strength of Green formula [1, Chapter 1, s.3.4]

$$u, v \in H^1(\Omega_\varepsilon) \Longrightarrow (Au, v)_\varepsilon = (u, A^*v)_\varepsilon - \Big\langle \frac{\partial u}{\partial n_A}, v \Big\rangle_\varepsilon + \Big\langle u, \frac{\partial v}{\partial n_{A^*}} \Big\rangle_\varepsilon \tag{2.23}$$

we obtain

$$\langle g_{1,\Gamma}^*, v \rangle_0 = \Big\langle \frac{\partial g_1^*}{\partial n_A}, v \Big\rangle_\varepsilon = -(Ag_1^*, v)_\varepsilon + \Big\langle g_1^*, \frac{\partial v}{\partial n_{A^*}} \Big\rangle_\varepsilon.$$

Hence, the correlation (2.22) can be written in the form

$$(A^*v, f_2^* + \lambda g_1^*) + \left\langle \frac{\partial v}{\partial n_{A^*}}, g_{2,\Gamma}^* + \lambda g_1^* \right\rangle_0 + \left\langle \frac{\partial v}{\partial n_{A^*}}, g_{2,\gamma}^* + \lambda g_1^* \right\rangle_{\varepsilon\gamma} = (v, \lambda A g_1^*)_\varepsilon.$$

Whereas $\lambda A g_1^* \in H^1(\Omega_\varepsilon)$, then again, applying s. 2 of Theorem 5.1 from [8, Chapter 2, s. 5] we obtain, that

$$f_2^* + \lambda g_1^* \in H^3(\Omega_\varepsilon),\ g_{2,\Gamma}^* + \lambda g_1^* \in H^{5/2}(\Gamma),\ g_{2,\gamma}^* + \lambda g_1^* \in H^{1/2}(\varepsilon\gamma).$$

Subject to the theorem on traces this provides:

$$f_2^* \in H^3(\Omega_\varepsilon),\ g_{2,\Gamma}^* \in H^{5/2}(\Gamma),\ g_{2,\gamma}^* \in H^{1/2}(\varepsilon\gamma).$$

Now, if we take in (2.21), (2.22) $u, v \in \overset{\circ}{H}{}^2(\Omega_\varepsilon)$, we obtain, that $0 = (u, A^* f_1^* - f_2^*)_\varepsilon$ and $0 = (v, A f_2^*)$, where from on the strength of the density $\overset{\circ}{H}{}^2(\Omega_\varepsilon)$ in $L_2(\Omega_\varepsilon)$ the following equalities result

$$A^* f_1^* = 0, \quad A f_2^* = 0, \quad x \in \Omega_\varepsilon. \tag{2.24}$$

If we apply in (2.21) and (2.22) Green formula (2.23) and if we consider equalities (2.24), we obtain, that

$$\begin{aligned} \forall\, u \in H^2(\Omega_\varepsilon) \quad 0 \quad &= \quad \left\langle \frac{\partial u}{\partial n_A}, g_{1,\Gamma}^* - f_1^* \right\rangle_0 + \left\langle \frac{\partial u}{\partial n_A}, g_{1,\gamma}^* - f_1^* \right\rangle_{\varepsilon\gamma} + \\ &\quad + \left\langle u, \frac{\partial f_1^*}{\partial n_{A^*}} \right\rangle_0, \end{aligned}$$

$$\begin{aligned} \forall\, v \in H^2(\Omega_\varepsilon) \quad 0 \quad &= \quad \left\langle v, \lambda g_{1,\Gamma}^* + \frac{\partial f_2^*}{\partial n_A} \right\rangle_0 + \left\langle \frac{\partial v}{\partial n_{A^*}}, g_{2,\Gamma}^* - f_2^* \right\rangle_0 + \\ &\quad + \left\langle v, \frac{\partial f_2^*}{\partial n_A} \right\rangle_{\varepsilon\gamma} + \left\langle \frac{\partial v}{\partial n_{A^*}}, g_{2,\gamma}^* - f_2^* \right\rangle_{\varepsilon\gamma}, \end{aligned}$$

where from on the strength of surjectiveness of the mapping of the trace we obtain

$$\begin{aligned} &g_{1,\Gamma}^* - f_1^* = 0,\ \frac{\partial f_1^*}{\partial n_{A^*}} \text{ on } \Gamma, \quad g_{1,\gamma}^* - f_1^* \text{ on } \varepsilon\gamma, \\ &\lambda g_{1,\Gamma}^* + \frac{\partial f_2^*}{\partial n_A},\ g_{2,\Gamma}^* - f_2^* \text{ on } \Gamma, \quad \frac{\partial f_2^*}{\partial n_A},\ g_{2,\gamma}^* - f_2^* \text{ on } \varepsilon\gamma, \end{aligned} \tag{2.25}$$

which subject to the equalities (2.24) provides

$$\begin{cases} A f_2^* = 0, & A^* f_1^* - f_2^* = 0, \quad x \in \Omega_\varepsilon, \\ \dfrac{\partial f_2^*}{\partial n_A} + \lambda f_1^* = 0, & \dfrac{\partial f_1^*}{\partial n_{A^*}} = 0, \qquad\quad x \in \Gamma, \\ \dfrac{\partial f_2^*}{\partial n_A} = 0, & \dfrac{\partial f_1^*}{\partial n_{A^*}} = 0, \qquad\quad x \in \varepsilon\gamma. \end{cases}$$

Let us note, that (f_2^*, f_1^*) satisfies the homogeneous problem (2.16), and thereby, as it has already been shown, $f_2^* = f_1^* = 0$. Therefore, on the strength of (2.25) all the remaining elements are also equal to zero.

