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ON RESOLVENTS OF PERIODIC OPERATORS WITH DISTANT PERTURBATIONS

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Dedicated to Arlen Mikhailovich Il'in

Abstract. We consider distant perturbations of an abstract periodic operator. An unperturbed operator is introduced as a closed operator on a Sobolev space defined on a periodic domain in a multidimensional space. We impose certain conditions for the unperturbed operator being a natural generalization of the ellipticity and periodicity conditions for differential operators. The perturbations are described by abstract relatively bounded operators being localized in a certain sense. We study the case when the distances between the domains, where the perturbations are localized, increases unboundedly. The main obtained result is the explicit representation for the resolvent of the perturbed operator.

Keywords: resolvent, periodic operator, distant perturbations.

1. Introduction

There is quite a number of papers devoted to operators with distant perturbations (see, for instance, [1]–[14]). The greater part of them is devoted to study of Schrodinger operator with real potentials (see, for example, [4]–[10], [12]–[14]). At that, the potentials were localized on finite domains and the between these domains were supposed to tend to infinity, which explains the name of such perturbations, "distant". The main attention in the cited articles was paid to the study of asymptotic behavior of eigenvalues and eigenfunctions. But there are just a few papers devoted to studying the behavior of the resolvent of the operators with distant perturbations (see, for instance, [5], [6, Ch. 8, §8.6], [10], [15], [16]). Let us dwell on these papers.

In [5] the Laplace operator in the space $\mathbb{R}^3$ with three distant potentials was considered. The potentials satisfied two conditions, the first of them ensured which relative compactness and the second described analytic properties of the potentials. Strong resolvent convergence of a perturbed operator to the unperturbed operator was proven there. The expansion of the resolvent of the perturbed operator in the Neumann series converging in the strong resolvent sense was provided. The Schrödinger operator with two distant perturbations in $\mathbb{R}^3$ was considered in [6, Ch. 8, §8.6]. The perturbations were two real potential decaying at infinity. The convergence of the resolvent of a unitary transformation of some matrix operator was proven, where the operator was constructed on the base of an initial operator with distant perturbations. At the same time, the unitary transformation was constructed there in a special way and depended itself on the distance between distant potentials. In [10] the behavior of the resolvent of a Schrodinger operator with two distant perturbations in the space $\mathbb{R}^3$ was considered. The

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perturbations were given by real-valued functions from Rollnik class. It is supposed that the function $V(x)$ belongs to the Rollnik class, if

$$\int\limits_{\mathbb{R}^3\times\mathbb{R}^3} \frac{|V(x)||V(y)|}{|x-y|^2}dxdy<\infty.$$

It was proven that the difference of the resolvents of perturbed and unperturbed operators tends to zero.

The most general results were obtained in the papers [15], [16]. Here a periodic differential elliptic operator of an even order with finite number of distant perturbations in multidimensional space was considered. The perturbing operators were abstract localized operators, whose localization was described in terms of special weight functions. In these papers an explicit formula for the resolvent of the perturbed operator was obtained. On the basis of this formula a uniform resolvent convergence of a perturbed operator to some limiting one was proven, and also the representation for the resolvent in the form of a uniformly converging asymptotic series was obtained.

In the present paper we consider an abstract operator with distant perturbations in an arbitrary domain. The unperturbed operator, unlike papers [15], [16], is not considered to be differential. This operator is introduced as an operator in $L_2(\Omega)$ on some periodic domain Ω in a multidimensional space. We replace the ellipticity conditions in the papers [15], [16] by certain apriori estimates, and the periodicity condition is replaced by the commuting of our operator with a shift operator along the domain Ω. One more difference from [15],[16] is that the domain Ω is quite arbitrary, while in [15], [16] this domain was a multidimensional space. The class of unperturbed operators considered in the present paper is rather wide. The unperturbed operator can be a differential operator of an arbitrary order in various periodic domains and an integral-differential operator (see the third section). Distant perturbations are defined by analogy with [15], [16]. Our general result is of the same type as in the cited papers, i.e., we obtain an explicit representation for the resolvent of the perturbed operator under the assumption that the distance between the domains, where the perturbing operators are localized, tends to infinity. On the basis of this representation we prove the uniform resolvent convergence of a perturbed operator to some limiting operator. We also present complete asymptotic series converging in the uniform operator norm. The core of the proof of the main result is a generalization of the scheme presented in the papers [15], [16].

