ISSN 2304-0122 Ufa Mathematical Journal. Volume 4. 2 (2012). Pp. 28-64.

UDC 517.928:517.984

PERTURBATION OF AN ELLIPTIC OPERATOR BY A NARROW POTENTIAL IN AN n-DIMENSIONAL DOMAIN

A.R. BIKMETOV, R.R. GADYL'SHIN

Abstract. We study a discrete spectrum of an elliptic operator of the second order in an n-dimensional domain, $n \geqslant 2$, perturbed by a potential depending on two parameters, one of the parameters describes the length of the support of the potential and the inverse of the other corresponds to the magnitude of the potential. We give the relation between these parameters, under which the generalized convergence of the perturbed operator to the unperturbed one holds. Under this relation we construct the asymptotics w.r.t. small parameters of the eigenvalues of the perturbed operators.

Keywords: Elliptic operator, perturbation, matching of asymptotic expansions

1. Introduction

Let the domain $\Omega \subset \mathbb{R}^n$, $n \geqslant 2$, and, Ω can also coincide with $\mathbb{R}^n$, $a_{ij}(x)$, $a(x)$ be locally integrated functions in Ω such that

$$\int\limits_{\Omega} a(x)|u(x)|^2 dx \geqslant c(a)\|u\|^2_{L_2(\Omega)}, \qquad c(a) > 0, \tag{1.1}$$

for any functions u from $L_2(\Omega)$, for which this integral exists, $a_{ij} = a_{ji}$,

$$\alpha_1|\xi|^2 \leqslant \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \qquad \alpha_1 > 0, \quad \forall x \in \Omega, \quad \forall \xi = (\xi_1, \dots, \xi_n). \tag{1.2}$$

Since

$$\mathfrak{h}_0(u, v) := \sum_{i,j=1}^{n} \left(a_{ij}\frac{\partial u}{\partial x_i}, \frac{\partial v}{\partial x_j} \right)_{L_2(\Omega)} + (au, v)_{L_2(\Omega)} \tag{1.3}$$

is on the strength of (1.1) and (1.2) are of sesquilinear positive symmetrical form, then we consider it as a scalar product in the Hilbert space $\widetilde{W}_2^1(\Omega)$ of all functions for which

$$\|u\|_{\widetilde{W}_2^1(\Omega)} := \sqrt{\mathfrak{h}_0(u, u)} < \infty.$$

Since $\widetilde{W}_2^1(\Omega) \subset L_2(\Omega)$ on the strength of (1.1) and (1.2), then the quadric quantic

$$\mathfrak{h}_0[u] := \mathfrak{h}_0(u, u) \tag{1.4}$$

is closed in $L_2(\Omega)$ (see, for instance, [1, chapter VI, Theorem 1.1]). And though the subset of the functions from $C^\infty(\Omega)$, equal to zero in the neighbourhood of the border $\partial\Omega$ (if $\Omega \neq \mathbb{R}^n$) and with big x (if Ω is an unbounded domain), is apparently a subset $\widetilde{W}_2^1(\Omega)$ and is dense in $L_2(\Omega)$, then the quadric quantic $\mathfrak{h}_0$ is densely determined in $L_2(\Omega)$. Consequently (see, for instance,

A.R. Bikmetov, R.R. Gadyl'shin, Perturbation of an elliptic operator by a narrow potential in an n-dimensional domain.

The work was suppotred by grants of RFFR (12-01-00445) and FTP (02.740.110612).
Submitted on May 10, 2012.

[1, chapter VI, Theorems 2.1,2.6]), there is an associated with $\mathfrak{h}_0$ selfconjugated operator $\mathcal{H}_0$ in $L_2(\Omega)$ with the domain of definition

$$\mathcal{D}(\mathcal{H}_0) \subset \mathcal{D}(\mathfrak{h}_0) = \widetilde{W}_2^1(\Omega)$$

(i.e. such that $(\mathcal{H}_0 u, v)_{L_2(\Omega)} = \mathfrak{h}_0(u, v)$ for any u, $v \in \mathcal{D}(\mathcal{H}_0)$).

Everywhere below without limiting generality we consider, that the origin of coordinates lies in Ω. Let us denote

$$\mathfrak{h}_{\mu,\varepsilon}[u] := \mathfrak{h}_0[u] + \mu^{-1} (V_\varepsilon u, u)_{L_2(\Omega)}\,, \tag{1.5}$$

where

$$0 < \varepsilon \ll 1, \qquad \mu(\varepsilon) > 0,$$

V_ε is a family of equally bounded by ε functions from $L_\infty(\Omega)$, which carriers lie in n-dimensional sphere of the radius $\gamma\varepsilon$ with the centre in the origin of coordinates for some $\gamma > 0$.

Though the quadric quantic $\mu^{-1} (V_\varepsilon u, u)_{L_2(\Omega)}$ is apparently bounded on $L_2(\Omega)$, then the quadric quantic $\mathfrak{h}_{\mu,\varepsilon}$ is closed and densely determined in $L_2(\Omega)$, moreover, $\mathcal{D}(\mathfrak{h}_{\mu,\varepsilon}) = \widetilde{W}_2^1(\Omega)$. Let us denote the selfconjugated operator associated with the quadric quantic $\mathfrak{h}_{\mu,\varepsilon}[u]$ by $\mathcal{H}_{\mu,\varepsilon}$.

Remark 1.1. *If $\Omega \neq \mathbb{R}^n$, then we denote by $\widetilde{W}_{2,0}^1(\Omega)$ the closure by the norm $\widetilde{W}_2^1(\Omega)$ of the subset of the functions from $\widetilde{W}_2^1(\Omega)$, reducing to zero in the neighbourhood $\partial\Omega$. It is easy to see, that the quadric quantics $\mathfrak{h}_0$ and $\mathfrak{h}_{\mu,\varepsilon}$ determined in $\widetilde{W}_{2,0}^1(\Omega)$ by the equalities (1.3), (1.4) and (1.5), correspondingly, are symmetrical, closed and densely determined in $L_2(\Omega)$. For the selfconjugated operators associated with these forms, we retain the notation $\mathcal{H}_0$.*

In the first part of the paper we prove the convergence of characteristic constants of the operator $\mathcal{H}_{\mu,\varepsilon}$ to characteristic constants of the operator $\mathcal{H}_0$ (when the latter ones exist), when

$$\mu^{-1}\beta_n(\varepsilon) = o\,(1)\,, \tag{1.6}$$

where $\beta_2(\varepsilon) = \varepsilon^2 \left|\ln \varepsilon\right|$, $\beta_n(\varepsilon) = \varepsilon^2$ when $n \geqslant 3$.

It is clear, that if, for instance,

$$\begin{aligned} &V_\varepsilon(x) = V\left(\frac{x}{\varepsilon}\right), \quad V \in C_0^\infty(\Omega), \\ &a_{ij},\, a \in C^\infty(\mathbb{R}^n), \quad \text{if } \Omega = \mathbb{R}^n, \\ &a_{ij},\, a \in C^\infty(\overline{\Omega}), \quad \partial\Omega \in C^\infty, \quad \text{if } \Omega \neq \mathbb{R}^n, \end{aligned} \tag{1.7}$$

then the operators $\mathcal{H}_0$ and $\mathcal{H}_{\mu,\varepsilon}$ are expanded by Friedrichs differential operators H_0 and $H_{\mu,\varepsilon}$ in $L_2(\Omega)$, determined correspondingly as

$$H_0 u := -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right) + a(x)u, \qquad H_{\mu,\varepsilon} = H_0 u + \mu^{-1}V_\varepsilon(x)u \tag{1.8}$$

on the functions, satisfying with $\Omega \neq \mathbb{R}^n$ to the supplementary bounded to the Neumann conditions

$$\frac{\partial u}{\partial \nu} := \left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right)\cos(x_i, \mathbf{n}) = 0, \quad x \in \partial\Omega,$$

where $\mathbf{n}$ is an outer normal to $\partial\Omega$, if the operators $\mathcal{H}_0$ and $\mathcal{H}_{\mu,\varepsilon}$ are associated with quadric quantics determined on $\widetilde{W}_2^1(\Omega)$ and to the boundary Dirichlet condition

$$u = 0, \quad x \in \partial\Omega,$$

if the operators $\mathcal{H}_0$ and $\mathcal{H}_{\mu,\varepsilon}$ are associated with quadric quantics determined on $\widetilde{W}_{2,0}^1(\Omega)$.

In the general second part of the paper with the satisfied conditions (1.7) we construct complete asymptotic expansions of characteristic constants of the operator $\mathcal{H}_{\mu,\varepsilon}$, converging to the characteristic constants of the operator $\mathcal{H}_0$ as in the case of a simple limiting characteristic constant of the operator $\mathcal{H}_0$, as in the case of the twofold one. Though for the rigorous substantiation of the constructed asymptotics we have to impose a stricter (unlike (1.6)) restriction:

$$\mu^{-1}\beta_n(\varepsilon) = O\left(\varepsilon^{\tau}\right), \tag{1.9}$$

where $\tau > 0$ is any number.

As it is seen from the further derivation of complete asymptotics of characteristic constants, for their formal construction it would be sufficient to demand only an infinite differentiability of the functions $a_{ij}(x)$, and $a(x)$ in the neighbourhood of zero. Stricter conditions (1.7) are imposed only for the purpose to avoid insignificant but bulky detailing in notations and proofs.

Let us note, that boundary-value problems for the Laplace operator in bounded domains with similar perturbations depending on one parameter, were considered in [2], [3]. In [2] for the three-dimensional domain there was proved convergence of characteristic constants in the case $\mu = \varepsilon^{\tau}$, $\tau < 2$ and there was constructed the asymptotics of the characteristic constant of the perturbed boundary-value problem, reducing to a simple characteristic constant of the boundary problem. In [3] for n-dimensional bounded domain there was proved the convergence of the characteristic constant of the perturbed operator in the case when $\mu = \varepsilon^{\tau}$, $\tau < 1$, and the characteristic constant of the boundary operator is simple and we have constructed its binomial asymptotics. In both papers during the proof of convergence compactness of the embedding W_2^1 into L_2 for the bounded domains was significant. As it has already been mentioned above, the asymptotics were constructed only for the case of a simple characteristic constant of the boundary problem. Moreover, for the problem in the three-dimensional domain, considered in [2], it was supplementary assumed, that, firstly, the eigenfunction of the boundary problem does not reduce to zero in the point of compression of the carrier of the perturbed potential, and secondly, the average value (integral) of this potential is not equivalent to zero. In [3] during construction of binomial asymptotics there removed two last restrictions, but there was imposed a stricter (in comparison with [2]) condition on the growth of the perturbing potential ($\tau < 1$). As it is shown below (see remark 2.1), the influence of the equivalence to zero of the average value of the perturbing potential on the first term of the theory of perturbations is significantly different for the cases $\tau < 1$ and $\tau > 1$. In the conclusion of the section we note, that suchlike perturbations of a differential operator of the second order in the one-dimensional case were considered in [4],[5],[6].

2. Formulation of general statements

In the next section we prove

Theorem 2.1. *Let the condition (1.6) hold. Then there takes place the convergence $\mathcal{H}_{\mu,\varepsilon} \to \mathcal{H}_0$ when $\varepsilon \to 0$ in the general sense (resolvent convergence).*

It results from this theorem and [1, chapter IV, Theorem 3.16]

Corollary 1. *Let λ_0 be a characteristic constant of the operator $\mathcal{H}_0$ of the order m and there holds the equality (1.6). Then when $\varepsilon \to 0$ to λ_0 converge characteristic constants $\lambda^{\mu,\varepsilon,j}$ of the operator $\mathcal{H}_{\mu,\varepsilon}$, the total order of which is also equal to m, and for the corresponding projector $\mathcal{P}_{\mu,\varepsilon}$ there is convergence by the norm to the projector $\mathcal{P}_0$, corresponding to the characteristic constant λ_0.*

The general contents of the paper which is the rest of the article devoted to is the proof of the method of matching of asymptotic decompositions [7], [8] of the formulated below Theorems 2.2–2.4, with the satisfying the supplementary conditions of smoothness (1.7), a stricter demand (1.9) to the relationship of the parameters ε and μ and not restricting the generality condition $a_{ij}(0) = \delta_i^j$, where δ_i^j is Kronecker delta.

Before we proceed to formulation of general theorems, let us introduce some notations:

$$\langle g\rangle := \int\limits_{\mathbb{R}^n} g(x)\,dx, \qquad \langle g\rangle_i := \int\limits_{\mathbb{R}^n} x_i g(x)\,dx, \qquad \langle g\rangle_{ij} := \int\limits_{\mathbb{R}^n} x_i x_j g(x)\,dx,$$

$$\mathcal{G}_2(x) = \frac{1}{2\pi}\ln r \quad \text{when } n=2, \qquad \mathcal{G}_n(x) = -\frac{1}{(n-2)|S_n|} r^{-n+2} \quad \text{when } n \geqslant 3,$$

$$z_0^{(1)}(x) = \int\limits_{\mathbb{R}^n} \mathcal{G}_n(x-y)V(y)dy.$$

Here and further $|S_n|$ is the area of a singular sphere in $\mathbb{R}^n$. Assume $\delta(n)=0$ with odd n and $\delta(n)=1$ with odd n.