The last statement of the theorem is the result of the property of elliptical boundary value problems for one unknown function. ∎

3. A boundary value problem and a priori estimates of approximation error

Now we are intended to show, that ultimate for the problem (2.4), (2.5) will be the following problem

$$\begin{cases} Az_0 = f(x), & A^*p_0 - z_0 = z_d, \quad x \in \Omega_\varepsilon, z_0, p_0 \in H^1(\Omega), \\ \dfrac{\partial z_0}{\partial n_A} + \lambda_0 p_0 = g(x), & \dfrac{\partial p_0}{\partial n_{A^*}} = 0, \qquad x \in \Gamma \end{cases} \tag{3.1}$$

$$\big(\lambda_0 \in (0,\nu]\big) \wedge \Big(\lambda_0|||p_0|||_0 \leqslant 1\Big) \wedge \Big((\nu - \lambda_0)\cdot(1 - \lambda_0|||p_0|||_0) = 0\Big). \tag{3.2}$$

This problem coincides with the system of optimality for the problem (1.1) — (1.3) in the domain with "glued" cavity, that is with substitution Ω_ε on Ω and $\mathcal{U}_\varepsilon(r)$ on $\mathcal{U}(r) := \{u \in L_2(\Gamma) : |||u|||_0 \leqslant r\}$.

Theorem 2. *Assume λ_ε, z_ε and the solution of the problem* (2.4), (2.5). *Then when $\varepsilon \to 0$*

$$\lambda_\varepsilon \longrightarrow \lambda_0, \ ||z_\varepsilon - z_0||_{\varepsilon,1} \longrightarrow 0, \ ||p_\varepsilon - p_0||_{\varepsilon,1} \longrightarrow 0.$$

Proof. Let us assume the contrary. Then we obtain $\eta > 0$, and sequence ε_m such that

$$|\lambda_m - \lambda_0| + ||z_m - z_0||_{m,1} + ||p_m - p_0||_{m,1} \geqslant \eta, \tag{3.3}$$

where $\lambda_m := \lambda_{\varepsilon_m}$, $z_m := z_{\varepsilon_m}$, $p_m := p_{\varepsilon_m}$, and $||\cdot||_{m,1}$ is the norm in $H^1(\Omega_{\varepsilon_m})$.

Whereas $0 < \lambda_m \leqslant \nu$ and $\lambda_m|||p_m|||_0 \leqslant 1$, then we can consider without bounding of generality, that

$$\lambda_m \longrightarrow \overline{\lambda}, \ \lambda_m|||p_m|||_0 \longrightarrow \overline{\mu}, \ (\nu - \overline{\lambda})\cdot(1 - \overline{\mu}) = 0. \tag{3.4}$$

If $\overline{\lambda} = 0$, then $\overline{\mu} = 1$ and, it denotes, $|||p_m|||_0 \longrightarrow \infty$, that contradicts correlations (2.8) and (2.11). Hence, $\nu \geqslant \overline{\lambda} > 0$.

Let $\overline{z}$, $\overline{p}$ be the solution of the problem

$$\begin{cases} A\overline{z} = f(x), & A^*\overline{p} - \overline{z} = z_d, \quad x \in \Omega_\varepsilon, \overline{z}, p_0 \in H^1(\Omega), \\ \dfrac{\partial \overline{z}}{\partial n_A} + \overline{\lambda}\overline{p} = g(x), & \dfrac{\partial \overline{p}}{\partial n_{A^*}} = 0, \qquad x \in \Gamma. \end{cases}$$

Let us note, that solubility of this problem with all right parts with the needed degree of smoothness is obtained by analogy with the proof of the Theorem 1. Therewith on the strength of the conditions on f and g the following embeddings hold $\overline{z}, \overline{p} \in C^\infty(\overline{\Omega})$.

Therefore $\widehat{z}_m := z_m - \overline{z}$, $\widehat{p}_m := p_m - \overline{p}$, satisfy the following system

$$\begin{cases} A\widehat{z}_m = 0, & A^*\widehat{p}_m - \widehat{z}_m = 0, \quad x \in \Omega_\varepsilon, \\ \dfrac{\partial \widehat{z}_m}{\partial n_A} + \overline{\lambda}\widehat{p}_m = (\overline{\lambda} - \lambda_m)p_m, & \dfrac{\partial \widehat{p}_m}{\partial n_{A^*}} = 0, \qquad x \in \Gamma, \\ \dfrac{\partial \widehat{z}_m}{\partial n_A} = -\dfrac{\partial \overline{z}}{\partial n_A}, & \dfrac{\partial \widehat{p}_m}{\partial n_{A^*}} = -\dfrac{\partial \overline{p}}{\partial n_{A^*}}, \quad x \in \varepsilon\gamma. \end{cases}$$

On the strength of (2.17) we obtain

$$||\widehat{z}||_{\varepsilon_m,1}, ||\widehat{p}||_{\varepsilon_m,1} = \mathcal{O}\Big(|\overline{\lambda} - \lambda_m|\cdot|||p_m|||_0 + \Big|\Big|\Big|\frac{\partial \overline{z}}{\partial n_A}\Big|\Big|\Big|_m + \Big|\Big|\Big|\frac{\partial \overline{p}}{\partial n_{A^*}}\Big|\Big|\Big|_m\Big), \varepsilon \to 0, \tag{3.5}$$

where $|||\cdot|||_m$ is the norm in $L_2(\varepsilon_m\gamma)$.

But in $|\overline{\lambda} - \lambda_m|\cdot|||p_m|||_0 \to 0$ due to boundedness $\{\,|||p_m|||_0\,\}$ and (3.4).