Let us describe the structure of the paper. In the next section we state the problem and describe the main result. In the third section we give examples of the unperturbed and perturbing operators, weight functions, and various types of domains. In the fourth section we prove the main result.

2. Formulation of the problem and the main result

Let $(x_1,\ldots,x_d)$ be Cartesian coordinates in $\mathbb{R}^d$, $d\geqslant 1$, and $(e_1,\ldots,e_\ell)$ be a set of linearly independent vectors in $\mathbb{R}^d$, where $\ell\leqslant d$. We denote by Γ the group of all integer combinations of the form $z_1e_1+\ldots+z_\ell e_\ell$, $z_i\in\mathbb{Z}$. By Ω we indicate a domain in $\mathbb{R}^d$ with sufficiently smooth boundary being invariant w.r.t. to the shifts by the elements of the group Γ.

Let $X_i\in\Gamma$, $i=1,\ldots k$ be some parameters. Throughout below we suppose $\tau(X)\to\infty$.

In $L_2(\Omega;\mathbb{C}^n)$ we consider an abstract operator $\mathcal{H}_0$ whose domain $\mathfrak{D}(\mathcal{H}_0)$ is a subspace of the Hilbert space $W_2^m(\Omega;\mathbb{C}^n)$, where $m\in\mathbb{N}$. We suppose that this operator satisfies the following conditions,

A1. For any $u\in\mathfrak{D}(\mathcal{H}_0)$ the inequality

$$\|u\|_{W_2^m(\Omega;\mathbb{C}^n)}\leqslant C_1\big(\|\mathcal{H}_0u\|_{L_2(\Omega;\mathbb{C}^n)}+\|u\|_{L_2(\Omega;\mathbb{C}^n)}\big)\leqslant C_2\|u\|_{W_2^m(\Omega;\mathbb{C}^n)},$$

holds true, where C_1, C_2 are some constants independent of u.

A2. The identity
$$\mathcal{S}(-X_i)\mathcal{H}_0\mathcal{S}(X_i)=\mathcal{H}_0,\quad i=1,\ldots,k,$$
holds true, where $\mathcal{S}(X_i)$ is the shift operator acting by the rule $(\mathcal{S}(X_i)u)(\cdot)=u(\cdot-X_i)$.

The former of these conditions should be understood as a generalization in a certain sense of an ellipticity condition for differential operators, and the second as a generalization of periodicity. We also note that the condition A1 implies immediately the closedness of the operator $\mathcal{H}_0$.

We introduce functions $\xi_i,\eta_i\in C^m(\overline{\Omega})$, $i=1,\ldots,k$, satisfying the following conditions,

A3. There exists a positive function $\varphi\in C^m(\overline{\Omega})$ such that the estimates
$$|\xi_i(x)|\leqslant C\varphi(x),\quad \partial^\alpha\varphi(x)\leqslant C\varphi(x),\quad x\in\overline{\Omega},\quad i=1,\ldots,k,\quad |\alpha|\leqslant m,$$
hold true, where C is some constant independent of x, $\alpha\in\mathbb{Z}_+^d$ is an arbitrary multi-index,

A4. The functions φ, η_i and all their derivatives up to the order m tends to zero on infinity.

In what follows we call these functions weight functions. We suppose that for the operator $\mathcal{H}_0$ one more assumption holds,

A5. For any $u\in\mathfrak{D}(\mathcal{H}_0)$ and quite small ε the following estimate holds:
$$\|(\varphi^{-\varepsilon}\mathcal{H}_0\varphi^{\varepsilon}-\mathcal{H}_0)u\|_{L_2(\Omega;\mathbb{C}^n)}\leqslant\varsigma(\varepsilon)\|u\|_{W_2^m(\Omega;\mathbb{C}^n)},$$
where the function $\varsigma(\varepsilon)$ is independent of u and $\varsigma(\varepsilon)\to0$ as $\varepsilon\to0$.