Theorem 2.2. *Let the condition (1.9)hold, then λ_0 is a simple characteristic constant of the operator $\mathcal{H}_0$, ψ_0 is the corresponding normalized in $L_2(\Omega)$ eigenfunction.*

Therefore, if $\psi_0(0)\neq 0$, then the characteristic constant $\lambda^{\mu,\varepsilon}$ of the operator $\mathcal{H}_{\mu,\varepsilon}$, converging to λ_0, possesses the asymptotics

$$\begin{aligned}\lambda^{\mu,\varepsilon} = \lambda_0 + \varepsilon^n\mu^{-1}\sum_{j=0}^{\infty}\sum_{i=2j}^{\infty}\lambda_{n+i,j+1}\varepsilon^i\mu^{-j}\\ + d(n)\varepsilon^{2n}\mu^{-2}\ln\varepsilon\sum_{p=0}^{\infty}\sum_{j=p}^{\infty}\sum_{i=2j+(n-2)p}^{\infty}\lambda_{2n+i,j+2,p+1}\varepsilon^i\mu^{-j}\ln^p\varepsilon,\end{aligned} \tag{2.1}$$

where

$$\lambda_{n,1} = \psi_0^2(0)\left\langle V\right\rangle, \tag{2.2}$$

$$\lambda_{n+2,2} = -\psi_0^2(0)\left\|\nabla z_0^{(1)}\right\|^2_{L_2(\mathbb{R}^n)}. \tag{2.3}$$

If, therewith, $a_{ij}(x)\equiv\delta_j^i$ (i.e. $H_0=-\Delta+a(x)$), then

$$\lambda_{n+1,1} = (n-2)\psi_0(0)\sum_{m=1}^{n}\left\langle V\right\rangle_m\frac{\partial\psi_0}{\partial x_m}(0), \qquad n\geqslant 3, \tag{2.4}$$

$$\lambda_{3,1} = \psi_0(0)\sum_{m=1}^{2}\left\langle V\right\rangle_m\frac{\partial\psi_0}{\partial x_m}(0), \qquad n=2. \tag{2.5}$$

Remark 2.1. *It results from the Theorem, that if $\langle V\rangle = 0$, then*

$$\lambda^{\mu,\varepsilon} = \lambda_0 + \varepsilon^{n+1}\mu^{-1}\left(\lambda_{n+1,1}+o(1)\right), \quad \text{if } \varepsilon = o(\mu),$$
$$\lambda^{\mu,\varepsilon} = \lambda_0 + \varepsilon^{n+2}\mu^{-2}\left(\lambda_{n+2,2}+o(1)\right), \quad \text{if } \mu = o(\varepsilon).$$

Hence, when $\langle V\rangle = 0$ the order of the infinitesimality of the first term of the theory of perturbations for $\lambda^{\mu,\varepsilon}$ notably differs for the cases $\varepsilon=o(\mu)$ and $\mu=o(\varepsilon)$.

Theorem 2.3. *Let the conditions of the Theorem 2.2 hold.*

Then, if $\psi_0(0) = 0$, then the characteristic constant $\lambda^{\mu,\varepsilon}$ of the operator $\mathcal{H}_{\mu,\varepsilon}$ converging to λ_0 possesses the asymptotics

$$\begin{aligned}\lambda^{\mu,\varepsilon} =&\lambda_0 + \varepsilon^{n+2}\mu^{-1}\sum_{j=0}^{\infty}\sum_{i=2j}^{\infty}\lambda_{n+2+i,j+1}\varepsilon^i\mu^{-j}\\&+ d(n)\varepsilon^{2n+2}\mu^{-2}\ln\varepsilon\sum_{p=0}^{\infty}\sum_{j=p}^{\infty}\sum_{i=2j+(n-2)p}^{\infty}\varepsilon^i\mu^{-j}\ln^p\varepsilon\lambda_{2n+i+2,j+2,p+1},\end{aligned}\tag{2.6}$$

where

$$\lambda_{n+2,1} = \nabla\psi_0(0)\mathcal{V}\nabla\psi_0(0),\tag{2.7}$$

and $\mathcal{V}$ is a symmetrical $n\times n$-matrix with the components $\langle V\rangle_{km}$.

Let λ_0 be a twofold characteristic constant of the operator $\mathcal{H}_0$. It results from the corollary 1, that for the converging to λ_0 characteristic constants of the operator $H_{\mu,\varepsilon}$ the following cases are possible: either it is two simple characteristic constants, or it is one twofold characteristic constant, or for different ε one of these variants takes place. And even, if two simple characteristic constants $\lambda^{\mu,\varepsilon,1}$ and $\lambda^{\mu,\varepsilon,2}$ converge to λ_0, it is impossible to state, that the corresponding normalized in $L_2(\Omega)$ eigenfunctions $\psi^{\mu,\varepsilon,j}$ have the limit. The corollary 1 only guarantees, that from any sequence $\varepsilon_k \to 0$ we can single out the subsequence $\varepsilon_{k_m} \to 0$ such that there takes place the convergence $\psi^{\mu,\varepsilon,j} \to \psi_0^{(j)}$ in $L_2(\Omega)$, where $\psi_0^{(j)}$ are orthonormalized in $L_2(\Omega)$ eigenfunctions of the operator $\mathcal{H}_0$, corresponding to λ_0. Though, these limits can change in dependence of the choice of the subsequence $\varepsilon_{k_m} \to 0$.

In the paper we consider the case of the most general statement:

$$|\psi_0^{(1)}(0)| + |\psi_0^{(2)}(0)| \neq 0.\tag{2.8}$$

Then, obviously, these eigenfunctions can be chosen so, that

$$\psi_0^{(1)}(0) \neq 0, \qquad \psi_0^{(2)}(0) = 0.\tag{2.9}$$

We plan to prove the following

Theorem 2.4. *Let the following condition hold (1.9), $\langle V\rangle \neq 0$, λ_0 is a twofold characteristic constant of the operator $\mathcal{H}_0$, $\psi_0^{(1)}$ and $\psi_0^{(2)}$ are the corresponding orthonormalized in $L_2(\Omega)$ eigenfunctions, which satisfy the condition (2.8) and which are chosen in compliance with (2.9).*

Hence, there exist two simple characteristic constants $\lambda^{\mu,\varepsilon,1}$ and $\lambda^{\mu,\varepsilon,2}$ of the operator $\mathcal{H}_{\mu,\varepsilon}$, converging to λ_0, and they possess the asymptotics

$$\begin{aligned}\lambda^{\mu,\varepsilon,1} =& \lambda_0 + \varepsilon^{n}\mu^{-1}\sum_{j=0}^{\infty}\sum_{i=2j}^{\infty}\lambda^{(1)}_{n+i,j+1}\varepsilon^i\mu^{-j}\\&+ d(n)\varepsilon^{2n}\mu^{-2}\ln\varepsilon\sum_{p=0}^{\infty}\sum_{j=p}^{\infty}\sum_{i=2j+(n-2)p}^{\infty}\varepsilon^i\mu^{-j}\ln^p\varepsilon\lambda^{(1)}_{2n+i,j+2,p+1},\end{aligned}\tag{2.10}$$

$$\begin{aligned}\lambda^{\mu,\varepsilon,2} =&\lambda_0 + \varepsilon^{n+2}\mu^{-1}\sum_{j=0}^{\infty}\sum_{i=2j}^{\infty}\lambda^{(2)}_{n+2+i,j+1}\varepsilon^i\mu^{-j}\\&+ d(n)\varepsilon^{2n+2}\mu^{-2}\ln\varepsilon\sum_{p=0}^{\infty}\sum_{j=p}^{\infty}\sum_{i=2j+(n-2)p}^{\infty}\varepsilon^i\mu^{-j}\ln^p\varepsilon\lambda^{(2)}_{2n+i+2,j+2,p+1},\end{aligned}\tag{2.11}$$

where

$$\lambda_{n,1}^{(1)} = \left(\psi_0^{(1)}(0)\right)^2 \langle V\rangle , \tag{2.12}$$

$$\lambda_{n+2,1}^{(2)} = \nabla\psi_0^{(2)}(0)\widetilde{\mathcal{V}}\nabla\psi_0^{(2)}(0), \tag{2.13}$$

$\widetilde{\mathcal{V}}$ *is a symmetrical* $n\times n$*-matrix with the components*

$$\langle V\rangle_{mi} - (n-2)\frac{\langle V\rangle_m \langle V\rangle_i}{\langle V\rangle}, \quad n\geqslant 3, \qquad \langle V\rangle_{mi} - \frac{\langle V\rangle_m \langle V\rangle_i}{\langle V\rangle}, \quad n=2,$$

and the corresponding eigenfunctions $\psi^{\mu,\varepsilon,s}$ *converge to* $\psi_0^{(s)}$ *in* $L_2(\Omega)$.

It results from the theorem, in particular, that if the condition (2.8) and $\langle V\rangle \neq 0$ is satisfied then the twofold characteristic constant λ_0 with the considered perturbation splits into two simple characteristic constants, and the corresponding eigenfunctions converge to the eigenfunctions of the operator $\mathcal{H}_0$, chosen in relation to (2.9).

In the paper there were also constructed complete asymptotic expansions of the corresponding eigenfunctions.

3. Proof of Theorem 2.1

It results from the definition of quadric quantics $\mathfrak{h}_0$ and $\mathfrak{h}_{\mu,\varepsilon}$ and the function V, that firstly, these forms are bounded below, and secondly, the following estimation holds:

$$|(\mathfrak{h}_{\mu,\varepsilon}-\mathfrak{h}_0)[u]| = \mu^{-1}\left|\int\limits_{\Omega} V_\varepsilon(x)|u(x)|^2dx\right| \leqslant C\mu^{-1}\int\limits_{|x|<\gamma\varepsilon}|u(x)|^2dx, \tag{3.1}$$

where $C>0$ is a constant independent of ε.

Let B be a n-dimensional sphere with the centre in the origin of coordinates and the radius equal to three. Without limiting generality, we consider, that $\overline{B}\subset\Omega$. It correspondingly results from ([9, Ch. 3, Lemma 5.1]) and [10] for $n\geqslant 3$ and $n=2$, that for any function $v\in C_0^\infty(B)$ the following inequality holds:

$$\int\limits_{|x|<\gamma\varepsilon}|v(x)|^2dx \leqslant C_1(\gamma)\beta_n(\varepsilon)\int\limits_{B}|\nabla v(x)|^2dx, \tag{3.2}$$

where the constant C_1 does not depend on ε. Let $\chi(t)$ be an infinitely differentiated patch function, identically equal to the unit when $t<1$ and to the zero when $t>2$.

Since $\widetilde{W}_2^1(\Omega)\subset W_2^1(\Omega)$ in the strength of (1.1), (1.2), then for any function $u\in\widetilde{W}_2^1(\Omega)$ according to (3.2), (1.1), (1.2) we sequentially obtain, that

$$\begin{aligned}\int\limits_{|x|<\gamma\varepsilon}|u(x)|^2dx = \int\limits_{|x|<\gamma\varepsilon}|u(x)\chi(|x|)|^2dx \leqslant C_1\beta_n(\varepsilon)\int\limits_{B}|\nabla(u(x)\chi(|x|))|^2dx\\ \leqslant C_2\beta_n(\varepsilon)\int\limits_{\Omega}\left(|\nabla u(x)|^2+|u(x)|^2\right)dx \leqslant C_3\beta_n(\varepsilon)\mathfrak{h}_0[u],\end{aligned}$$

where C_2, C_3 are some constants independent of u. It results from this inequality and the inequality (3.1), that

$$|(\mathfrak{h}_{\mu,\varepsilon}-\mathfrak{h}_0)[u]| \leqslant C_3C\mu^{-1}\beta_n(\varepsilon)\mathfrak{h}_0[u]$$

for any function $u\in\widetilde{W}_2^1(\Omega)=\mathcal{D}(\mathfrak{h}_0)=\mathcal{D}(\mathfrak{h}_{\mu,\varepsilon})$. Since the quadric quantics $\mathfrak{h}_0$ and $\mathfrak{h}_{\mu,\varepsilon}$ are densely defined in $L_2(\mathbb{R})$, bounded below and closed, and $\mu^{-1}\beta_n(\varepsilon)\to 0$ when $\varepsilon\to 0$ on the strength of (1.6), then it results from the latter estimate and [1, Chapter VI, Theorem 3.6], that the statement of the theorem under consideration holds.

4. Auxiliary statements

It is considered in the text below, that the conditions (1.7) on the function V_ε hold and that the coefficients of the differential expression H_0 determined in (1.8), and there exists a not limiting generality supposition, that $a_{ij}(0) = \delta_i^j$, where δ_i^j is the Kronecker delta.