Whereas $\overline{z}, \overline{p} \in C^\infty(\overline{\Omega})$, then

$$\Big|\Big|\Big|\frac{\partial \overline{z}}{\partial n_A}\Big|\Big|\Big|_m, \quad \Big|\Big|\Big|\frac{\partial \overline{p}}{\partial n_{A^*}}\Big|\Big|\Big|_m = \mathcal{O}(\varepsilon_m).$$

On the strength of this and the correlations (3.5) $|||p_m|||_0 \longrightarrow |||\overline{p}|||_0$ and, thereby, $\overline{\lambda}$, $\overline{z}$, $\overline{p}$ there is the solution of the problem (3.1), (3.2), having only one solution. Therefore $\overline{\lambda} = \lambda_0$, $\overline{z} = z_0$ and $\overline{p} = p_0$, that contradicts to (3.3). ∎

In the sequel we are intended to assume, that

$$\lambda_0 < \nu, \text{ and, thereby, } \lambda_0|||p_0|||_0 = 1, \tag{3.6}$$

therefore, in the maximum problem bounds are in essence.

Then, on the strength of theorem 2 with all sufficiently low $\varepsilon > 0$ the condition (2.5) takes the form

$$\lambda_\varepsilon|||p_\varepsilon|||_0 = 1. \tag{3.7}$$

Theorem 3. *Let $u_{\varepsilon,r}$ be the solution of the problem* (1.1) — (1.3) *with $\mathcal{U} = \mathcal{U}_\varepsilon(r)$, $r \in [r_1; r_2]$, satisfying the condition $|||u_{\varepsilon,r}|||_0 = r$. Therefore*

$$\exists\, K > 0\, \exists\, \varepsilon_0 > 0\, \forall\, r, r' \in (r_1; r_2)\, \forall\, \varepsilon \in (0; \varepsilon_0) \quad |||u_{\varepsilon,r} - u_{\varepsilon,r'}|||_0 \leqslant K \cdot |r - r'|.$$

Proof. Let $\overset{\circ}{z}_\varepsilon$ be the solution of the problem (1.1) when $u \equiv 0$, and the operator $\mathcal{F}_\varepsilon : L_2(\Gamma) \to L_2(\Omega_\varepsilon)$ sets the solution of the problem (1.1) in the correspondence of the function $u \in L_2(\Gamma)$ as a function from $L_2(\Omega)$. Hence in the point $u_{\varepsilon,r}$ we obtain the minimum function of the function $||\overset{\circ}{z}_\varepsilon + \mathcal{F}_\varepsilon u - z_d||^2_\varepsilon + \nu^{-1}|||u|||^2_0$ on $\mathcal{U}_\varepsilon(r)$ which is a closed sphere with the radius r in $L_2(\Gamma)$. Then on the strength of Lagrange principle $u_{\varepsilon,r}$ is a point of the local minimum and for

$$||\overset{\circ}{z}_\varepsilon + \mathcal{F}_\varepsilon u - z_d||^2_\varepsilon + \nu^{-1}|||u|||^2_0 + \mu|||u|||^2_0, \quad \mu > 0.$$

Thereby there is $\mu_{\varepsilon,r}$ such that $\mathcal{F}^*_\varepsilon\big(\overset{\circ}{z}_\varepsilon + \mathcal{F}_\varepsilon u_{\varepsilon,r} - z_d\big) + \big(\nu^{-1} + \mu_{\varepsilon,r}\big)u_{\varepsilon,r} = 0$ or

$$u_{\varepsilon,r} = \big(\mathcal{F}^*_\varepsilon\mathcal{F}_\varepsilon + \big(\nu^{-1} + \mu_{\varepsilon,r}\big)I\big)^{-1}\mathcal{F}^*_\varepsilon\big(z_d - \overset{\circ}{z}_\varepsilon\big), \tag{3.8}$$

where $\mathcal{F}^*_\varepsilon : L_2(\Omega_\varepsilon) \to L_2(\Gamma)$ is an operator, conjugated to $\mathcal{F}_\varepsilon$, and I is an identical operator in $L_2(\Gamma)$.

Applying spectral presentation of the selfconjugated operator $\mathcal{F}^*_\varepsilon\mathcal{F}_\varepsilon$ (see, for example, [11, ch. 4, § 4]) and introducing symbols $w_\varepsilon := \mathcal{F}^*_\varepsilon\big(z_d - \overset{\circ}{z}_\varepsilon\big)$, from (3.8) we obtain

$$\begin{aligned} u_{\varepsilon,r} &= \int_0^{M_\varepsilon} \big(\sigma + \nu^{-1} + \mu_{\varepsilon,r}\big)^{-1}\, d\, I_\sigma w_\varepsilon, \\ |||u_{\varepsilon,r}|||_0 &= \int_0^{M_\varepsilon} \big(\sigma + \nu^{-1} + \mu_{\varepsilon,r}\big)^{-2}\, d\, |||I_\sigma w_\varepsilon|||_0, \end{aligned} \tag{3.9}$$

$$|||u_{\varepsilon,r} - u_{\varepsilon,r'}|||^2_0 = \int_0^{M_\varepsilon} \frac{\big(\mu_{\varepsilon,r} - \mu_{\varepsilon,r'}\big)^2 d\, |||I_\sigma w_\varepsilon|||^2_0}{\big(\sigma + \nu^{-1} + \mu_{\varepsilon,r}\big)^2\big(\sigma + \nu^{-1} + \mu_{\varepsilon,r'}\big)^2} \tag{3.10}$$

(here $\{I_\sigma\}$ are ortoprojectors, generated by operators $\mathcal{F}^*_\varepsilon\mathcal{F}_\varepsilon : L_2(\Gamma) \to L_2(\Gamma)$, and $M_\varepsilon = ||\mathcal{F}^*_\varepsilon\mathcal{F}_\varepsilon|| + \varepsilon = ||\mathcal{F}_\varepsilon||^2 + \varepsilon$).