Let $\mathcal{L}_i^0$, $i=1,\ldots k$ be arbitrary operators bounded as the operators from $W_2^m(\Omega;\mathbb{C}^n)$ into $L_2(\Omega;\mathbb{C}^n)$. We introduce the operators $\mathcal{L}_i=\xi_i\mathcal{L}_i^0\eta_i$ in $L_2(\Omega;\mathbb{C}^n)$ with the domain $W_2^m(\Omega;\mathbb{C}^n)$. As distance perturbations, we regard the operators of the form $\sum\limits_{i=1}^{k}\mathcal{S}(-X_i)\mathcal{L}_i\mathcal{S}(X_i)$.

In $L_2(\Omega;\mathbb{C}^n)$ we define a perturbed operator
$$\mathcal{H}_X:=\mathcal{H}_0+\sum_{i=1}^{k}\mathcal{S}(-X_i)\mathcal{L}_i\mathcal{S}(X_i)$$
with the domain $\mathfrak{D}(\mathcal{H}_0)$.

The aim of the present paper is to study the behavior of the resolvent of the perturbed operator as $\tau(X)\to\infty$. To formulate the main result, we need auxiliary notations.

We consideration the family of operators $\mathcal{H}_i:=\mathcal{H}_0+\mathcal{L}_i$ in $L_2(\Omega;\mathbb{C}^n)$ with the domain $\mathfrak{D}(\mathcal{H}_0)$. We assume that

A5. Operators $\mathcal{H}_i$ are closed.

We denote by $\sigma(\cdot)$ the spectrum of the operator, by $\|\bullet\|_{Y_1\to Y_2}$ the norm of a linear operator acting from the space Y_1 to the space Y_2, and by I the identical operator.

Let us formulate the main result.

Theorem 1. *Let the set* $M:=\mathbb{C}\setminus\bigcup\limits_{i=0}^{k}\sigma(\mathcal{H}_i)$ *be nonempty. Then for sufficiently great* $\tau(X)$ *the operator* $\mathcal{H}_X$ *is closed. For any* $\lambda\in M$ *and sufficiently great* $\tau(X)$ *the resolvent of the perturbed operator is well-defined and the representation*
$$(\mathcal{H}_X-\lambda)^{-1}=\Big[\sum_{i=1}^{k}\mathcal{S}(-X_i)(\mathcal{H}_i-\lambda)^{-1}\mathcal{S}(X_i)-(k-1)(\mathcal{H}_0-\lambda)^{-1}\Big](\mathrm{I}+\mathcal{P}_X)^{-1},$$
$$\mathcal{P}_X:=\sum_{\substack{i,j=1\\ i\neq j}}^{k}\mathcal{S}(-X_i)\mathcal{L}_i\mathcal{S}(X_i)\Big[\mathcal{S}(-X_j)(\mathcal{H}_j-\lambda)^{-1}\mathcal{S}(X_j)-(\mathcal{H}_0-\lambda)^{-1}\Big],\tag{1}$$
holds true, where $\|\mathcal{P}_X\|_{L_2(\Omega;\mathbb{C}^n)\to L_2(\Omega;\mathbb{C}^n)}\to0$ *when* $\tau(X)\to\infty$.

Let us discuss the main result of the present paper. The assumption that the set M is nonempty is quite natural and holds for a rather large class of operators. For example, this set is apriori nonempty, if the operators $\mathcal{H}_i$, $i=0,\ldots,k$ are self-adjoint. The set M is also nonempty, if we suppose that the operators $\mathcal{H}_i$ or the operators $-\mathcal{H}_i$, $i=0,\ldots,k$, are m-sectorial.

The main and the most important result of the present paper is the explicit formula for the resolvent of the perturbed operator $\mathcal{H}_X$ provided in the theorem. As it follows from this formula, the form of the resolvent is actually determined by the operator $\mathcal{P}_X$. The latter operator is a kind of universal characteristic of the resolvent of the perturbed operator $\mathcal{H}_X$. As one can see in the formula for the operator $\mathcal{P}_X$, it is a sum of terms which can be interpreted as pairwise interaction of the operators $\mathcal{L}_i$. Therefore, the problem of finding the resolvent of the perturbed operator $\mathcal{H}_X$ is reduced to determining the operator $\mathcal{P}_X$. If we know the latter, we can not only deduce the explicit formula for the resolvent of the perturbed operator $\mathcal{H}_X$, but also to obtain the complete asymptotic expansion of the resolvent of the perturbed operator $\mathcal{H}_X$. In order to do it, it is sufficient to expand the operator $(\mathrm{I}+\mathcal{P}_X)^{-1}$ in the formula for the resolvent into the Neumann series.