Also below $r = |x|$, we denote homogeneous polynomials of the degree k by $P_k(x)$, $Q_k(x)$ and $R_k(x)$, homogeneous harmonic polynomials of the degree k by $Y_k(x)$, $Z_k(x)$, and homogeneous polynomials of the degree j relative to the differentiating symbol $D = (D_1, \dots, D_n)$, $D_q = \partial/\partial x_q$, which coefficients are homogeneous polynomials of the degree i by $Q_{i,j}(x, D)$. For the whole j by $T_j(x)$ we consider homogeneous functions of the degree k, presented in the form $R_{j+k}(x)r^{-k}$ at least for some whole k.

In these notations for the differential expression H_0 when $r \to 0$ the following presentation holds

$$H_0 = -\Delta + \sum_{i=1}^{\infty} Q_{i,2}(x, D) + \sum_{i=0}^{\infty} Q_{i,1}(x, D) + \sum_{i=0}^{\infty} Q_{i,0}(x, D). \tag{4.1}$$

Let us denote by $\widetilde{\mathcal{A}}_0$ the set of series of the form

$$\mathcal{E}(x) = \Phi_0(x) + \sum_{j=1}^{\infty} \Phi_j(x), \tag{4.2}$$

where

$$\begin{aligned}
\Phi_0(x) =& b \ln r + c, \qquad \Phi_j(x) = r^{-2j} P_{3j}(x) + \ln r R_j(x) \quad \text{when } n = 2, j \geqslant 1,\\
\Phi_0(x) =& b r^{2-n} \quad \text{when } n \geqslant 3,\\
\Phi_j(x) =& r^{2-n-2j} P_{3j}(x) \quad \text{when } n \geqslant 4, \quad 1 \leqslant j \leqslant n-3,\\
\Phi_j(x) =& r^{2-n-2j} P_{3j}(x) + \delta(n) \ln r R_{j+2-n}(x) + (1 - \delta(n)) Q_{j+2-n}(x)\\
& \quad \text{when } n \geqslant 3, \ j \geqslant n-2,
\end{aligned}$$

and b, c are arbitrary numbers. Let us remind, that $\delta(n) = 0$ with odd n and $\delta(n) = 1$ with even n.

For the whole $m \geqslant 1$ we denote by $\widetilde{\mathcal{A}}_m$ the set of series of the form (4.2), where

$$\begin{aligned}
\Phi_0(x) =& Z_m(x) r^{-2m+2-n} \quad \text{when } n \geqslant 2,\\
\Phi_j(x) =& \sum_{s=0}^{2j-1} Z_{m+3j-2s}(x) r^{-2m+2-n-2j+2s}\\
& \quad \text{when } n \geqslant 3, \quad m \geqslant 1, \quad 1 \leqslant j \leqslant n+m-3\\
& \quad \text{and when } n = 2, \quad m \geqslant 2, \quad 1 \leqslant j \leqslant m-1,\\
\Phi_m(x) =& P_{4m}(x) r^{-4m} \quad \text{when } n = 2, \quad j = m,\\
\Phi_j(x) =& P_{m+3j}(x) r^{-2m+2-n-2j} + \delta(n) \ln r \, R_{j-m-n+2}(x)\\
& \quad + (1 - \delta(n)) Q_{j-m-n+2}(x)\\
& \quad \text{when } n \geqslant 3, \quad j \geqslant n+m-2 \quad \text{and when } n = 2, \quad j \geqslant m+1.
\end{aligned}$$

Let us denote the set of series presented in the form of the sum of series from $\widetilde{\mathcal{A}}_j$ when $j \leqslant m$ by $\widetilde{\mathcal{A}}^m$.

Lemma 4.1. *Assume $\mathcal{F} \in \widetilde{\mathcal{A}}^k$. Then there is a series $\mathcal{E} \in \widetilde{\mathcal{A}}_k$, possessing the dominant term $\Phi_0(x) = Z_k(x)r^{-2k+n-2}$ when $k \geqslant 1$, where Z_k is an arbitrary harmonic polynomial, and the dominant terms $\Phi_0(x) = b\ln r + c$ when $n = 2$, $k = 0$ and $\Phi_0(x) = br^{2-n}$ when $n \geqslant 3$, $k = 0$, where b, c are arbitrary constants, such that the following equalities hold:*

$$\Delta\Phi_0 = 0, \qquad \Delta\Phi_1 = \left(Q_{1,2}(x,D) + Q_{0,1}(x,D)\right)\Phi_0,$$

$$\Delta\Phi_j = \sum_{i=2}^{j}\left(Q_{i,2}(x,D) + Q_{i-1,1}(x,D) + Q_{i-2,0}(x,D)\right)\Phi_{j-i} + \left(Q_{1,2}(x,D) + Q_{0,1}(x,D)\right)\Phi_{j-1} - \lambda_0\Phi_{j-2} - \widetilde{\Phi}_{j-2} \qquad \text{for } j \geqslant 2,$$

where Φ_q, $\widetilde{\Phi}_q$ are members of the series $\mathcal{E}$ and $\mathcal{F}$ correspondingly.

Validity of this statement is shown in the proof of Therem 1.1 from [11].

Let us denote by $\mathcal{A}_k$ the set of functions $u \in C^\infty(\mathbb{R}^n\backslash\{0\})$ when $\Omega = \mathbb{R}^n$ and the set of functions $u \in C^\infty(\overline{\Omega}\backslash\{0\})$ when $\Omega \neq \mathbb{R}^n$, possessing in the zero the differentiating asymptotics from $\widetilde{\mathcal{A}}_k$and such that $u\varkappa$ belongs to the domain of the definition of the operator $\mathcal{H}_0$ for any patch function $\varkappa \in C^\infty(\Omega)$, identically equal to zero in the neighbourhood of the origin of coordinates,and such that $\operatorname{supp}(1-\varkappa) \subset \Omega$. We denote by $\mathcal{A}^m$ the set of functions presented in the form of sums of functions from $\mathcal{A}_j$ when $j \leqslant m$.

Lemma 4.2. *Assume $n+k \geqslant 3$, $F \in \mathcal{A}^k$. Hence there exists the function $E \in \mathcal{A}_k$ possessing the dominant term of the asymptotics in the zero $\Phi_0(x) = Z_k(x)r^{-2k+n-2}$ when $k \geqslant 1$, where Z_k is any required harmonic polynomial, and the dominant term of the asymptotics in the zero $\Phi_0(x) = br^{2-n}$ when $k = 0$, where b is any required constant, such that*

$$H_0E = \lambda_0E + F + \Lambda\psi_0 \quad \text{in } \Omega\backslash\{0\} \tag{4.3}$$

with some number Λ, if λ_0 is a simple characteristic constant of the operator $\mathcal{H}_0$, and the equations

$$H_0E = \lambda_0E + F + \Lambda^{(1)}\psi_0^{(1)} + \Lambda^{(2)}\psi_0^{(2)} \quad \text{in } \Omega\backslash\{0\} \tag{4.4}$$

with some numbers $\Lambda^{(k)}$, if λ_0 is a twofold characteristic constant of the operator $\mathcal{H}_0$.

Proof. Let us denote by $\mathcal{F} \in \widetilde{\mathcal{A}}^k$ the asymptotic expansion in the zero of the function $F(x)$, by $\mathcal{E} \in \widetilde{\mathcal{A}}_k$ - the series satisfying the statement of Lemma 4.1, and by $\mathcal{E}_N(x)$ - a partial sum of the series $\mathcal{E}(x)$ up to the terms $O\left(r^N \ln r\right)$ inclusive, $N \geqslant 4$. We search for the function $E(x)$ in the form

$$E_N(x) = (1-\varkappa(x))\mathcal{E}_N(x) + \widetilde{E}_N(x), \tag{4.5}$$

where $\widetilde{E}_N \in \mathcal{D}(\mathcal{H}_0)$.

Let us consider the case when λ_0 is a simple characteristic constant. From (4.3) and (4.4) on the strength of Lemma 4.1 we obtain the equation on $\widetilde{E}_N$:

$$\mathcal{H}_0\widetilde{E}_N = \lambda_0\widetilde{E}_N + \widetilde{F}_N + \Lambda(N)\psi_0, \tag{4.6}$$

where $\widetilde{F}_N \in L_2(\mathbb{R}^n) \cap C^{N-3}(\mathbb{R}^n)$, if $\Omega = \mathbb{R}^n$, and $\widetilde{F}_N \in L_2(\Omega) \cap C^{N-1}(\overline{\Omega})$, if $\Omega \neq \mathbb{R}^n$. It results from the needed and sufficient condition of the resolvability of this equation, that when

$$\Lambda(N) = -\left(\widetilde{F}_N, \psi_0\right)_{L_2(\Omega)}$$

the equation (4.6) has the solution $\widetilde{E}_N \in \mathcal{D}(\mathcal{H}_0)$, and it results from the theorems of increasing smoothness for the solutions of elliptical boundary-value problems, that $\psi_0 \in C^\infty(\mathbb{R}^n)$, $\widetilde{E}_N \in C^{N-1}(\mathbb{R}^n)$, if $\Omega = \mathbb{R}^n$, and $\psi_0 \in C^\infty(\overline{\Omega})$, $\widetilde{E}_N \in C^{N-1}(\overline{\Omega})$, if $\Omega \neq \mathbb{R}^n$.

Let us show, that $\Lambda(N)$ does not depend on N. Let us denote

$$E_{N,M}(x) := E_N(x) - E_M(x), \qquad N < M.$$

Then by the construction, firstly, $E_{N,M} \in \mathcal{D}(\mathcal{H}_0)$, and secondly,

$$\mathcal{H}_0 E_{N,M} = \lambda_0 E_{N,M} + (\Lambda(N) - \Lambda(M))\, \psi_0.$$

Whence it implies, that, firstly, $\Lambda(N) = \Lambda(M)$ (i.e. $\Lambda(N)$ does not depend on N), and secondly, $E_{N,M}(x) = b_{N,M}\psi_0(x)$. It is easy to see, that if when $N \geqslant 5$ of the function E_N is normalized by the condition $(E_{N,4}, \psi_0)_{L_2(\Omega)} = 0$, then they also do not depend on N. Therefore it results from (4.5) and the arbitrary choice of N, that $E \in \mathcal{A}_k$.

Validity of the statement of the lemma for the case when λ_0 is a simple characteristic constant of the operator $\mathcal{H}_0$, has been proved.

By analogy we show the validity of lemma for the case when λ_0 is a twofold characteristic constant of the operator $\mathcal{H}_0$. □

Lemma 4.3. *Let $n = 2$, $F \in \mathcal{A}_0$, b be any constant. then there is the function $E \in \mathcal{A}_0$ possessing the dominant term of the asymptotics in the zero $\Phi_0(x) = b \ln r + d$, satisfying the equation (4.3) with some number Λ, if λ_0 is a simple characteristic constant of the operator $\mathcal{H}_0$, meanwhile, if $\psi_0(0) \neq 0$, then the constant d can be chosen whatever and satisfying the equation (4.4) with some numbers $\Lambda^{(k)}$, if λ_0 is a twofold characteristic constant of the operator $\mathcal{H}_0$, meanwhile, in the nonsingular case (2.8) the constant d can be chosen whatever.*

Proof. The proof of this statement is completely analogous to that of Lemma 4.2. An opportunity of choice of the constant d arbitrary (under the conditions $\psi_0(0) \neq 0$ and (2.8)) apparently results from the fact that the functions E are defined with precision to the summand $C\psi_0(x)$ for any C in the case when λ_0 is a simple characteristic constant of the operator $\mathcal{H}_0$, and with precision to the arbitrary linear combination of the eigenfunctions $\psi_0^{(s)}(x)$ in case when λ_0 is a twofold characteristic constant of the operator $\mathcal{H}_0$. □

Lemma 4.4. *there exist functions $E_0 \in \mathcal{A}_0$ when $n \geqslant 3$ and $E_1, \dots E_n \in \mathcal{A}_1$ when $n \geqslant 2$, possessing with $r \to 0$ the asymptotics*

$$\begin{aligned} E_0(x) =& r^{-n+2} + O\left(r^{-n+3}\right) \quad \text{when } n \geqslant 3,\\ E_m(x) =& x_m r^{-n} + O\left(r^{-n+2}\right) \quad \text{when } n \geqslant 2,\ j = 1, ..., m \end{aligned}$$

and satisfying in $\Omega\backslash\{0\}$ the equations

$$H_0 E_q = \lambda_0 E_q + \Lambda_q \psi_0 \quad \text{in } \Omega\backslash\{0\}, \tag{4.7}$$

where

$$\Lambda_0 = -\,|S_n|(n-2)\psi_0(0) \quad \text{when } n \geqslant 3, \tag{4.8}$$

$$\Lambda_m = -\,|S_n| \frac{\partial \psi_0}{\partial x_m}(0) \quad \text{when } n \geqslant 2,\ m = 1, ..., n, \tag{4.9}$$

if λ_0 is a simple characteristic constant of the operator $\mathcal{H}_0$, and satisfying in $\Omega\backslash\{0\}$ the equations

$$H_0 E_q = \lambda_0 E_q + \Lambda_q^{(1)} \psi_0^{(1)} + \Lambda_q^{(2)} \psi_0^{(2)},$$

where the eigenfunctions $\psi_0^{(s)}(x)$ are orthonormalized in compliance with (2.9),

$$\begin{aligned} \Lambda_0^{(1)} =& -\,|S_n|(n-2)\psi_0^{(1)}(0), \qquad \Lambda_0^{(2)} = 0 \quad \text{when } n \geqslant 3,\\ \Lambda_m^{(s)} =& -\,|S_n| \frac{\partial \psi_0^{(s)}}{\partial x_m}(0) \quad \text{when } n \geqslant 2,\ \ m = 1, ..., n,\ s = 1, 2, \end{aligned} \tag{4.10}$$

if λ_0 is a twofold characteristic constant of the operator $\mathcal{H}_0$.