Let us consider the function

$$F(\mu) := \int_0^{M_\varepsilon} \big(\sigma + \nu^{-1} + \mu\big)^{-2}\, d\, |||I_\sigma w_\varepsilon|||^2_0.$$

Hence $r^2 = |||u_{\varepsilon,r}|||^2_0 \overset{(3.9)}{=} F(\mu_{\varepsilon,r}) \leqslant \nu^2|||w_\varepsilon|||^2_0$, i.e.

$$r \leqslant \nu|||w_\varepsilon|||_0. \tag{3.11}$$

Let us note, that whereas $\left(\sigma+\nu^{-1}+\mu\right)^{-2}$ strongly decreases as a function from μ, then $F(\cdot)$ also strongly decreases. Therefore $F(\cdot)$ possesses a reverse function. Moreover,

$$|F'(\mu)| = 2\left|\int\limits_0^{M_\varepsilon}\left(\sigma+\nu^{-1}+\mu\right)^{-3} d\,|||I_\sigma w_\varepsilon|||_0^2\right| \geqslant \frac{2|||w_\varepsilon|||_0^2}{\left(M_\varepsilon+\nu^{-1}+\mu\right)^3}. \tag{3.12}$$

Then

$$|||u_{\varepsilon,r}-u_{\varepsilon,r'}|||_0 \overset{(3.10)}{\leqslant} \nu^2\left|\mu_{\varepsilon,r}-\mu_{\varepsilon,r'}\right|\cdot|||w_\varepsilon|||_0 = \nu^2|||w_\varepsilon|||_0\cdot\left|F^{-1}(r^2)-F^{-1}(r'^2)\right| =$$
$$= \nu^2|||w_\varepsilon|||_0\cdot\left|(F^{-1})'(\widetilde{r})\right|\cdot|r^2-r'^2| = \nu^2|||w_\varepsilon|||_0\cdot\left|F'(\widetilde{\mu})\right|^{-1}\cdot|r^2-r'^2| \overset{(3.12)}{\leqslant}$$
$$\leqslant \nu^2|||w_\varepsilon|||_0\cdot|r-r'|\cdot|r+r'|\frac{(M_\varepsilon+\nu^{-1}+\widetilde{\mu})^3}{2|||w_\varepsilon|||_0^2} \overset{(3.11)}{\leqslant} \frac{\nu^3 2r_2|r-r'|(M_\varepsilon+\nu^{-1}+\mu_1)^3}{2r_1},$$

(here $\mu_1 := F^{-1}(r_1^2)$).

Let us estimate $||\mathcal{F}_\varepsilon||$, and, consequently, M_ε. Assume $|||u|||_0 \leqslant 1$ and $z := \mathcal{F}_\varepsilon u$. Then, according to the definition $\mathcal{F}_\varepsilon$

$$Az = 0,\ x\in\Omega_\varepsilon, \quad \frac{\partial z}{\partial n_A} = u,\ x\in\Gamma \quad \frac{\partial\widetilde{z}}{\partial n_A} = 0,\ x\in\varepsilon\gamma,$$

therefore $||z||_\varepsilon = \mathcal{O}\left(|||u|||_0\right) = \mathcal{O}(1)$. ■

Now let us prove the general approximation theorem

Theorem 4. *The functions* $f_{i,m} \in C^\infty(\overline{\Omega})$, $g_{i,\Gamma,m} \in C^\infty(\Gamma)$, $g_{i,\gamma,m} \in C^\infty(\varepsilon\gamma)$ $(i=1,2)$, *but* $\lambda_m(\varepsilon)$ *and* $h_m(\varepsilon)$ *be some functions from* ε *and* $\lambda_m \in (0;\nu]$ *with all sufficiently low* $\varepsilon>0$. *If*

$$||f_{i,\varepsilon,m}||_\varepsilon,\ |||g_{i,\Gamma,m}|||_0,\ |||g_{i,\gamma,m}|||_{\varepsilon\gamma},\ |h_m(\varepsilon)| = \mathcal{O}\left(\varepsilon^m\right),\ \varepsilon\to 0, \tag{3.13}$$

and z_m, p_m *is the solution of the problem*

$$\begin{cases} Az_m = f(x)+f_{1,\varepsilon,m}(x), & x\in\Omega_\varepsilon,\\ A^*p_m - z_m = f_{2,\varepsilon,m}(x), & z_m, p_m \in H^1(\Omega_\varepsilon),\\ \dfrac{\partial z_m}{\partial n_A}+\lambda_m p_m = g(x)+g_{1,\Gamma,m}(x), & \dfrac{\partial p_m}{\partial n_{A^*}} = g_{2,\Gamma,m}(x), \quad x\in\Gamma\\ \dfrac{\partial z_m}{\partial n_A} = g_{1,\gamma,m}(x), & \dfrac{\partial p_m}{\partial n_{A^*}} = g_{2,\gamma,m}(x), \quad x\in\varepsilon\gamma, \end{cases}$$

$$\lambda_m|||p_m|||_0 = 1+h_m, \tag{3.14}$$

Then for $z_{\varepsilon,m} := z_\varepsilon - z_m$, $p_{\varepsilon,m} := p_\varepsilon - p_m$, $\lambda_{\varepsilon,m} := \lambda_\varepsilon - \lambda_m$, *where* z_ε, p_ε, λ_ε, *is the solution of the problem* (2.4), (3.7), *the following asymptotic estimates hold:*

$$\begin{aligned} ||z_{\varepsilon,m}||_{H^2(\Omega_\varepsilon)},\ ||p_{\varepsilon,m}||_{H^2(\Omega_\varepsilon)},\ |\lambda_{\varepsilon,m}| &= \mathcal{O}\left(\varepsilon^m\right),\ \varepsilon\to 0,\\ ||z_{\varepsilon,m}||_{C(\overline{\Omega_\varepsilon})},\ ||p_{\varepsilon,m}||_{C(\overline{\Omega_\varepsilon})} &= \mathcal{O}\left(\varepsilon^m\right),\ \varepsilon\to 0. \end{aligned} \tag{3.15}$$