3. Examples

In the present section we provide the examples of the unperturbed operator $\mathcal{H}_0$ and examples of the domains Ω. Numerous examples of weight functions and perturbing operators $\mathcal{L}_i^0$ were in details discussed in the third section of the paper [15] for the case $\Omega=\mathbb{R}^d$. These examples are easily extended on the case of an arbitrary domain Ω. Note, that like in [15], the class of weight functions is rather wide. In particular, the decay can be exponential, power, and even logarithmic.

As the unperturbed operator we can consider different differential operators with boundary conditions of the first, second, and the third kind. For example, a second-order differential operator, matrix and magnetic Schrödinger operator, operator of the elasticity theory, two- and three-dimensional Pauli operator, and also an operator with δ-potential. The main restriction for the boundary conditions is periodicity. A detailed description of these examples in the whole space $\mathbb{R}^d$ is described in the third section of the paper [15]. Their generalization for different domains is not complicated and this is why we are not intended to dwell on it.

As examples of the domains Ω we can take a multidimensional space, $\Omega=\mathbb{R}^d$, $d\in\mathbb{N}$, a domain like layer or strip, $\Omega=\omega\times\mathbb{R}^p$, ω is a bounded domain in $\mathbb{R}^q$. The next example is periodically curved strips (cf. Fig. 1), or periodically twisted multidimensional cylinders (cf. Fig. 2). It is also possible to deal with the domains with periodic perforation (cf. Fig. 3).

As an example of a non-differential operator $\mathcal{H}_0$ we can take an integral-differential operator

$$\mathcal{H}_0 u := \left(\sum_{\substack{\beta,\gamma\in\mathbf{Z}_+^d\\ |\beta|=|\gamma|=p}} \frac{\partial^\beta}{\partial x^\beta}\mathrm{A}_{\beta\gamma}\frac{\partial^\gamma}{\partial x^\gamma} + \sum_{\substack{\beta\in\mathbf{Z}_+^d\\ |\beta|\leqslant 2p-1}} \mathrm{A}_\beta\frac{\partial^\beta}{\partial x^\beta}\right) u + \int\limits_\Omega \mathrm{F}(\cdot,y,\cdot-y)u(y)\,\mathrm{d}y,$$

where $p\in\mathbb{N}$. It is supposed that the differential part of the operator $\mathcal{H}_0$ satisfies the ellipticity condition

$$\nu\sum_{\substack{\beta\in\mathbf{Z}_+^d\\ |\beta|=p}}|\xi_\beta|^2\leqslant \operatorname{Re}\sum_{\substack{\beta,\gamma\in\mathbf{Z}_+^d\\ |\beta|=|\gamma|=p}}(\mathrm{A}_{\beta\gamma}(x)\xi_\beta,\xi_\gamma)_{\mathbb{C}^n}\,,\quad \xi_\beta\in\mathbb{C}^n,$$

ν is some constant, independent of x and ξ_β, $m\in\mathbb{N}$, the functions $\mathrm{A}_{\beta\gamma}\in C^p(\overline{\Omega})$, $\mathrm{A}_\beta(\overline{\Omega})$ are periodic w.r.t. to the shifts by the elements of the group Γ, i.e.,

$$\mathrm{A}_{\beta\gamma}(x+\rho)=\mathrm{A}_{\beta\gamma}(x),\quad \mathrm{A}_\beta(x+\rho)=\mathrm{A}_\beta(x),\quad x\in\Omega,\quad \rho\in\Gamma.$$