Proof. The statements of the Lemma being proved except for the explicit formulae (4.8)–(4.10) are a partial case of Lemma 4.2. Therefore we should only show the validity of the equalities (4.8)–(4.10).

Let us first obtain the equality (4.8). For positive s we denote $\chi_q(t) := \chi(tq^{-1})$, $\widetilde{\chi}_q(t) := 1 - \chi_q(t)$, $\widetilde{E}(x) := E_0(x)\widetilde{\chi}_q(r)$, where $\chi(t)$ is an infinitely differentiating patch function which is identically equal to the unit when $t < 1$and to the zero when $t > 2$. It is apparent, that $\widetilde{E} \in \mathcal{D}(\mathcal{H}_0)$ for any sufficiently small q, and on the strength of (4.7) the following equality holds:

$$\mathcal{H}_0\widetilde{E} - \lambda_0\widetilde{E} = \Lambda_0\psi_0\widetilde{\chi}_q - 2\sum_{i,j=1}^{n} a_{ij}\frac{\partial\widetilde{\chi}_q}{\partial x_j}\frac{\partial E_0}{\partial x_i} - E_0\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial\widetilde{\chi}_q}{\partial x_j}\right).$$

On the strength of the condition of resolvability of this equation (orthogonality in $L_2(\Omega)$ the right side of the eigenfunction ψ_0) and the definition $\widetilde{\chi}_q$ we obtain:

$$\Lambda_0(\widetilde{\chi}_q\psi_0, \psi_0) = -2\left(\sum_{i,j=1}^{n} a_{ij}\frac{\partial\chi_q}{\partial x_j}\frac{\partial E_0}{\partial x_i}, \psi_0\right) - \left(E_0\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial\chi_q}{\partial x_j}\right), \psi_0\right).$$

Considering the asymptotics in the zero of the functions $a_{i,j}(x)$, $E_0(x)$ and $\psi_0(x)$, passing in the integrals in the right side of the latter equality to the expanded in q^{-1} times variable and rushing q to the zero, we obtain, that

$$\begin{aligned}\Lambda_0 =& -\psi_0(0)\left(2\int\limits_{r<2}\nabla r^{2-n}\nabla\chi(r)dx + \int\limits_{r<2} r^{2-n}\Delta\chi(r)dx\right)\\ =& -\psi_0(0)\int\limits_{r<2}\nabla r^{2-n}\nabla\chi(r)dx.\end{aligned} \tag{4.11}$$

Integrating in parts with small $t > 0$ we have:

$$\int\limits_{t<r<2}\nabla r^{2-n}\nabla\chi(r)dx = (n-2)|S_n|.$$

Passing in the latter equality to the limit when $t \to 0$ on the strength of (4.11) we obtain the validity of the equality (4.8).

By analogy we prove the equalities (4.9) and (4.10). □

Lemma 4.5. *Assume $n = 2$. Then there is the function $E_0 \in \mathcal{A}_0$, possessing with $r \to 0$ the asymptotics*

$$\begin{aligned}E_0(x) &= -\ln r + O(r\ln r), && \text{if } \psi_0(0) \neq 0,\\ E_0(x) &= -\ln r + c(\Omega) + O(r\ln r), && \text{if } \psi_0(0) = 0,\end{aligned} \tag{4.12}$$

and satisfying in $\Omega\backslash\{0\}$ the equation

$$H_0E_0 = \lambda_0E_0 + \Lambda_0\psi_0, \quad \text{where } \Lambda_0 = -2\pi\psi_0(0),$$

if λ_0 is a simple characteristic constant of the operator $\mathcal{H}_0$, and possessing with $r \to 0$ the asymptotics (4.12) and satisfying in $\Omega\backslash\{0\}$ the equation

$$H_0E_0 = \lambda_0E_0 + \Lambda_0^{(1)}\psi_0^{(1)} + \Lambda_0^{(2)}\psi_0^{(2)},$$

where the eigenfunctions $\psi_0^{(m)}(x)$ are orthonormalized in compliance with (2.9),

$$\Lambda_0^{(1)} = -2\pi\psi_0^{(1)}(0), \qquad \Lambda_0^{(2)} = 0,$$

if λ_0 is a twofold characteristic constant of the operator $\mathcal{H}_0$.

Proof. Subject to Lemma 4.3 the proof of this statement is completely analogous to the proof of Lemma 4.4. The absence of the constant $c(\Omega)$ in (4.12) apparently results from the fact that the function E_0 is defined with precision to the summand $C\psi_0(x)$ for any C in case when λ_0 is a characteristic constant of the operator $\mathcal{H}_0$, and with the precision to the arbitrary linear combination of the eigenfunctions $\psi_0^{(s)}(x)$ in case when λ_0 is a twofold characteristic constant of the operator $\mathcal{H}_0$. □

It results from Lemmas 4.2, 4.4, 4.5, that

Corollary 2. *Let λ_0 be a twofold characteristic constant of the operator $\mathcal{H}_0$ and the eigenfunctions $\psi_0^{(s)}(x)$ be orthonormalized in compliance with (2.9). Then for any $Z_k(x)$, $k \geqslant 1$, $F \in \mathcal{A}^k$ there is the solution $E \in \mathcal{A}^k$ of the equation*

$$H_0 E = \lambda_0 E + F + \Lambda\psi_0^{(2)},$$

in $\Omega\backslash\{0\}$ with some constant, possessing the dominant term of the asymptotics in the zero $\Phi_0(x) = Z_k(x) r^{-2k+n-2}$.

Let us denote

$$z_m^{(1)}(x) = \int\limits_{\mathbb{R}^n} \mathcal{G}_n(x-y) y_m V(y) dy \quad \text{when } m = 1, \dots, n.$$

It results from the definition of the functions$z_0^{(1)}, \dots, z_n^{(1)}$, that

Lemma 4.6. *The functions $z_0^{(1)}, \dots, z_n^{(1)} \in C^\infty(\mathbb{R}^n)$ satisfy the equations*

$$\Delta z_0^{(1)} = V, \qquad \Delta z_m^{(1)} = x_m V, \quad m = 1, \dots, n$$

in $\mathbb{R}^n$ and possess with $r \to \infty$ differentiating asymptotics

$$z_q^{(1)}(x) = -\, c_{q,0}^{(1)} \ln r + \left(c_{q,1}^{(1)} x_1 r^{-2} + c_{q,2}^{(1)} x_2 r^{-2} \right) + \sum_{i=2}^{\infty} Y_i^{(1,q)}(x) r^{-2i} \quad \textit{when } n = 2,$$

$$z_q^{(1)}(x) = c_{q,0}^{(1)} r^{2-n} + \sum_{m=1}^{n} c_{q,m}^{(1)} x_m r^{-n} + \sum_{i=2}^{\infty} Y_i^{(1,q)}(x) r^{-2i-n+2} \qquad \textit{when } n \geqslant 3,$$

where

$$c_{0,0}^{(1)} = -\frac{\langle V \rangle}{2\pi} \quad \textit{when } n = 2, \qquad c_{0,0}^{(1)} = -\frac{\langle V \rangle}{(n-2)|S_n|} \qquad \textit{when } n \geqslant 3,$$

$$c_{0,m}^{(1)} = c_{m,0}^{(1)} = -\frac{\langle V \rangle_m}{|S_n|}, \qquad c_{p,m}^{(1)} = -\frac{\langle V \rangle_{pm}}{|S_n|} \quad \textit{when } p, m = 1, \dots, n,$$

and $Y_i^{(1,q)}(x)$ are homogeneous harmonic polynomials of the order i.

When $k \geqslant 2$ we recurrently define the following functions:

$$z_q^{(k)}(x) = \int\limits_{\mathbb{R}^n} \mathcal{G}_n(x-y) V(y) z_q^{(k-1)}(y) dy, \quad \text{when } q = 0, 1, \dots, n.$$

Lemma 4.7. *The functions $z_0^{(k)}, \dots, z_n^{(k)} \in C^\infty(\mathbb{R}^n)$, $k \geqslant 2$ satisfy in $\mathbb{R}^n$ the equations*

$$\Delta z_q^{(k)} = V z_q^{(k-1)}$$

and possess with $r\to\infty$ differentiating asymptotics

$$z_p^{(k)}(x) = -c_{p,0}^{(k)}\ln r + \left(c_{p,1}^{(k)}x_1r^{-2} + c_{p,2}^{(k)}x_2r^{-2}\right) + \sum_{i=2}^{\infty} Y_i^{(k,p)}(x)r^{-2i} \quad \textit{when } n=2,$$

$$z_p^{(k)}(x) = c_{p,0}^{(k)}r^{2-n} + \sum_{m=1}^{n} c_{p,m}^{(k)}x_mr^{-n} + \sum_{i=2}^{\infty} Y_i^{(k,p)}(x)r^{-2i-n+2} \quad \textit{when } n\geqslant 3,$$

where

$$\begin{aligned} c_{0,0}^{(2)} &= \frac{1}{2\pi}\left\|\nabla z_0^{(1)}\right\|^2_{L_2(\mathbb{R}^2)} && \textit{when } n=2,\\ c_{0,0}^{(2)} &= \frac{1}{(n-2)|S_n|}\left\|\nabla z_0^{(1)}\right\|^2_{L_2(\mathbb{R}^n)} && \textit{when } n\geqslant 3. \end{aligned} \tag{4.13}$$

Proof. The validity of the Lemma statement except for the equalities (4.13) results directly from the definition of the functions $z_q^{(k)}(x)$.

Let us show the validity of (4.13). From the definition $z_0^{(2)}(x)$ and $z_0^{(1)}(x)$ we sequentially obtain

$$c_{0,0}^{(2)} = -\frac{\left\langle Vz_0^{(1)}\right\rangle}{2\pi} \quad \text{when } n=2, \qquad c_{0,0}^{(2)} = -\frac{\left\langle Vz_0^{(1)}\right\rangle}{(n-2)|S_n|} \quad \text{when } n\geqslant 3,$$

$$c_{0,0}^{(2)} = -\frac{\left\langle z_0^{(1)}\Delta z_0^{(1)}\right\rangle}{2\pi} \quad \text{when } n=2, \qquad c_{0,0}^{(2)} = -\frac{\left\langle z_0^{(1)}\Delta z_0^{(1)}\right\rangle}{(n-2)|S_n|} \quad \text{when } n\geqslant 3.$$

Integrating in parts the right sides of two equalities we obtain the validity (4.13). □

When $j\geqslant 0$ we denote by $\widetilde{\mathcal{B}}_j$ a set of series of the form

$$\sum_{i=0}^{\infty} T_{j-i}(x) + \delta(n)\ln r\sum_{s=0}^{j} P_{j-s}(x).$$

We denote by $\mathcal{B}_j$ a set of functions from $C^\infty(\mathbb{R}^n)$ possessing at infinity differentiated asymptotics from $\widetilde{\mathcal{B}}_j$. It results from this definition, that $z_j^{(p)}\in\mathcal{B}_0$.