Proof. Let us take $z_{m,1}$ and $p_{m,1}$ that is the solution of the boundary value problem

$$\begin{cases} Az_{m,1} = f_{1,\varepsilon,m}(x), & x\in\Omega_\varepsilon,\\ A^*p_{m,1} - z_{m,1} = f_{2,\varepsilon,m}(x), & z_{m,1}, p_{m,1}\in H^1(\Omega_\varepsilon),\\ \dfrac{\partial z_{m,1}}{\partial n_A}+\lambda_m p_{m,1} = g_{1,\Gamma,m}(x), & \dfrac{\partial p_{m,1}}{\partial n_{A^*}} = g_{2,\Gamma,m}(x), \quad x\in\Gamma,\\ \dfrac{\partial z_{m,1}}{\partial n_A} = g_{1,\gamma,m}(x), & \dfrac{\partial p_{m,1}}{\partial n_{A^*}} = g_{2,\gamma,m}(x), \quad x\in\varepsilon\gamma. \end{cases}$$

Then on the strength of the estimates (2.17), (3.13) and the inequality $0 < \lambda_m \leqslant \nu$ we obtain, that

$$||z_{m,1}||_{\varepsilon,1}, ||p_{m,1}||_{\varepsilon,1} = \mathcal{O}\big(\varepsilon^m\big), \ \varepsilon \to 0. \tag{3.16}$$

Now the pair of functions $z_{m,2} := z_m - z_{m,1}$ and $p_{m,2} := p_m - p_{m,1}$ satisfies the following boundary value problem

$$\begin{cases} Az_{m,2} = f(x), & x \in \Omega_\varepsilon, \\ A^* p_{m,2} - z_{m,2} = 0, & z_{m,2}, p_{m,2} \in H^1(\Omega_\varepsilon), \\ \dfrac{\partial z_{m,2}}{\partial n_A} + \lambda_m p_{m,2} = g(x), & \dfrac{\partial p_{m,2}}{\partial n_{A^*}} = 0, \quad x \in \Gamma, \\ \dfrac{\partial z_{m,2}}{\partial n_A} = g_{1,\gamma,m}(x), & \dfrac{\partial p_{m,2}}{\partial n_{A^*}} = 0, \quad x \in \varepsilon\gamma. \end{cases}$$

It denotes, that the function $z_{m,2}(\cdot)$ is the solution of the problem of the optimal control (1.1) — (1.3) with $\mathcal{U} = \mathcal{U}_\varepsilon(r_m)$, where $r_m = \lambda_m |||p_{m,2}|||_0$ with optimal control $u_m = -\lambda_m p_{m,2}\big|_\Gamma$.

But on the strength of (3.14) and (3.16)

$$\lambda_m^2 |||p_{m,2}|||_0^2 = \lambda_m^2 |||p_m - p_{m,1}|||_0^2 = \lambda_m^2 \big(|||p_m|||_0^2 - 2\langle p_m, p_{m,1}\rangle_0 + |||p_{m,1}|||_0^2 \big) =$$
$$= 1 + \mathcal{O}\big(\varepsilon^m\big),$$

therefore and $\lambda_m |||p_{2,m}||| = 1 + \mathcal{O}\big(\varepsilon^m\big)$ when $\varepsilon \to 0$. Hereof with respect to the theorem 3 for $u_\varepsilon = -\lambda_\varepsilon p_\varepsilon\big|_\Gamma$ and $u_m = -\lambda_m p_{2,m}\big|_\Gamma$ with consideration of the equality (3.7) we obtain

$$|||u_\varepsilon - u_m|||_0 = \mathcal{O}\big(\varepsilon^m\big), \ \varepsilon \to 0. \tag{3.17}$$

Let us now consider the functions $z_{\varepsilon,m,2} := z_\varepsilon - z_{m,2}$, $z_{\varepsilon,m,2} := z_\varepsilon - z_{m,2}$ They satisfy the boundary value problem

$$\begin{cases} Az_{\varepsilon,m,2} = 0, & x \in \Omega_\varepsilon, \\ A^* p_{\varepsilon,m,2} - z_{m,2} = 0, & z_{\varepsilon,m,2}, p_{\varepsilon,m,2} \in H^1(\Omega_\varepsilon), \\ \dfrac{\partial z_{\varepsilon,m,2}}{\partial n_A} = u_\varepsilon(x) - u_m(x), & \dfrac{\partial p_{\varepsilon,m,2}}{\partial n_{A^*}} = 0, \quad x \in \Gamma \\ \dfrac{\partial z_{\varepsilon,m,2}}{\partial n_A} = 0, & \dfrac{\partial p_{m,2}}{\partial n_{A^*}} = 0, \quad x \in \varepsilon\gamma, \end{cases}$$

Thereby for any φ, $\psi \in H^1(\Omega_\varepsilon)$ the following correlations hold

$$\begin{aligned} 0 &= \pi_\varepsilon(\nabla z_{\varepsilon,m,2}, \nabla\varphi) + (a_0 z_{\varepsilon,m,2}, \varphi)_\varepsilon - \langle u_\varepsilon - u_m, \varphi\rangle_0, \\ (z_{\varepsilon,m,2}, \psi) &= \pi_\varepsilon(\nabla\psi, \nabla p_{\varepsilon,m,2}) + (a_0 p_{\varepsilon,m,2}, \psi)_\varepsilon. \end{aligned}$$