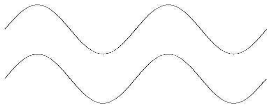

Fig. 1: Periodically curved stripe

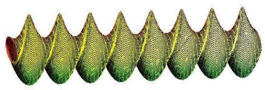

Fig. 2: Twisted cylinder

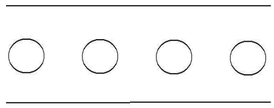

Fig. 3: Periodically perforated domain

The function $\mathrm{F}(x,y,z)$ is periodic by x and y w.r.t. to the shifts by the elements of the group Γ, is compactly supported w.r.t. z, and satisfies the condition

$$\int\limits_{\Omega} \max_{x,y} |\mathrm{F}(x,y,z)| dz = \int\limits_{\Omega} f(z) dz < \infty, \tag{2}$$

where $f(z) := \max\limits_{x,y} |\mathrm{F}(x,y,z)|$ is some compactly supported function. We observe that the conditions (2) and the fact that the function $\mathrm{F}(x,y,z)$ is compactly supported w.r.t. the variable z are rather weak restrictions, and the class of possible functions $\mathrm{F}(x,y,x-y)$ is quite wide.

Let us check the conditions (A1) — (A3) for the operator $\mathcal{H}_0$. In accordance with Lemmas 2 and 4 in the paper [15], the conditions (A1), (A2), (A5) hold for the differential part of the operator $\mathcal{H}_0$. Consider the integral part of the unperturbed operator $\mathcal{H}_0$. Due to Cauchy-Schwarz inequality, the estimate (2) and the compactness of the support of the function $F(x,y,z)$ w.r.t. the variable z, the inequalities

$$M(x) := \left| \int\limits_{\Omega} \mathrm{F}(x,y,x-y) u(y) dy \right|^2 \leqslant \int\limits_{\Omega} f(t) dt \int\limits_{\Omega} f(x-y) \big| u(y) \big|^2 dy,$$

$$\int\limits_{\Omega} M(x) dx \leqslant C \big\| u \big\|^2_{L_2(\Omega;\mathbb{C}^n)},$$

hold true, where C is some constant independent of u. Thus, the integral part of the unperturbed operator $\mathcal{H}_0$ acts from $L_2(\Omega;\mathbb{C}^n)$ into $L_2(\Omega;\mathbb{C}^n)$, and the condition (A1) holds true. Since the function $\mathrm{F}(x,y,z)$ is periodic w.r.t. the variables x and y, then the condition (A2) for the integral part of the unperturbed operator $\mathcal{H}_0$ also holds true. To prove the conditions (A5), it is sufficient to estimate the integral

$$N_\varepsilon(x) = \int\limits_{\Omega} \Big[\varphi^\varepsilon(y) \varphi^{-\varepsilon}(x) - 1 \Big] F(x,y,x-y) u(y) dy.$$

In order to do it, we additionally suppose that the function φ satisfies the condition

$$K_1 \leqslant \frac{\varphi(x-t)}{\varphi(x)} \leqslant K_2, \quad x \in \Omega, \quad t \in \Pi, \tag{3}$$

where Π is a compact set, K_1, K_2 are some positive numbers independent of x and t, and for all x, y the support of the function $F(x,y,\cdot)$ is lies inside Π. This condition is rather weak. In particular, it holds for all the examples of the weight functions described in the third section of the paper [15].

Applying Cauchy-Schwarz inequality, making the change of the variable $x - y = t$, and also taking into consideration the condition (3) and the fact that the function $f(z)$ is compactly supported, we obtain

$$\begin{aligned}\Big|N_\varepsilon(x)\Big|^2 &\leqslant \int\limits_\Omega \Big[\varphi^\varepsilon(y)\varphi^{-\varepsilon}(x)-1\Big]^2 f(x-y)dy \int\limits_\Omega f(x-y)\big|u(y)\big|^2dy\\ &\leqslant \int\limits_\Omega \Big[\Big(\frac{\varphi(x-t)}{\varphi(x)}\Big)^\varepsilon-1\Big]^2 f(t)dt \int\limits_\Omega f(x-y)\big|u(y)\big|^2dy\\ &\leqslant \varepsilon K_3 \int\limits_\Omega f(x-y)\big|u(y)\big|^2dy,\end{aligned}$$

where K_3 is a constant independent of x. According to the last estimate, the inequality

$$\int\limits_\Omega \big|N_\varepsilon(x)\big|^2dx \leqslant \varepsilon K_4\|u\|^2_{L_2(\Omega;\mathbb{C}^n)}, \tag{4}$$

holds true, where K_4 is a constant independent of ε and u. It follows from the inequality (4) that the condition (A5) holds true for the integral part of the unperturbed operator $\mathcal{H}_0$. Therefore, the integral-differential operator $\mathcal{H}_0$ is an example of a non-differential unperturbed operator.