Lemma 4.8. *Assume $S\in\mathcal{B}_q$, and the series $\widetilde{V}\in\widetilde{\mathcal{B}}_{q+2}$ is the asymptotic solution of the equation*

$$\Delta V = S \quad \textit{in} \quad \mathbb{R}^n, \tag{4.14}$$

when $r\to\infty$. Then there is the solution $V\in\mathcal{B}_{q+2}$ of this equation possessing at infinity the asymptotics

$$V(\xi) = \widetilde{V}(x) + \sum_{i=0}^{\infty} Z_i(x)r^{-2i-n+2} \quad \textit{when } n\geqslant 3,$$

$$V(x) = \widetilde{V}(x) + b\ln r + \sum_{i=1}^{\infty} Z_i(x)r^{-2i} \quad \textit{when } n=2.$$

Proof. Let us denote by $\widetilde{V}_N$ a partial sum of the series $\widetilde{V}$ up to the terms of the order r^{-N-n} inclusive. The solution of the equation (4.14) we search for in the form

$$V_N(x) = \widetilde{V}_N(x)(1-\chi(r)) + w_N(x). \tag{4.15}$$

Substituting (4.15) into (4.14) we obtain the equation for w_N:

$$\Delta w_N = S_N, \quad x\in\mathbb{R}^n, \tag{4.16}$$

where

$$S_N = S - (1-\chi)\Delta\widetilde{V}_N + 2\sum_{i=1}^{n}\frac{\partial\chi}{\partial x_i}\frac{\partial\widetilde{V}_N}{\partial x_i}.$$

Consequently, $S_N(x) = O(r^{-N-n-3})$ when $r \to \infty$. Then the function

$$w_N(x) = \int\limits_{\mathbb{R}^n} \mathcal{G}_n(x-y)S_N(y)dy$$

is the solution of the equation (4.16) and when $r \to \infty$ possesses the asymptotics

$$\begin{aligned} w_N(x) =& b\ln r + \sum_{i=1}^{N+1} Z_i(x)r^{-2i} + o(r^{-N-2}) \quad \text{when } n=2,\\ w_N(x) =& \sum_{i=0}^{N+1} Z_i(x)r^{-n-2i+2} + o(r^{-N-n}) \qquad \text{when } n \geqslant 3. \end{aligned}$$

It results from here and from (4.15), that when $r \to \infty$

$$\begin{aligned} V_N(x) =& \widetilde{V}_N(x) + b\ln r + \sum_{i=1}^{N+1} Z_i(x)r^{-2i} + o(r^{-N-2}) \quad \text{when } n=2,\\ V_N(x) =& \widetilde{V}_N(x) + \sum_{i=0}^{N+1} Z_i(x)r^{-n-2i+2} + o(r^{-N-n}) \qquad \text{when } n \geqslant 3. \end{aligned} \tag{4.17}$$

The difference $V_{N_1} - V_{N_2}$ is a harmonic in $\mathbb{R}^n$ function reducing at infinity. Consequently, $V_{N_1} - V_{N_2} = 0$, i.e. V_N does not depend on N. Therefore the validity of the statement of the Lemma under consideration results from (4.17) on the strength of the arbitrary choice of N. □

5. Derivation of the structure of the eigenfunction internal expansion in case of an odd-dimensional domain

Below in this and three more sections λ_0 is a simple characteristic constant of the operator $\mathcal{H}_0$. In this case it results from the corollary 1, that for the normalized in $L_2(\Omega)$ eigenfunction $\psi_{\mu,\varepsilon}$, corresponding to the characteristic constant $\lambda_{\mu,\varepsilon} \underset{\varepsilon\to 0}{\longrightarrow} \lambda_0$, the convergence $\psi^{\mu,\varepsilon} \to \psi_0$ in $L_2(\Omega)$ takes place. Therefore outside the neighbourhood of the origin of coordinates (where the perturbation of the operator $\mathcal{H}_{\mu,\varepsilon}$ is concentrated) the approximation $\psi^{ex}(x,\mu,\varepsilon)$ (*outer expansion*) of the function $\psi^{\mu,\varepsilon}$ is natural to be searched in the form $\psi^{ex}(x,\mu,\varepsilon) \approx \psi_0(x)$. In the neighbourhood of the origin of coordinates the approximation ψ^{in} (*internal expansion*) of the function $\psi_{\mu,\varepsilon}$ is also natural to search in the form of the expansion by the functions depending on the variable $\xi = x\varepsilon^{-1}$, corresponding to the argument of the perturbing potential $V\left(\frac{x}{\varepsilon}\right)$.

The Taylor series of the function ψ_0 in the zero has the form:

$$\psi_0(x) = \sum_{k=0}^{\infty} P_k(x), \qquad r \to 0, \tag{5.1}$$

where

$$P_0(x) = \psi_0(0), \quad P_1(x) = \sum_{m=1}^{n}\frac{\partial\psi_0}{\partial x_m}(0)x_m, \tag{5.2}$$

and, on the strength of the equation

$$H_0\psi_0 = \lambda_0\psi_0 \tag{5.3}$$

and the equality (4.1) the following equalities hold

$$\begin{aligned}\Delta P_0 =&0, \qquad \Delta P_1 = \left(Q_{1,2}(x,D) + Q_{0,1}(x,D)\right) P_0 = 0,\\ \Delta P_k =&\sum_{i=2}^{k} \left(Q_{i,2}(x,D) + Q_{i-1,1}(x,D) + Q_{i-2,0}(x,D)\right) P_{k-i}\\ &+ \left(Q_{1,2}(x,D) + Q_{0,1}(x,D)\right) P_{k-1} - \lambda_0 P_{k-2} \qquad \text{when } k \geqslant 2.\end{aligned} \tag{5.4}$$

Remark 5.1. *Everywhere below we denote by $P_k(x)$ only members of the Taylor series in the zero of the function $\psi_0(x)$, and by $P_k^{(s)}(x)$ - those of the functions $\psi_0^{(s)}(x)$.*

Let us denote $\rho = |\xi|$. Rewriting the right side (5.1) in the variable ξ subject to (5.2) we obtain:

$$\psi^{ex}(x,\mu,\varepsilon) \approx \psi_0(x) = \psi_0(0) + \varepsilon \sum_{j=1}^{n} \frac{\partial \psi_0}{\partial x_m}(0)\xi_m + \sum_{k=2}^{\infty} \varepsilon^k P_k(\xi), \qquad \rho\varepsilon^{-1} = r \to 0.$$

Therefore, following the method of matching of asymptotic expansions [7] we obtain, that the internal expansion should be searched in the form

$$\psi^{in}(\xi,\mu,\varepsilon) \approx \psi_0^{in}(\xi,\varepsilon) = v_{0,0}(\xi) + \varepsilon v_{1,0}(\xi) + \sum_{k=2}^{\infty} \varepsilon^k v_{k,0}(\xi), \tag{5.5}$$

where

$$\begin{aligned} v_{0,0}(\xi) \sim&\psi_0(0), \quad v_{1,0}(\xi) \sim \sum_{m=1}^{n} \frac{\partial \psi_0}{\partial x_m}(0)\xi_m, \quad \rho \to \infty,\\ v_{k,0}(\xi) \sim&P_k(\xi), \quad k \geqslant 2, \quad \rho \to \infty.\end{aligned} \tag{5.6}$$

Substituting $\lambda^{\mu,\varepsilon} = \lambda_0$, (4.1) and (5.5) into the equation

$$H_{\mu,\varepsilon}\psi^{\mu,\varepsilon} = \lambda^{\mu,\varepsilon}\psi^{\mu,\varepsilon}, \tag{5.7}$$

changing to the variable ξ and equalling the coefficients with similar degrees ε and μ, we obtain a recurrent system of equations for $v_{k,0}$:

$$\begin{aligned} \varepsilon^{-2}: &\quad \Delta_\xi v_{0,0} = 0,\\ \varepsilon^{-1}: &\quad \Delta_\xi v_{1,0} = \left(Q_{1,2}(\xi,D_\xi) + Q_{0,1}(\xi,D_\xi)\right) v_{0,0},\\ \varepsilon^{k-2}: &\quad \Delta_\xi v_{k,0} = \sum_{i=2}^{k} \left(Q_{i,2}(\xi,D_\xi) + Q_{i-1,1}(\xi,D_\xi) + Q_{i-2,0}(\xi,D_\xi)\right) v_{k-i,0}\\ &\qquad + \left(Q_{1,2}(\xi,D_\xi) + Q_{0,1}(\xi,D_\xi)\right) v_{k-1,0} - \lambda_0 v_{k-2,0}, \quad k \geqslant 2\end{aligned} \tag{5.8}$$

and supplementary demands for these functions:

$$\varepsilon^i\mu^{-1}: \quad V(\xi)v_{i,0}(\xi) = 0, \quad i \geqslant 0. \tag{5.9}$$

Remark 5.2. *Here Δ_ξ denotes the Laplace operator by the variable ξ. Similarly, the symbol of differentiation D_ξ denotes, that differentiation is made by the variable ξ. Though below in the equations for the coefficients of internal expansions the Laplace operator and the symbol of differentiation are applied only in this sense, for simplicity of the notations in Δ_ξ and D_ξ we omit this index ξ.*

On the strength of (5.4), (5.2) and (5.8) the functions

$$v_{0,0} \equiv \psi_0(0), \qquad v_{1,0}(\xi) = \sum_{m=1}^{n} \frac{\partial \psi_0}{\partial x_m}(0)\xi_m, \qquad v_{k,0}(\xi) = P_k(\xi), \quad k \geqslant 2, \tag{5.10}$$

are, apparently, solutions of the equations (5.8), satisfying the condition (5.6) (the condition of matching of asymptotic expansions).

Though, it is also apparent, that the conditions (5.9) are not satisfied. therefore, following the method of matching of asymptotic expansions we should supplement new terms into the internal expansion:

$$\psi^{in}(\xi,\mu,\varepsilon)\approx\psi_1^{in}(\xi,\mu,\varepsilon)=\psi_0^{in}(\xi,\varepsilon)+\mu^{-1}\left(\varepsilon^2 v_{2,1}(\xi)+\sum_{k=3}^{\infty}\varepsilon^k v_{k,1}(\xi)\right). \tag{5.11}$$

Substituting $\lambda^{\mu,\varepsilon}=\lambda_0$, (4.1) and (5.11) (instead of (5.5)) into the equation (5.7), changing to the variable ξ and equalling the coefficients with similar degrees ε and μ, we obtain a new recurrent system of equations (5.8), a new recurrent system of equations for the functions $v_{2+k,1}(\xi)$, the first two of which possess the form:

$$\mu^{-1}: \quad \Delta v_{2,1}=Vv_{0,0}, \tag{5.12}$$

$$\varepsilon\mu^{-1}: \quad \Delta v_{3,1}=\left(Q_{1,2}(\xi,D)+Q_{0,1}(\xi,D)\right)v_{2,1}+Vv_{1,0} \tag{5.13}$$

and supplementary demands for these functions (instead of the conditions (5.9)):

$$\varepsilon^i\mu^{-2}: \quad V(\xi)v_{i,1}(\xi)=0, \quad i\geqslant 2. \tag{5.14}$$

It is apparent, that the equalities (5.14) are not satisfied. And to substitute these equalities for the equations of the type (5.12), (5.13) in the internal expansion (5.11), we should supplement the summands $\mu^{-2}\varepsilon^{i+2}v_{i+2,2}$ (similarly to that with the equalities (5.10)). These new summands in their turn result in demands of the form (5.9), (5.14) when $\mu^{-3}\varepsilon^i$, $i\geqslant 4$, for which eliminating we should introduce the summands $\mu^{-3}\varepsilon^{i+2}v_{i+2,3}$ and etc. Therefore the internal expansion in natural to be searched in the form

$$\begin{aligned}\psi_{odd}^{in}(\xi,\mu,\varepsilon)=&\psi_{odd}^{in,1}(\xi,\mu,\varepsilon)=\sum_{i=0}^{\infty}\varepsilon^i v_{i,0}(\xi)\\&+\varepsilon^2\mu^{-1}\sum_{j=0}^{\infty}\sum_{i=2j}^{\infty}\varepsilon^i\mu^{-j}v_{2+i,j+1}(\xi), \qquad \text{if } \psi_0(0)\neq 0.\end{aligned} \tag{5.15}$$

Substituting $\lambda^{\mu,\varepsilon}=\lambda_0$, (4.1) and (5.15) (instead of (5.11)) into the equation (5.7), changing to the variable ξ and equalling the coefficients with $\varepsilon^k\mu^{-l}$, we obtain with $l=0$ the system of equations (5.8), and when $l=j+1\geqslant 1$ we obtain a recurrent system of equations for the functions $v_{2+k,j+1}(\xi)$, the first two of which (with the fixed $j\geqslant 0$) have the form

$$\varepsilon^{2j}\mu^{-j-1}: \ \Delta v_{2j+2,j+1}=Vv_{2j,j}, \tag{5.16}$$

$$\varepsilon^{2j+1}\mu^{-j-1}: \ \Delta v_{2j+3,j+1}=\left(Q_{1,2}(\xi,D)+Q_{0,1}(\xi,D)\right)v_{2j+2,j+1}+Vv_{2j+1,j}, \tag{5.17}$$

including, in particular, when $j=0$ the equations (5.12), (5.13).

In the strength of th the equalities (5.10) and the Lemmas 4.6, 4.7 the functions

$$v_{2j+2,j+1}(\xi)=\psi_0(0)z_0^{(j+1)}(\xi), \qquad j\geqslant 0, \tag{5.18}$$

are the solutions of the equations (5.16).

Remark 5.3 (the case $\psi_0(0)=0$). *If $\psi_0(0)=0$, then again on the strength of the equalities (5.10) and the Lemmas 4.6, 4.7 the functions*

$$\begin{gathered}v_{2j+2,j+1}(\xi)\equiv 0, \quad v_{2j+3,j+1}(\xi)=\sum_{m=1}^{n}\frac{\partial\psi_0}{\partial x_m}(0)z_m^{(j+1)}(\xi), \quad j\geqslant 0,\\ \text{if } \psi_0(0)=0,\end{gathered} \tag{5.19}$$

are the solutions of the equations (5.16), (5.17).