If we add into these correlations $\varphi = z_{\varepsilon,m,2}$ and $\psi = p_{\varepsilon,m,2}$ subject to (1.6) and (3.17), we obtain

$$||z_{\varepsilon,m,2}||_{\varepsilon,1}, ||p_{\varepsilon,m,2}||_{\varepsilon,1} = \mathcal{O}\big(\varepsilon^m\big), \ \varepsilon \to 0. \tag{3.18}$$

Whereas $z_{\varepsilon,m} = z_{\varepsilon,m,2} + z_{m,1}$, and $p_{\varepsilon,m} = p_{\varepsilon,m,2} + p_{m,1}$, then to obtain final estimates (3.15) for these functions, we should apply the inequality of the triangle for the corresponding norms and the obtained already estimates (3.16) and (3.18), Theorem 5.1 from [8, Chapter 2, s. 5] and the embedding theorem [12].

Let us prove the last remaining estimate for the value $|\lambda_{\varepsilon,m}|$.

It follows from the theorem 2 and the correlation (3.7) that $\lambda_0 |||p_0|||_0 = 1$. Whereas $|||p_\varepsilon|||_0 \to |||p_0|||_0$ when $\varepsilon \to 0$, then

$$|||p_\varepsilon|||_0^{-1} = \mathcal{O}(1), \ \varepsilon \to 0. \tag{3.19}$$

Finally, $|\lambda_{\varepsilon,m}|\cdot|||p_\varepsilon|||_0 = |||\lambda_\varepsilon p_\varepsilon - \lambda_m p_\varepsilon|||_0 \leqslant |||\lambda_\varepsilon p_\varepsilon - \lambda_m p_m|||_0 + \ |||\lambda_m p_m - \lambda_m p_\varepsilon|||_0 \overset{(3.17)}{=} \mathcal{O}\big(\varepsilon^m\big)$, that subject to (3.19) finally provides $|\lambda_{\varepsilon,m}| = \mathcal{O}\big(\varepsilon^m\big)$. ■

4. Construction of an asymptotic expansion

We are intended to search for an external expansion in the form of asymptotic series

$$\mathcal{Z}(x) = \sum_{k=0}^{\infty}\varepsilon^k\sum_{l=0}^{k-2} z_{k,l}(x)\ln^l\varepsilon, \qquad \mathcal{P}(x) = \sum_{k=0}^{\infty}\varepsilon^k\sum_{l=0}^{k-2} u_{k,l}(x)\ln^l\varepsilon,$$
$$\Lambda(\varepsilon) = \sum_{k=0}^{\infty}\varepsilon^k\sum_{l=0}^{k-2}\lambda_{k,l}\ln^l\varepsilon, \quad \varepsilon\to 0, \tag{4.1}$$

and an internal expansion for the function $v(\xi) := z(\varepsilon\xi)$ and $w(\xi) := p(\varepsilon\xi)$, where ξ is an internal variable ($x = \varepsilon\xi$), we search in the form

$$\mathcal{V}(\xi) = \sum_{i=0}^{\infty}\varepsilon^i\sum_{m=0}^{i-2} v_{i,m}(\xi)\ln^m\varepsilon, \qquad \mathcal{W}(\xi) = \sum_{i=0}^{\infty}\varepsilon^i\sum_{m=0}^{i-2} w_{i,m}(\xi)\ln^m\varepsilon. \tag{4.2}$$

As usual, we consider, that $z_{k,i} = 0$, $p_{k,l} = 0$, $\lambda_{k,l} = 0$ when $l > k-3$ and $v_{i,m} = 0$, $w_{i,m} = 0$ when $m > i-2$.

The functions $z_{0,0}(x)$, $p_{0,0}(x)$ and the number $\lambda_{0,0}$ are the solution of the boundary value problem (3.1), (3.6) $z_0(x)$, $p_0(x)$ and λ_0. Thereby, as it has already been noted, $z_0(x), p_0(x) \in C^\infty(\overline{\Omega})$.

For the series (4.1) and (4.2) the following condition of matching holds true [2]:

$$\forall\, n,m\in\mathbb{N} \quad \mathcal{A}_{m,\xi}\mathcal{A}_{n,x}\mathcal{Z} = \mathcal{A}_{n,x}\mathcal{A}_{m,\xi}\mathcal{V}, \quad \mathcal{A}_{m,\xi}\mathcal{A}_{n,x}\mathcal{P} = \mathcal{A}_{n,x}\mathcal{A}_{m,\xi}\mathcal{W}, \tag{4.3}$$

where $\mathcal{A}_{n,x}$ ($\mathcal{A}_{m,\xi}$) is the sampling operator of minor total of the asymptotic expansion of the function from ε, x (ε, ξ) with the precision to $o\big(\varepsilon^n\big)$ ($o\big(\varepsilon^m\big)$), thereby asymptotic expansions of the function of the form $b(x/\varepsilon)$ are applied when $\xi = x/\varepsilon\to\infty$ (and the function of the form $b(\varepsilon\xi)$ when $x = \varepsilon\xi\to 0$).