4. Proof of the main result

To prove Theorem 1, we need two auxiliary lemmas.

Lemma 1. *Let $\lambda\in M$, ξ be one of the functions ξ_i, $i=1,\dots k$, η one be of the functions η_j, $j=1,\dots k$, X be one of the vectors X_i-X_j, $i\neq j$. Then for all η_j, $j=1,\dots,k$, as $X\to\infty$*

$$\|\eta\mathcal{S}(X)(\mathcal{H}_0-\lambda)^{-1}\xi\|_{L_2(\Omega;\mathbb{C}^n)\to W_2^m(\Omega;\mathbb{C}^n)}\to 0 \tag{5}$$

holds true.

Proof. For each $f\in L_2(\Omega;\mathbb{C}^n)$ we let $u:=(\mathcal{H}_0-\lambda)^{-1}\xi f$. We seek the function u, being the solution of the equation

$$(\mathcal{H}_0-\lambda)u=\xi f, \tag{6}$$

as $u=\varphi^\varepsilon v$, where $\varepsilon>0$ is a sufficiently small constant. Substituting $u=\varphi^\varepsilon v$ into (6), we obtain,

$$(\mathcal{H}_0-\lambda)u=(\mathcal{H}_0-\lambda)\varphi^\varepsilon v=\mathcal{H}_0\varphi^\varepsilon v-\lambda\varphi^\varepsilon v=\xi f.$$

Let us divide the last equation by φ^ε, and then add and subtract $\mathcal{H}_0 v$ in the left hand side,

$$\begin{aligned}&\varphi^{-\varepsilon}\mathcal{H}_0\varphi^\varepsilon v-\lambda v+\mathcal{H}_0v-\mathcal{H}_0v=(\mathcal{H}_0+(\varphi^{-\varepsilon}\mathcal{H}_0\varphi^\varepsilon-\mathcal{H}_0)-\lambda)v\\ &=(\mathrm{I}+(\varphi^{-\varepsilon}\mathcal{H}_0\varphi^\varepsilon-\mathcal{H}_0)(\mathcal{H}_0-\lambda)^{-1})(\mathcal{H}_0-\lambda)v=\varphi^{-\varepsilon}\xi f\end{aligned} \tag{7}$$

In view of (A1), (A5), the norm of the operator $(\varphi^{-\varepsilon}\mathcal{H}_0\varphi^\varepsilon-\mathcal{H}_0)(\mathcal{H}_0-\lambda)^{-1}$ as an operator from $L_2(\Omega;\mathbb{C}^n)$ in $L_2(\Omega;\mathbb{C}^n)$ is small for sufficiently small ε.

Since the operators $(\mathcal{H}_0-\lambda)$, $(\mathrm{I}+(\varphi^{-\varepsilon}\mathcal{H}_0\varphi^\varepsilon-\mathcal{H}_0)(\mathcal{H}_0-\lambda)^{-1})$ are invertible, then the equation (7) is solvable, and its solution can be presented as

$$v=\big(\mathcal{H}_0-\lambda)^{-1}(\mathrm{I}+(\varphi^{-\varepsilon}\mathcal{H}_0\varphi^\varepsilon-\mathcal{H}_0)(\mathcal{H}_0-\lambda)^{-1}\big)^{-1}\varphi^{-\varepsilon}\xi f.$$

By the last equation and the condition (A3) we obtain the estimate

$$\begin{aligned}\|v\|_{W_2^m(\Omega;\mathbb{C}^n)}&=\|(\mathcal{H}_0-\lambda)^{-1}(\mathrm{I}+(\varphi^{-\varepsilon}\mathcal{H}_0\varphi^\varepsilon-\mathcal{H}_0)(\mathcal{H}_0-\lambda)^{-1})^{-1}\varphi^{-\varepsilon}\xi f\|_{L_2(\Omega;\mathbb{C}^n)}\\ &\leqslant C\,\|f\|_{L_2(\Omega;\mathbb{C}^n)},\end{aligned}$$

where C are some constants independent of v and f.