It results from (5.10), (5.19) and (5.15), in particular, that

$$\begin{aligned}\psi^{in}_{odd}(\xi,\mu,\varepsilon) =&\psi^{in,2}_{odd}(\xi,\mu,\varepsilon) = \sum_{i=1}^{\infty}\varepsilon^i v_{i,0}(\xi)\\ &+\varepsilon^3\mu^{-1}\sum_{j=0}^{\infty}\sum_{i=2j}^{\infty}\varepsilon^i\mu^{-j}v_{3+i,j+1}(\xi), \qquad \text{if } \psi_0(0)=0.\end{aligned} \tag{5.20}$$

Remark 5.4 (on evenness n)**.** *Let us emphasize, that the described above algorithm does not depend on oddness n. The further matching of the internal and the external asymptotic expansions of the eigenfunctions of the operator $\mathcal{H}_{\mu,\varepsilon}$, presented below, shows, that the internal asymptotic expansion really possesses the form (5.15), (5.20), (5.10), (5.18), (5.19) for odd n, but possesses a more bulky structure for even n, unlike (5.15), (5.20). The case of an even n is studied below in the section 10.*

6. Derivation of the structure of the external asymptotic expansion of the eigenfunction and the asymptotics of the characteristic constant in case of an odd-dimensional domain

Temporarily we consider unidentified coefficients in (5.15) and (5.20) equal to zero, i.e. we suppose, that

$$\begin{aligned}\psi^{in,1}_{odd}(\xi,\mu,\varepsilon) =&\sum_{i=0}^{\infty}\varepsilon^i v_{i,0}(\xi)+\sum_{j=0}^{\infty}\varepsilon^{2j+2}\mu^{-j-1}v_{2j+2,j+1}(\xi), \qquad \psi_0(0)\neq 0,\\ \psi^{in,2}_{odd}(\xi,\mu,\varepsilon) =&\sum_{i=1}^{\infty}\varepsilon^i v_{i,0}(\xi)+\sum_{j=0}^{\infty}\varepsilon^{2j+3}\mu^{-j-1}v_{2j+3,j+1}(\xi), \qquad \psi_0(0)= 0.\end{aligned} \tag{6.1}$$

Then, substituting coefficients in (6.1) the coefficients $v_{i,0}$, $v_{2j+2,j+1}$ and $v_{2j+3,j+1}$ into their asymptotics when $\rho\to\infty$ and rewriting the obtained sum in variables x, subject to the equalities (5.10), (5.18), (5.19) and statements of Lemmas 4.6, 4.7 we obtain, that

$$\begin{aligned}\psi^{in,1}_{odd}(\xi,\mu,\varepsilon) =&\sum_{k=0}^{\infty}P_k(x)+\varepsilon^n\mu^{-1}\sum_{j=0}^{\infty}\sum_{i=2j}^{\infty}\varepsilon^i\mu^{-j}\varphi^{(1)}_{n+i,j+1}(x)\\ &+\delta^2_n d_1(\mu,\varepsilon)\ln\varepsilon, \qquad \psi_0(0)\neq 0,\\ \psi^{in,2}_{odd}(\xi,\mu,\varepsilon) =&\sum_{k=1}^{\infty}P_k(x)+\varepsilon^{n+1}\mu^{-1}\sum_{j=0}^{\infty}\sum_{i=2j}^{\infty}\varepsilon^i\mu^{-j}\varphi^{(2)}_{n+i+1,j+1}(x)\\ &+\delta^2_n d_2(\mu,\varepsilon)\ln\varepsilon, \qquad \psi_0(0)= 0,\end{aligned} \tag{6.2}$$

where (we should remind) δ^q_p is the Kronecker delta,

$$d_1(\mu,\varepsilon) =\varepsilon^2\mu^{-1}\psi_0(0)\sum_{j=0}^{\infty}\varepsilon^{2j}\mu^{-j}c^{(j+1)}_{0,0},$$

$$d_2(\mu,\varepsilon) =\varepsilon^3\mu^{-1}\sum_{j=0}^{\infty}\varepsilon^{2j}\mu^{-j}\sum_{m=1}^{2}\frac{\partial\psi_0}{\partial x_m}(0)c^{(j+1)}_{m,0},$$

$$\begin{aligned}\varphi^{(1)}_{2+2j,j+1}(x) =& -\psi_0(0)c^{(j+1)}_{0,0}\ln r, \qquad j\geqslant 0, \quad n=2,\\ \varphi^{(2)}_{2+2j+1,j+1}(x) =& -\sum_{m=1}^{2}\frac{\partial\psi_0}{\partial x_m}(0)c^{(j+1)}_{m,0}\ln r, \quad j\geqslant 0, \quad n=2,\end{aligned} \tag{6.3}$$

$$
\begin{aligned}
\varphi^{(1)}_{n+2j,j+1}(x) =&\psi_0(0)c^{(j+1)}_{0,0}r^{-n+2}, \qquad j\geqslant 0, \quad n\geqslant 3,\\
\varphi^{(2)}_{n+2j+1,j+1}(x) =&\sum_{m=1}^{n}\frac{\partial\psi_0}{\partial x_m}(0)c^{(j+1)}_{m,0}r^{-n+2}, \quad j\geqslant 0, \quad n\geqslant 3,
\end{aligned}
\tag{6.4}
$$

and $\varphi^{(s)}_{n+i+q-1,j+1}(x)$ with the remaining low indexes are finite sums of homogeneous functions of the order not less than $(-n-i+2j+2)$.

Remark 6.1 (on the case $n=2$)**.** *As for the summands $d_q(\mu,\varepsilon)\ln\varepsilon$ in (6.2) when $n=2$, we plan to consider them below in the Remark 7.2. Until then we ignore them.*

Following the method of matching of asymptotic expansions and considering the equalities (6.2), (6.3), (6.4) and Remark 6.1, the external expansions is searched in the form

$$
\begin{aligned}
\psi^{ex}_{odd}(x,\mu,\varepsilon) =&\psi^{ex,1}_{odd}(x,\mu,\varepsilon)\\
=&\psi_0(x)+\varepsilon^n\mu^{-1}\sum_{j=0}^{\infty}\sum_{i=2j}^{\infty}\varepsilon^i\mu^{-j}\psi_{n+i,j+1}(x), \quad \psi_0(0)\neq 0,\\
\psi^{ex}_{odd}(x,\mu,\varepsilon) =&\psi^{ex,2}_{odd}(x,\mu,\varepsilon)\\
=&\psi_0(x)+\varepsilon^{n+1}\mu^{-1}\sum_{j=0}^{\infty}\sum_{i=2j}^{\infty}\varepsilon^i\mu^{-j}\psi_{n+i+1,j+1}(x), \quad \psi_0(0)= 0,
\end{aligned}
\tag{6.5}
$$

where, in particular,

$$
\begin{aligned}
\psi_{2+2j,j+1}(x)\sim&-\psi_0(0)c^{(j+1)}_{0,0}\ln r, \qquad j\geqslant 0, \quad n=2, \quad \psi_0(0)\neq 0,\\
\psi_{3+2j,j+1}(x)\sim&-\sum_{m=1}^{2}\frac{\partial\psi_0}{\partial x_m}(0)c^{(j+1)}_{m,0}\ln r, \quad j\geqslant 0, \quad n=2, \quad \psi_0(0)=0,\\
\psi_{n+2j,j+1}(x)\sim&\psi_0(0)c^{(j+1)}_{0,0}r^{-n+2}, \qquad j\geqslant 0, \quad n\geqslant 3, \quad \psi_0(0)\neq 0,\\
\psi_{n+2j+1,j+1}(x)\sim&\sum_{m=1}^{n}\frac{\partial\psi_0}{\partial x_m}(0)c^{(j+1)}_{m,0}r^{-n+2}, \quad j\geqslant 0, \quad n\geqslant 3, \quad \psi_0(0)=0,
\end{aligned}
\tag{6.6}
$$

when $r\to 0$.

Since the external expansion should describe behaviour of the eigenfunction almost in all the domain Ω (except fort he small neighbourhood of the zero), then by analogy with (6.5) (and subject to Remark 6.1) the asymptotics of the characteristic constant is natural to be searched in the form

$$
\lambda_{odd}(\mu,\varepsilon)=\lambda^1_{odd}(\mu,\varepsilon)=\lambda_0+\varepsilon^n\mu^{-1}\sum_{j=0}^{\infty}\sum_{i=2j}^{\infty}\varepsilon^i\mu^{-j}\lambda_{n+i,j+1}, \quad \psi_0(0)\neq 0, \tag{6.7}
$$

$$
\begin{aligned}
\lambda_{odd}(\mu,\varepsilon)=&\lambda^2_{odd}(\mu,\varepsilon)\\
=&\lambda_0+\varepsilon^{n+1}\mu^{-1}\sum_{j=0}^{\infty}\sum_{i=2j}^{\infty}\varepsilon^i\mu^{-j}\lambda_{n+i+1,j+1}, \quad \psi_0(0)=0.
\end{aligned}
\tag{6.8}
$$

Remark 6.2 (on the structure of the asymptotics of the characteristic constant)**.** *For the odd n the series (6.7) has the form (2.1), but in the critical case $\psi_0(0)=0$ the form of the series (6.8) differs from the form of the series (2.6). For the series (6.8) to possess the form (2.6) we need only the equalities $\lambda_{n+2j+1,j+1}=0$. Consideration of this equality satisfying are presented below in Remark 7.1.*

7. Derivation of equations for the coefficients of asymptotic expansions in case of an odd-dimensional domain

Since the external expansion is considered outside the neighbourhood of the origin of coordinates and $H_0 = H_{\mu,\varepsilon}$ outside the neighbourhood of the origin of coordinates, then substituting into the equation

$$H_0\psi^{\mu,\varepsilon} = \lambda^{\mu,\varepsilon}\psi^{\mu,\varepsilon} \tag{7.1}$$

the series (6.5), (6.7), (6.8) and equalling the coefficients with similar degrees ε and μ, we obtain a fortiori satisfying equation (5.3) and a recurrent system of equations in $\Omega\backslash\{0\}$ for the remaining coefficients of the external expansion (6.5):

$$\begin{aligned}
\varepsilon^{n+i}\mu^{-1}: \quad & (H_0-\lambda_0)\,\psi_{n+i,1} = \lambda_{n+i,1}\psi_0, \quad i \geqslant 0,\\
\varepsilon^{n+i+2j}\mu^{-1-j}: \quad & (H_0-\lambda_0)\,\psi_{n+i+2j,j+1} = \lambda_{n+i+2j,j+1}\psi_0, \quad 0 \leqslant i \leqslant n-3,\\
& (H_0-\lambda_0)\psi_{n+i+2j,j+1} = \lambda_{n+i+2j,j+1}\psi_0\\
& \qquad + \sum_{k=0}^{i-n+2}\sum_{s=0}^{j-1}\lambda_{n+k+2s,s+1}\psi_{i-k+2(j-s),j-s},\\
& \qquad i \geqslant n-2, \quad j \geqslant 1,
\end{aligned} \tag{7.2}$$

where

$$\psi_{n+2j,j+1}(x) = \lambda_{n+2j,j+1} = 0, \qquad \text{if } \psi_0(0) = 0, \tag{7.3}$$

on the strength of (6.5) and (6.8).

Remark 7.1 (on the structure of asymptotics of the characteristic constant in the case $\psi_0(0) = 0$)**.** *From (7.2), (7.3) we obtain the following equality:*

$$H_0\psi_{n+2j+1,j+1} = \lambda_0\psi_{n+2j+1,j+1} + \lambda_{n+2j+1,j+1}\psi_0, \qquad \psi_0(0) = 0, \tag{7.4}$$

when $j \geqslant 0$. On the strength of Lemmas 4.4, 4.5 the functions

$$\psi_{n+2j+1,j+1}(x) = \sum_{m=1}^{n}\frac{\partial\psi_0}{\partial x_m}(0)c_{m,0}^{(j+1)}E_0(x), \quad j \geqslant 0, \qquad \psi_0(0) = 0, \tag{7.5}$$

possess the asymptotics (6.6) and are solutions of the equations (7.4) when

$$\lambda_{n+2j+1,j+1} = 0, \qquad j \geqslant 0, \qquad \text{if } \psi_0(0) = 0. \tag{7.6}$$

Subject to the equalities (7.6), firstly, the series (6.8) already takes the form (2.6) for the odd n, and secondly, in the equations (7.2) the condition (7.3) is substituted by the following:

$$\psi_{2+2j,j+1}(x) = \lambda_{2+2j,j+1} = \lambda_{3+2j,j+1} = 0 \qquad \text{when } \psi_0(0) = 0 \tag{7.7}$$

for the coefficients of the external expansion.

Certainly, even from the position of construction of complete formal asymptotic expansions of characteristic constants and the corresponding eigenfunctions the equalities (7.7) still remain expected and reliable. The verification of validity of the equality (7.7) in this sense is presented in the next section 8 with the construction of complete formal asymptotic expansions (see, for instance, the conclusion of the equality (8.13)).