The functions $z_{k,l}(x)$, $pu_{k,l}(x)$ and the numbers $\lambda_{k,l}$ are the solutions of the problems

$$\begin{cases} Az_{k,l}(x) = 0, \quad x\in\Omega\setminus O, \\ A^*p_{k,l} - z_{k,l} = 0, \quad z_{k,l}, p_{k,l}\in C^\infty(\overline{\Omega}\setminus\{O\}), \\ \dfrac{\partial z_{k,l}}{\partial n_A} + \lambda_0 p_{k,l}(x) = -\lambda_{k,l}p_0(x) + g_{k,l}(x), \quad \dfrac{\partial p_{k,l}}{\partial n_{A*}} = 0 \quad x\in\partial\Omega, \end{cases} \tag{4.4}$$

where $g_{k,l}(x) = -\displaystyle\sum_{s=1}^{k-1}\sum_{\sigma}\lambda_{s,\sigma}p_{k-s,l-\sigma}(x)$ and they are completely denoted by the solutions of the previous equations (here $\sigma : s-3\ge\sigma\ge 0$, $k-l-3\ge s-\sigma$, $l\ge\sigma$).

To obtain analogous equations for $v_{i,m}(\xi)$ and $w_{i,m}(\xi)$ it is necessary to expand operators A, A^*, ∂/n_A and ∂/n_{A^*} in the neighbourhood of the point O in the series when $x\to 0$. On the strength of (1.6) when $x \longrightarrow 0$ we obtain

$$A = -\Delta - \sum_{i=1}^{\infty} Q_{i,2}(x,D) - \sum_{i=0}^{\infty} Q_{i,1}(x,D) - \sum_{i=0}^{\infty} Q_{i,0}(x),$$
$$A^* = -\Delta - \sum_{i=1}^{\infty} Q^*_{i,2}(x,D) - \sum_{i=0}^{\infty} Q^*_{i,1}(x,D) - \sum_{i=0}^{\infty} Q_{i,0}(x),$$
$$\frac{\partial}{\partial n_A} = \frac{\partial}{\partial n} + \sum_{i=1}^{\infty} q_{i,1}(x,D), \quad \frac{\partial}{\partial n_{A^*}} = \frac{\partial}{\partial n} + \sum_{i=1}^{\infty} q^*_{i,1}(x,D),$$

where $Q_{i,j}(x,D)$, $Q^*_{i,j}(x,D)$, $q_{i,j}(x,D)$ and $q^*_{i,j}(x,D)$ are polynomials from $x = (x_1, x_2, x_3)$ and $D = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2} \frac{\partial}{\partial x_3}\right)$ are homogeneous degree of i on x and the degree of j on D (thereby the operator D comes before multiplication). Let us note, that $\sum\limits_{i=0}^{\infty} Q_{i,0}(x)$ is Maclaurin series of the function $a_0(x)$.

Substituting these expansions in the system for the functions v and w, we obtain functions $v_{i,m}$ and $w_{i,m}$ for the following problems

$$\left\{\begin{array}{rcl} \Delta v_{0,0}(\xi) &=& 0, \qquad \Delta v_{0,0}(\xi) = 0, \\ \Delta v_{1,0}(\xi) &=& (Q_{1,2}(\xi,D) + Q_{0,1}(\xi,D))v_{0,0}(\xi), \\ \Delta w_{1,0}(\xi) &=& (Q^*_{1,2}(\xi,D) + Q^*_{0,1}(\xi,D))w_{0,0}(\xi), \\ \Delta v_{i,m}(\xi) &=& \sum\limits_{s=1}^{i} \Big(Q_{s,2}(\xi,D) + Q_{s-1,1}(\xi,D) + \\ && +Q_{s-2,0}(\xi)\Big) v_{i-s,m}(\xi) - f_{1,i-2,m}(\xi), \\ \Delta w_{i,m}(\xi) &=& \sum\limits_{s=1}^{i} \Big(Q^*_{s,2}(\xi,D) + Q^*_{s-1,1}(\xi,D) + \\ && +Q_{s-2,0}(\xi)\Big) w_{i-s,m}(\xi) + v_{i-2,m} - f_{2,i-2,m}(\xi), \end{array}\right. \quad \xi \notin \omega \tag{4.5}$$

with boundary conditions

$$\left\{\begin{array}{l} \frac{\partial v_{0,0}}{\partial n} = 0, \quad \frac{\partial w_{0,0}}{\partial n} = 0, \quad \frac{\partial v_{1,0}}{\partial n} = 0, \quad \frac{\partial w_{1,0}}{\partial n} = 0, \\ \frac{\partial v_{i,m}}{\partial n} = -\sum\limits_{s=1}^{i} q_{s,i} v_{i-s,m}, \quad \frac{\partial w_{i,m}}{\partial n} = -\sum\limits_{s=1}^{i} q_{s,i} w_{i-s,m}, \end{array}\right. \quad \xi \in \omega. \tag{4.6}$$

Here $f_{1,i-2,m}$ and $f_{2,i-2,m}$ are generated by expansions when $x \to 0$ of the functions $f(x)$ and $z_d(x)$, correspondingly.

The supplementary condition (3.7) takes the following form

$$\lambda_0 \langle p_0, p_{k,l} \rangle_0 + \lambda_{k,l} |||p_0|||_0^2 = \delta_{k,l}, \tag{4.7}$$

where the numbers $\delta_{k,l}$ are denoted by the previous $p_{k,l}$ and $\lambda_{k,l}$.

Uppermost, we note, that $v_{0,0} = z_0(0)$ and $w_{0,0} = p_0(0)$, however, on the strength of (4.3) $v_{1,0}$ and $w_{1,0}$ are not constants, thereby these functions are not bounded when $\xi \to \infty$. In its turn this generates unboundedness of other functions $z_{k,l}$, $p_{k,l}$, $v_{i,m}$, $w_{i,m}$. Thereby the present problem is bisingular. In [3] there are classes of functions unbounded when $x \to 0$ and when $\xi \to \infty$, correspondingly, in which the problem, analogous to the one considered here is solvable. In these classes functions and problems (4.4) — (4.6) are solvable. The proof of this fact almost word by word repeats the proofs from [3, § 3].