By the conditions (A3), (A4) the inequality

$$\sum_{\substack{\beta\in\mathbf{Z}_+^d\\|\beta|=m}}\left\|\frac{\partial^\beta}{\partial x^\beta}\left(\eta_i\mathcal{S}(X)u\right)\right\|^2_{L_2(\Omega;\mathbb{C}^n)}=\sum_{\substack{\beta\in\mathbf{Z}_+^d\\|\beta|=m}}\left\|\frac{\partial^\beta}{\partial x^\beta}\left(\eta_i\mathcal{S}(X)\varphi^\varepsilon\mathcal{S}(X)v\right)\right\|^2_{L_2(\Omega;\mathbb{C}^n)}$$

$$=\sum_{\substack{\beta\in\mathbf{Z}_+^d\\|\beta|=m}}\sum_{\substack{\rho\in\mathbf{Z}_+^d\\0\leqslant|\rho|\leqslant|\beta|}}\sum_{\substack{\alpha\in\mathbf{Z}_+^d\\0\leqslant|\alpha|\leqslant|\rho|}}\left\|C_{\beta\rho\alpha}\frac{\partial^\alpha\eta_i}{\partial x^\alpha}\frac{\partial^{\rho-\alpha}\mathcal{S}(X)\varphi^\varepsilon}{\partial x^{\rho-\alpha}}\frac{\partial^{\beta-\rho}\mathcal{S}(X)v}{\partial x^{\beta-\rho}}\right\|^2_{L_2(\Omega;\mathbb{C}^n)}$$
$$\leqslant C\sum_{\substack{\beta\in\mathbf{Z}_+^d\\|\beta|=m}}\sum_{\substack{\rho\in\mathbf{Z}_+^d\\0\leqslant|\rho|\leqslant|\beta|}}\sum_{\substack{\alpha\in\mathbf{Z}_+^d\\0\leqslant|\alpha|\leqslant|\rho|}}\left\|\frac{\partial^\alpha\eta_i}{\partial x^\alpha}\mathcal{S}(X)\varphi^\varepsilon\frac{\partial^{\beta-\rho}\mathcal{S}(X)v}{\partial x^{\beta-\rho}}\right\|^2_{L_2(\Omega;\mathbb{C}^n)}$$
$$\leqslant\widetilde{C}(X)\|v\|^2_{W_2^m(\Omega;\mathbb{C}^n)}$$

holds, where

$$\widetilde{C}(X):=C\max_{\overline{\Omega}}\sum_{\substack{\beta\in\mathbf{Z}_+^d\\|\beta|=2}}\sum_{\substack{\rho\in\mathbf{Z}_+^d\\0\leqslant|\rho|\leqslant|\beta|}}\sum_{\substack{\alpha\in\mathbf{Z}_+^d\\0\leqslant|\alpha|\leqslant|\rho|}}\left|\frac{\partial^\alpha\eta_i}{\partial x^\alpha}\mathcal{S}(X)\varphi^\varepsilon\right|\to 0\quad\text{for}\quad X\to\infty,$$

C is some constant independent of X and u. □

By the easily checkable identity

$$\begin{aligned}\mathcal{S}(-X_j)(\mathcal{H}_j-\lambda)^{-1}\mathcal{S}(X_j)-(\mathcal{H}_0-\lambda)^{-1}\\=\mathcal{S}(-X_j)\big((\mathcal{H}_j-\lambda)^{-1}-(\mathcal{H}_0-\lambda)^{-1}\big)\mathcal{S}(X_j)\\=-\mathcal{S}(-X_j)(\mathcal{H}_0-\lambda)^{-1}\mathcal{L}_j(\mathcal{H}_j-\lambda)^{-1}\mathcal{S}(X_j)\end{aligned}$$

and the definition of the operator $\mathcal{P}_X$ we deduce

$$\begin{aligned}\mathcal{P}_X&=-\sum_{\substack{i,j=1\\i\neq j}}^k\mathcal{S}(-X_i)\mathcal{L}_i\mathcal{S}(X_i-X_j)\left(\mathcal{H}_0-\lambda\right)^{-1}\mathcal{L}_j\left(\mathcal{H}_j-\lambda\right)^{-1}\mathcal{S}(X_j)\\&=-\sum_{\substack{i,j=1\\i\neq j}}^k\mathcal{S}(-X_i)\xi_i\mathcal{L}_i^0\eta_i\mathcal{S}(X_i-X_j)\left(\mathcal{H}_0-\lambda\right)^{-1}\xi_j\mathcal{L}_j^0\eta_j\left(\mathcal{H}_j-\lambda\right)^{-1}\mathcal{S}(X_j).\end{aligned}$$

Applying no Lemma 1, we arrive at the following statement.