Remark 7.2 (on evenness n)**.** *Let us again emphasize, that the presented above algorithm still does not depend on evenness-oddness $n \geqslant 3$. The further matching of the internal and the external asymptotic expansions of the eigenfunctions of the operator $\mathcal{H}_{\mu,\varepsilon}$, presented below, demonstrates, that the external asymptotic expansion really possesses the form (6.5) for odd n. Though for even n the situations complicates. For instance, to match in (6.2)the summands,* *containing* $\ln\varepsilon$ *when* $n=2$, *in the internal expansions (5.15) for* $\psi^{in,1}_{odd}(\xi,\mu,\varepsilon)$ *and (5.20) for* $\psi^{in,2}_{odd}(\xi,\mu,\varepsilon)$ *it is necessary to supplement summands containing* $\ln\varepsilon$*:*

$$\ln\varepsilon\, d_1(\mu,\varepsilon),\quad \ln\varepsilon\, d_2(\mu,\varepsilon)$$

correspondingly. A similar situation occurs at the following stages of matching of asymptotic expansions and for even $n\geqslant 4$, *as, for instance, the asymptotics in the zero of the function* $E_0(x)$ *from the equalities (7.5) contains with even* n *logarithmic terms. The derivation of the structure of complete asymptotic expansions of characteristic constant is presented below in section 10.*

In the conclusion of the section let us derive equations for the coefficients of the internal expansion. Substituting series (5.15), (5.20), (6.7) and (6.8) into the equation

$$H_{\mu,\varepsilon}\psi^{\mu,\varepsilon}=\lambda^{\mu,\varepsilon}\psi^{\mu,\varepsilon},$$

changing in it to the internal variables ξ and writing the equalities with similar degrees ε and μ, we obtain for the coefficients of internal expansions equations (5.8), satisfied for the functions defined by the equalities (5.10) and the equations

$$\begin{aligned}
\Delta v_{2j+2,j+1} &= V v_{2j,j},\\
\Delta v_{2j+3,j+1} &= \left(Q_{1,2}(\xi,D)+Q_{0,1}(\xi,D)\right) v_{2j+2,j+1}+V v_{2j+1,j},\\
\Delta v_{i+4+2j,j+1} &= \sum_{q=2}^{i}\big(Q_{q,2}(\xi,D)+Q_{q-1,1}(\xi,D)\\
&\qquad\qquad + Q_{q-2,0}(\xi,D)\big)v_{i+4-q+2j,j+1}\\
&\quad+\left(Q_{1,2}(\xi,D)+Q_{0,1}(\xi,D)\right) v_{i+3+2j,j+1}+V v_{i+2j+2,j}\\
&\quad-\lambda_0 v_{i+2j+2,j+1},\quad i<n,\\
\Delta v_{i+4+2j,j+1} &= \sum_{q=2}^{i}\big(Q_{q,2}(\xi,D)+Q_{q-1,1}(\xi,D)\\
&\qquad\qquad + Q_{q-2,0}(\xi,D)\big)v_{i+4-q+2j,j+1}\\
&\quad+\left(Q_{1,2}(\xi,D)+Q_{0,1}(\xi,D)\right) v_{i+3+2j,j+1}+V v_{i+2j+2,j}\\
&\quad-\sum_{p=0}^{i-n}\sum_{l=0}^{j} v_{p+2l,l}\lambda_{i+2(j-l)-p+2,j-l+1}\\
&\quad-\lambda_0 v_{i+2j+2,j+1},\quad i\geqslant n,\quad j\geqslant 0,
\end{aligned}\tag{7.8}$$

where

$$v_{2j+2,j+1}(\xi)=\lambda_{n+2j+2,j+1}=\lambda_{n+1+2j+2,j+1}=0,\quad \text{if } \psi_0(0)=0. \tag{7.9}$$

on the strength of (5.19), (7.7).

8. Construction of complete formal asymptotic expansions in case of odd-dimensional domain

We determine operators $\mathcal{K}_{q,m}$ and $\mathcal{K}$ in the series $U(x,\varepsilon,\mu)$ of the form (6.5) the following way. We decompose coefficients of the series $U(x,\varepsilon,\mu)$ into the series when $r\to 0$ and change to the variables ξ. In the series obtained we save only terms of the form $\varepsilon^q\mu^{-m}\Phi(\xi)$. We denote this series by $\mathcal{K}_{q,m}(U(x,\varepsilon,\mu))$ and assume

$$\mathcal{K}=\sum_{q,m}\mathcal{K}_{q,m}.$$

We construct the coefficients of the asymptotics of the characteristic constant and the external expansion of the eigenfunction in the following form

$$\lambda_{n+i+2j,j+1} = \sum_{t=0}^{i} \Lambda^{(t)}_{n+i+2j,j+1}, \quad j \geqslant 0, \tag{8.1}$$

$$\psi_{n+i+2j,j+1}(x) = \sum_{t=0}^{i} \Psi^{(t)}_{n+i+2j,j+1}(x), \quad j \geqslant 0. \tag{8.2}$$

Let us denote

$$\Phi^{(N)}_{n+i+2j,j+1}(x) := \sum_{t=0}^{\min\{i,N\}} \Psi^{(t)}_{n+i+2j,j+1}(x).$$

In these notations $\Phi^{(N)}_{n+i+2j,j+1}(x) = \psi_{n+i+2j,j+1}(x)$ when $N \geqslant i$ on the strength of (8.2). We denote by $\Phi^{ex}_{odd,N}(x,\mu,\varepsilon)$ series of the form (6.5), where the coefficients $\psi_{n+i+2j,j+1}(x)$ are substituted for $\Phi^{(N)}_{n+i+2j,j+1}(x)$.

The validity of the following statement results from the definition $\mathcal{A}^m$, $\mathcal{A}_m$, $\widetilde{\mathcal{B}}_m$, $\mathcal{K}_{m,l}$, $\mathcal{K}$, $\Phi^{ex}_{odd,N}(x,\mu,\varepsilon)$ and (8.1), (8.2).

Lemma 8.1. *If the coefficients $\psi_{n+i+2j,j+1}(x)$ of the series (6.5) belong to $\mathcal{A}^i$, then*

$$\mathcal{K}(\psi^{ex,s}_{odd}(x,\mu,\varepsilon)) = \Psi^{in,s}_{odd}(\xi,\mu,\varepsilon),$$

where $\Psi^{in,s}_{odd}(\xi,\mu,\varepsilon)$ are series of the form (5.15), (5.20), in which the coefficients $v_{2+i,j+1}(\xi)$ are substituted for the series $V_{2+i,j+1}(\xi) \in \widetilde{\mathcal{B}}_{i-2j}$.

If $\Psi^{(t)}_{n+i+2j,j+1}(x) \in \mathcal{A}_{i-t}$, then the functions $\psi_{n+i+2j,j+1}(x)$, determined by the equality (8.2), belong to $\mathcal{A}^i$ the following equalities take place:

$$V_{2j+2+t,j+1}(\xi) = \widetilde{V}_{2j+2+t,j+1}(\xi) + \sum_{k=0}^{\infty} Z^{(2j+2+t,j+1)}_k(\xi)\rho^{-n+2-2k},$$

where

$$\begin{aligned}\widetilde{V}_{2j+2,j+1}(\xi) \equiv& 0,\\ \widetilde{V}_{2j+2+t,j+1}(\xi) =& \varepsilon^{-2j-2-t}\mu^{j+1}\mathcal{K}_{2j+2+t,j+1}\left(\Phi^{ex}_{odd,t-1}(x,\mu,\varepsilon)\right) \in \widetilde{\mathcal{B}}_t, \quad t \geqslant 1,\end{aligned}$$

(i.e. $\widetilde{V}_{2j+2+t,j+1}$ do not depend on $\Psi^{(m)}_{p,q}$ when $m \geqslant t-1$), and $Z^{(2j+2+t,j+1)}_k r^{-n+2-2k}$ is the dominant term of the asymptotics $\Psi^{(t)}_{n+2j-t+k,j+1}$ in the zero.

If, meanwhile, the functions $\Psi^{(t)}_{n+i+2j,j+1}(x)$ are in $\Omega\backslash\{0\}$ the solutions of the equations

$$\begin{aligned}(H_0-\lambda_0)\,\Psi^{(t)}_{n+i,1} =& \Lambda^{(t)}_{n+i,1}\psi_0, \quad i \geqslant 0,\\ (H_0-\lambda_0)\,\Psi^{(t)}_{n+i+2j,j+1} =& \Lambda^{(t)}_{n+i+2j,j+1}\psi_0, \quad 0 \leqslant i \leqslant n-3,\\ (H_0-\lambda_0)\,\Psi^{(t)}_{n+i+2j,j+1} =& \Lambda^{(t)}_{n+i+2j,j+1}\psi_0\\ &+ \sum_{k=0}^{i-n+2}\sum_{s=0}^{j-1}\sum_{p=0}^{t} \Lambda^{(p)}_{n+k+2s,s+1}\Psi^{(t-p)}_{i-k+2(j-s),j-s},\\ & i \geqslant n-2, \quad j \geqslant 1,\end{aligned} \tag{8.3}$$

then the functions $\psi_{n+i+2j,j+1}(x)$, determined by the equalities (8.2), are the solutions of the equations (7.2) where $\lambda_{n+i+2j,j+1}$, determined by the equalities (8.1), the series $\widetilde{V}_{2j+2+t,j+1}$ are formal asymptotic solutions of the equations (7.8) when $\rho \to \infty$, where in the first side of the function $v_{m,q}(\xi)$ are substituted by the series $V_{m,q}(\xi)$ when $q > 0$.

Theorem 8.1. *Let n be odd, λ_0 be a simple characteristic constant of the operator $\mathcal{H}_0$, ψ_0 be the corresponding normalized in $L_2(\Omega)$ eigenfunction.*

Therefore there exist series (2.1), (2.6), (6.5), (5.15) and (5.20) such that:
1) the equalities (2.2), (2.3), (2.4), (2.7) hold;
2) the functions $\psi_{n+2j+i,j+1} \in \mathcal{A}^i$ are the solutions of the equations (7.2), (7.7);
3) the functions $v_{i,0}$ are determined by the equalities (5.10), and the functions $v_{2j+2+i,j+1} \in \mathcal{B}_i$ are the solutions of the equations (7.8), (7.9);
4) the following equality holds

$$\mathcal{K}(\psi^{ex,s}_{odd}(x,\mu,\varepsilon)) = \psi^{in,s}_{odd}(\xi,\mu,\varepsilon), \qquad \rho \to \infty.$$

Proof. Subject to the statements of Lemma 8.1 to prove the theorem it is sufficient to show, that the correct choice at the t stage of the matching of the dominant terms of asymptotics in the zero of the functions $\Psi^{(t)}_{n+2j-t+k,j+1}(x)$, enables to achieve the existence of the series (5.15), (5.20) such that their coefficients $v_{2j+2+t,j+1}(\xi) \in \mathcal{B}_t$ are the solutions of the equations (7.8), (7.9) and possess with $\rho \to \infty$ the asymptotics $V_{2j+2+t,j+1}$ from the formulation of Lemma 8.1.

Let us start with the definition $\Psi^{(0)}_{n+2j+k,j+1}(x)$. As is has been shown above (see, (5.10), (5.8), (5.18), (5.16)) the functions

$$v_{0,0} \equiv \psi(0), \quad v_{2j+2,j+1}(\xi) = \psi_0(0) z_0^{(j+1)}(\xi) \in \mathcal{B}_0, \qquad j \geqslant 0, \tag{8.4}$$

are the solutions of the equations (5.8) and (7.8) (in the first line) and due to Lemmas 4.6, 4.7 they have with $\rho \to \infty$ the following asumptotics

$$V_{2j+2,j+1}(\xi) = \psi_0(0) \left(c^{(j+1)}_{0,0} \rho^{2-n} + \sum_{m=1}^{n} c^{(j+1)}_{0,m} \xi_m \rho^{-n} + \sum_{k=2}^{\infty} Y_k(\xi) \rho^{-2k-n+2} \right).$$

Whence on the strength of Lemma 8.1 we obtain the dominant terms of the asymptotics in the zero for the functions $\Psi^{(0)}_{n+2j+k,j+1}(x)$:

$$\begin{aligned} \Psi^{(0)}_{n+2j,j+1}(x) \sim& \psi_0(0) c^{(j+1)}_{0,0} r^{2-n}, \\ \Psi^{(0)}_{n+2j+1,j+1}(x) \sim& \psi_0(0) \sum_{m=1}^{n} c^{(j+1)}_{0,m} x_m r^{-n}, \\ \Psi^{(0)}_{n+2j+k,j+1}(x) \sim& \psi_0(0) Y_k(x) r^{-2k-n+2}, \qquad k \geqslant 2. \end{aligned} \tag{8.5}$$

On the strength of Lemma 4.2 there are the functions $\Psi^{(0)}_{n+2j+q,j+1}(x) \in \mathcal{A}_q$, possessing the required asymptotics in the zero and satisfying the equations (8.3) when some $\Lambda^{(0)}_{n+2j+q,j+1}$. Consequently, in particular, we verify the suppositions (2.1) and (6.5) for the case $\psi_0(0) \neq 0$.