Whereas the solution of the system (4.4) can be presented in the form

$$z_{k,l}(x) = \lambda_{k,l}\overline{z}_0(x) + \overline{Z}_{k,l}, \quad p_{k,l}(x) = \lambda_{k,l}\overline{p}_0(x) + \overline{P}_{k,l}, \tag{4.8}$$

where $\overline{z}_0, \overline{p}_0 \in C^\infty(\overline{\Omega})$ is the solution of the problem

$$\begin{cases} A\overline{z}_0 = 0, \quad A^*\overline{p}_0 = 0, & x \in \Omega, \\ \dfrac{\overline{z}_0}{\partial n_A} + \lambda_0\overline{p}_0 = -p_0, \quad \dfrac{\overline{p}_0}{\partial n_{A^*}} = 0, & x \in \Gamma, \end{cases} \tag{4.9}$$

and $\overline{Z}_{k,l}, \overline{P}_{k,l} \in C^\infty(\overline{\Omega} \setminus \{O\})$ is the solution of the heterogeneous system

$$\begin{cases} A\overline{Z}_{k,l} = 0, \quad A^*\overline{P}_{k,l} - \overline{Z}_{k,l} = 0, & x \in \Omega \setminus O, \\ \dfrac{\partial\overline{Z}_{k,l}}{\partial n_A} + \lambda_0\overline{P}_{k,l} = g_{k,l}(x), \quad \dfrac{\partial\overline{P}_{k,l}}{\partial n_{A*}} = 0, & x \in \partial\Omega, \end{cases}$$

Then the equations (4.7) take the form

$$\lambda_{k,l}\big(\lambda_0\langle p_0, \overline{p}_0\rangle_0 + |||p_0|||_0^2\big) = \overline{\delta}_{k,l}. \tag{4.10}$$

Lemma 4. *The following correlation holds*

$$\lambda_0\langle p_0, \overline{p}_0\rangle_0 + |||p_0|||_0^2 \neq 0. \tag{4.11}$$

Proof. If we multiply the first equality in the system (4.9) by $\overline{p}_0$ and apply Green formula (2.23) for the domain Ω, we obtain the equality

$$||\overline{z}_0||_0^2 + \lambda_0|||\overline{p}_0|||_0^2 = -\langle p_0, \overline{p}_0\rangle. \tag{4.12}$$

Assume now, that the correlation (4.11) is not valid. Then

$$-\langle p_0, \overline{p}_0\rangle = \lambda_0^{-1}|||p_0|||_0^2 \ \text{ and} \tag{4.13}$$

$$p_0 \perp \big(p_0 + \lambda_0^{-1}\overline{p}_0\big) \ \text{ in } \ L_2(\Gamma). \tag{4.14}$$

We obtain from the equalities (4.12) and (4.13) that

$$\lambda_0||\overline{z}_0||_0^2 + \lambda_0^2|||\overline{p}_0|||_0^2 = |||p_0|||_0^2. \tag{4.15}$$

On the other hand, on the strength of the correlation (4.14) and Pythagorean theorem

$$\lambda_0^2|||\overline{p}_0|||_0^2 = |||p_0|||_0^2 + |||p_0 + \lambda_0\overline{p}_0|||_0^2. \tag{4.16}$$

It results from the equalities (4.15) and (4.16) that $\overline{z}_0 = 0$ and $\big(p_0 + \lambda_0^{-1}\overline{p}_0\big)\big|_\Gamma = 0$. But then, on the strength of (4.9) $\overline{p}_0\big|_\Gamma = 0$ and, therefore, $p_0\big|_\Gamma = 0$, that contradicts the correlation (3.6). ▪

Construction of the functions $z_{k,l}(x)$, $p_{k,l}(x)$, $v_{i,m}(\xi)$, $w_{i,m}(\xi)$ and the numbers $\lambda_{k,l}$ is standard for the method of matching of asymptotic expansions [2]. The functions $z_{0,0}(\varepsilon\xi)$, $p_{0,0}(\varepsilon\xi)$ determine dominant terms of asymptotic expansions of the functions $v_{i,m}(\xi)$, $v_{i,m}(\xi)$ $(i > 0)$ when $\xi \to \infty$. If we denote the functions $v_{1,0}(\xi)$ and $w_{1,0}(\xi)$ by them, then from the expansion of the functions $v_{1,0}(x/\varepsilon)$ and $w_{1,0}(x/\varepsilon)$ when $x/\varepsilon \to \infty$ we obtain dominant terms of the asymptotic when $x \to 0$ of the functions $z_{k,l}(x)$, $p_{k,l}(x)$ $(k > 0)$. Having obtained $\overline{Z}_{k,l}(x)$, $\overline{P}_{k,l}(x)$ with the given asymptotic, from the equation (4.10) we obtain $\lambda_{1,0}(x)$. Now $z_{1,0}(x)$ and $p_{1,0}(x)$ are determined together with the following terms of expansion $v_{i,m}(\xi)$ and $v_{i,m}(\xi)$ $(i > 1)$, and etc.

The proof of the fact, that the constructed and matched series (4.1) and (4.2) in the sense of (4.3) are the asymptotic of the solution of the problem (2.4), (3.7), is conducted by analogy with [3, § 2,§ 5]). Thereby, the following theorem holds.

Theorem 5. *Let the conditions* (1.4)*,* (1.6)*,* (1.7) *and* (3.6) *hold. Then the solution of the problem* (2.4)*,* (3.7) *is expanded into asymptotic series of the form* (4.1)*,* (4.2) *equal in the domain* $C^\infty(\overline{\Omega} \setminus \{O\})$ (*in the sense of norms* $||\cdot||_{H^2(\Omega_\varepsilon)}$ *and* $||\cdot||_{C(\overline{\Omega_\varepsilon})}$).

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