Lemma 2. *For each $\lambda\in M$ as $\tau(X)\to+\infty$*

$$\|\mathcal{P}_X\|_{L_2(\Omega;\mathbb{C}^n)\to L_2(\Omega;\mathbb{C}^n)}\to 0$$

holds true.

We proceed to the proof of Theorem 1. By straightforward calculations we check that

$$\begin{aligned}\mathcal{T} :&= (\mathcal{H}_X-\lambda)\Big(\sum_{i=1}^{k}\mathcal{S}(-X_i)(\mathcal{H}_i-\lambda)^{-1}\mathcal{S}(X_i)-(k-1)(\mathcal{H}_0-\lambda)^{-1}\Big)\\ &= k\mathrm{I}+\sum_{\substack{i,j=1\\ i\neq j}}^{k}\mathcal{S}(-X_i)\mathcal{L}_i\mathcal{S}(X_i)(\mathcal{H}_0+\mathcal{S}(-X_j)\mathcal{L}_j\mathcal{S}(X_j)-\lambda)^{-1}-(k-1)\mathrm{I}\\ &-(k-1)\sum_{i=1}^{k}\mathcal{S}(-X_i)\mathcal{L}_i\mathcal{S}(X_i)(\mathcal{H}_0-\lambda)^{-1}\\ &= \mathrm{I}+\sum_{\substack{i,j=1\\ i\neq j}}^{k}\mathcal{S}(-X_i)\mathcal{L}_i\mathcal{S}(X_i)\big[(\mathcal{H}_0+\mathcal{S}(-X_j)\mathcal{L}_j\mathcal{S}(X_j)-\lambda)^{-1}-(\mathcal{H}_0-\lambda)^{-1}\big]\\ &= \mathrm{I}+\mathcal{P}_X.\end{aligned}$$

Due to Lemma 2 for sufficiently great $\tau(X)$ the operator $(\mathrm{I}+\mathcal{P}_X)$ is boundedly invertible. Together with the definition of the operator $\mathcal{T}$ it yields the validity of the required presentation for the resolvent of the perturbed operator; it is sufficient just to prove the triviality of the kernel of the operator $(\mathcal{H}_X-\lambda)$.

By analogy to the above calculations it is easy to show that

$$\left[\sum_{i=1}^{k}(\mathcal{H}_i-\lambda)^{-1}-(k-1)(\mathcal{H}_0-\lambda)^{-1}\right](\mathcal{H}_X-\lambda)=\mathrm{I}+\mathcal{Q}_X,$$

$$\mathcal{Q}_X=-\sum_{\substack{i,j=1\\ i\neq j}}^{k}\mathcal{S}(-X_j)(\mathcal{H}_j-\lambda_0)^{-1}\mathcal{L}_j\mathcal{S}(X_j-X_i)(\mathcal{H}_0-\lambda_0)^{-1}\mathcal{L}_i\mathcal{S}(X_i),$$

$$\big\|\mathcal{Q}_X\big\|_{W_2^m(\Omega;\mathbb{C}^n)\to W_2^m(\Omega;\mathbb{C}^n)}\to 0\quad\text{when}\quad \tau(X)\to\infty.$$

Hereof it is easy to obtain triviality of the kernel of the operator $(\mathcal{H}_X-\lambda)$ for sufficiently large $\tau(X)$.

By the obtained presentation for the resolvent and Banach theorem on the inverse operator we obtain an apriori estimate for the operator $\mathcal{H}_X$,

$$\|u\|_{W_2^m(\Omega;\mathbb{C}^n)}\leqslant C_1\big(\|\mathcal{H}_Xu\|_{L_2(\Omega;\mathbb{C}^n)}+\|u\|_{L_2(\Omega;\mathbb{C}^n)}\big),$$

which yields the closedness of this operator for sufficiently large $\tau(X)$.

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