Besides, firstly, the functions

$$\begin{aligned} \Psi^{(0)}_{n+2j,j+1}(x) =& \psi_0(0) c^{(j+1)}_{0,0} E_0(x), \\ \Psi^{(0)}_{n+2j+1,j+1}(x) =& \psi_0(0) \sum_{m=1}^{n} c^{(j+1)}_{0,m} E_m(x) \end{aligned} \tag{8.6}$$

possess the required asymptotics (8.5) and satisfy the equations (8.3) when

$$\Lambda^{(0)}_{n+2j,j+1} = \psi_0(0) c^{(j+1)}_{0,0} \Lambda_0, \qquad \Lambda^{(0)}_{n+2j+1,j+1} = \psi_0(0) \sum_{m=1}^{n} c^{(j+1)}_{0,m} \Lambda_m \tag{8.7}$$

on the strength of Lemma 4.4, and secondly, the following presentations are apparent

$$\begin{aligned} \Psi^{(0)}_{n+2j+k,j+1}(x) =& \psi_0(0) \widetilde{\Psi}^{(0)}_{n+2j+k,j+1}(x), \\ \Lambda^{(0)}_{n+2j+k,j+1} =& \psi_0(0) \widetilde{\Lambda}^{(0)}_{n+2j+k,j+1}, \qquad k \geqslant 2. \end{aligned} \tag{8.8}$$

Remark 8.1 (derivation of the formulae (2.2) and (2.3)). *On the strength of (8.1), (8.2), (8.6) and (8.7) we obtain, that*

$$\lambda_{n+2j,j+1} = \psi_0(0)c_{0,0}^{(j+1)}\Lambda_0, \quad \psi_{n+2j,j+1}(x) = \psi_0(0)c_{0,0}^{(j+1)}E_0(x) \in \mathcal{A}^0. \tag{8.9}$$

Substituting into these equalities for $\lambda_{n,1}$ and $\lambda_{n+2,2}$ values of the constants Λ_0, $c_{0,0}^{(1)}$ and $c_{0,0}^{(2)}$ from Lemmas 4.4, 4.6, 4.7 we obtain the equalities (2.2) and (2.3).

Remark 8.2 (the case $\psi_0(0) = 0$). *On the strength of (8.5)–(8.8), (8.9) and the presentations (8.1), (8.2) we sequentially obtain, that*

$$\begin{gathered}\Psi^{(0)}_{n+2j+k,j+1}(x) = \Lambda^{(0)}_{n+2j+k,j+1} = 0, \quad k \geqslant 1,\\ \lambda_{n+2j,j+1} = \psi_{n+2j,j+1}(x) = 0 \quad \text{if } \psi_0(0) = 0.\end{gathered} \tag{8.10}$$

Consequently, in particular, the presentation (6.5) is verified also for the case $\psi_0(0) = 0$, and on the strength of Lemma 8.1 the following equality holds

$$\widetilde{V}_{2j+3,j+1}(\xi) \equiv 0.$$

The next stage is $(t = 1)$. Due to Lemma 8.1 we obtain, that the series

$$\widetilde{V}_{2j+3,j+1}(\xi) = \varepsilon^{-2j-3}\mu^{j+1}\mathcal{K}_{2j+3,j+1}\left(\Phi^{ex}_{odd,0}(x,\mu,\varepsilon)\right) \in \widetilde{\mathcal{B}}_1$$

are asymptotic solutions when $\rho \to \infty$ of the second equations in (7.8), where in the right side the functions $v_{2q+2,q+1}$ are substituted for their asymptotics $V_{2q+2,q+1}$ when $\rho \to \infty$, and $v_{1,0} = P_1$. Due to Lemma 4.8 the are the functions $v_{2j+3,j+1} \in \mathcal{B}_1$, which are the solutions of the second equations in (7.8) and possess with $\rho \to \infty$ the asymptotics $V_{2j+3,j+1}$, such that

$$V_{2j+3,j+1}(\xi) = \widetilde{V}_{2j+3,j+1}(\xi) + \sum_{k=0}^{\infty} Z_k(\xi)\rho^{-n+2-2k}. \tag{8.11}$$

Whence due to Lemma 8.1 we obtain the dominant terms of the asymptotics in the zero for the functions $\Psi^{(1)}_{n+2j+1+k,j+1}(x)$:

$$\Psi^{(1)}_{n+2j+1+k,j+1}(x) \sim Z_k(x)r^{-n+2-2k}, \qquad k \geqslant 0. \tag{8.12}$$

On the strength of Lemma 4.2 there exist the functions $\Psi^{(1)}_{n+2j+1+k,j+1}(x) \in \mathcal{A}_k$, possessing the required asymptotics in the zero and satisfying the equations (8.3) when some $\Lambda^{(1)}_{n+2j+1+k,j+1}$.

And since at the previous stage there were $\Psi^{(0)}_{n+2j+k,j+1}(x) \in \mathcal{A}_k$ and $\Lambda^{(0)}_{n+2j+k,j+1}$ determined, then in compliance with (8.1), (8.2) the coefficients $\lambda_{n+2j+1,j+1}$ and $\psi_{n+2j+1,j+1}(x) \in \mathcal{A}^1$ are finally determined.

Remark 8.3 (the case $\psi_0(0) = 0$). *Let us note, that*

$$\Lambda^{(1)}_{n+2j+1,j+1} = 0, \quad \text{if } \psi_0(0) = 0$$

due to (8.12) and Lemma 4.4. It results from this simple equality and (8.10), (8.1), that

$$\lambda_{n+2j+1,j+1} = \lambda_{n+2j,j+1} = 0, \quad \text{if } \psi_0(0) = 0. \tag{8.13}$$

Therefore the presentation (2.6) has been verified.

To obtain at the next stage the equality (2.7) for $\lambda_{n+2,1}$ in the critical case $\psi_0(0) = 0$, let us note, that

$$v_{0,0}(\xi) = v_{2,1}(\xi) \equiv 0, \qquad v_{1,0}(\xi) = \sum_{j=1}^{n} \frac{\partial \psi_0}{\partial x_m}(0)\xi_m, \quad \text{if } \psi_0(0) = 0$$

(see, (5.10), (8.4)). Therefore the equation (7.8) for $v_{3,1}(\xi)$ (the second when $j=0$) takes the form

$$\Delta v_{3,1} = V v_{1,0} = V(\xi) \sum_{m=1}^{n} \frac{\partial \psi_0}{\partial x_m}(0)\xi_m. \tag{8.14}$$

Due to Lemma 4.6 the function

$$v_{3,1}(\xi) = \sum_{m=1}^{n} \frac{\partial \psi_0}{\partial x_m}(0) z_m^{(1)}(\xi) \tag{8.15}$$

is the solution of this equation and when $\rho \to \infty$ it possesses the following asymptotic expansion $V_{3,1}(\xi)$:

$$\begin{aligned} V_{3,1}(\xi) =& \rho^{2-n} \sum_{m=1}^{n} \frac{\partial \psi_0}{\partial x_m}(0) c_{m,0}^{(1)} \\ &+ \sum_{m=1}^{n}\sum_{k=1}^{n} \frac{\partial \psi_0}{\partial x_m}(0) c_{m,k}^{(1)} \xi_k \rho^{-n} + O(\rho^{-n}), \quad \textit{if } \psi_0(0)=0. \end{aligned} \tag{8.16}$$

It results from (8.11), (8.12)and (8.16), that

$$\Psi_{n+2,1}^{(1)}(x) \sim \sum_{m=1}^{n}\sum_{k=1}^{n} \frac{\partial \psi_0}{\partial x_m}(0) c_{m,k}^{(1)} x_k r^{-n}. \tag{8.17}$$

Due to Lemma 4.4 the function

$$\Psi_{n+2,1}^{(1)}(x) = \sum_{m=1}^{n}\sum_{k=1}^{n} \frac{\partial \psi_0}{\partial x_m}(0) c_{m,k}^{(1)} E_k(x)$$

possesses the asymptotics (8.17) in the zero and is the solution of the equation (8.3) when

$$\Lambda_{n+2,1}^{(1)} = \sum_{m=1}^{n}\sum_{k=1}^{n} \frac{\partial \psi_0}{\partial x_m}(0) c_{m,k}^{(1)} \Lambda_k. \tag{8.18}$$

Let us proceed to the next stage ($t=2$). Due to Lemma 8.1 we obtain, that the series

$$\widetilde{V}_{2j+4,j+1}(\xi) = \varepsilon^{-2j-4}\mu^{j+1}\mathcal{K}_{2j+4,j+1}\left(\Phi_{odd,1}^{ex}(x,\mu,\varepsilon)\right) \in \widetilde{\mathcal{B}}_2$$

are the asymptotic solutions when $\rho \to \infty$ of the equation (7.8), where the functions $v_{2j+3,j+1}(\xi)$ are substituted for their asymptotics $V_{2j+3,j+1}(\xi)$, and when $j>0$ and the functions and $v_{2j+2,j}(\xi)$ are substituted for their asymptotics $V_{2j+2,j}(\xi)$. Due to Lemma 4.8 there exist the functions $v_{2j+4,j+1}(\xi) \in \mathcal{B}_2$ which are the solutions of the equations (7.8) and possess when $\rho \to \infty$ the following asymptitics $V_{2j+4,j+1}(\xi)$:

$$V_{2j+4,j+1}(\xi) = \widetilde{V}_{2j+4,j+1}(\xi) + \sum_{k=0}^{\infty} Z_k^{(2j+4,j+1)}(\xi)\rho^{-n+2-2k}.$$

Whence on the strength of Lemma 8.1 we obtain the dominant terms of the asymptotics in the zero for the functions $\Psi_{n+2j+2+k,j+1}^{(2)}(x)$:

$$\Psi_{n+2j+2+k,j+1}^{(2)}(x) \sim Z_k^{(2j+4,j+1)}(x) r^{-n+2-2k}, \qquad k \geqslant 0.$$

On the strength of Lemma 4.2 there exist the functions $\Psi_{n+2j+2+k,j+1}^{(1)}(x) \in \mathcal{A}_k$, possessing the required asymptotics in the zero and satisfying the equations (8.3) when some $\Lambda_{n+2j+2+k,j+1}^{(2)}$.

Since we have already determined $\Psi_{n+2j+k,j+1}^{(0)}$, $\Psi_{n+2j+1+k,j+1}^{(1)} \in \mathcal{A}_k$ and $\Lambda_{n+2j+k,j+1}^{(0)}$, $\Lambda_{n+2j+1+k,j+1}^{(1)}$, then in compliance with (8.1), (8.2) the coefficients $\lambda_{n+2j+2,j+1}$ and $\psi_{n+2j+2,j+1} \in \mathcal{A}^2$ are also finally determined.

And on the same lines.

Remark 8.4 (derivation of the formula (2.7))**.** *Let us note, that*

$$\Lambda^{(2)}_{n+2,1} = 0, \quad \textit{if } \psi_0(0) = 0 \tag{8.19}$$

on the strength of (8.12) and Lemma 4.4. It results from (8.10), (8.18), (8.19) and (8.1), that

$$\lambda_{n+2,1} = \sum_{m=1}^{n}\sum_{k=1}^{n} \frac{\partial\psi_0}{\partial x_m}(0)c^{(1)}_{m,k}\Lambda_k, \quad \textit{if } \psi_0(0) = 0.$$

Substituting into the equality the values of the constants Λ_k and $c^{(1)}_{m,k}$ from Lemmas 4.4, 4.6 we obtain the equality (2.7).

Remark 8.5 (derivation of the formula (2.4))**.** *If $H_0 = -\Delta + a$, then the equation (7.8) for $v_{3,1}(\xi)$ once again takes the form (8.14). Its solution is determined by the equality (8.15) and possesses when $\rho \to \infty$ the asymptotics (8.16). It results from (8.11), (8.12) and (8.16), that*

$$\Psi^{(1)}_{n+1,1}(x) \sim r^{2-n} \sum_{m=1}^{n} \frac{\partial\psi_0}{\partial x_m}(0)c^{(1)}_{m,0}, \quad r \to 0.$$

Due to Lemma 4.4 the function

$$\Psi^{(1)}_{n+1,1}(x) = E_0(x) \sum_{m=1}^{n} \frac{\partial\psi_0}{\partial x_m}(0)c^{(1)}_{m,0}.$$

possesses in the zero the required asymptotics and is the solution of the equation (8.3) when

$$\Lambda^{(1)}_{n+1,1} = \Lambda_0 \sum_{m=1}^{n} \frac{\partial\psi_0}{\partial x_m}(0)c^{(1)}_{m,0}. \tag{8.20}$$

It results from (8.7), (8.20) and (8.1), that

$$\lambda_{n+1,1} = \psi_0(0)\sum_{m=1}^{n} c^{(1)}_{0,m}\Lambda_m + \Lambda_0 \sum_{m=1}^{n} \frac{\partial\psi_0}{\partial x_m}(0)c^{(1)}_{m,0}.$$

Substituting into this equality the values of the constants Λ_k and $c^{(1)}_{m,k}$ from Lemmas 4.4, 4.6 we obtain the equality (2.4).

The Theorem has been completely proved. □

Let us denote partial sums of the series $\psi^{ex,s}_{odd}(x,\mu,\varepsilon)$ and $\psi^{in,s}_{odd}(\xi,\mu,\varepsilon)$ up to the degrees M by ε inclusive by $\widehat{\psi}^{ex,s}_{odd,M}(x,\mu,\varepsilon)$ and $\widehat{\psi}^{in,s}_{odd,M}(\xi,\mu,\varepsilon)$, correspondingly. And by $\widehat{\lambda}^1_{odd,M}(\mu,\varepsilon)$ and $\widehat{\lambda}^2_{odd,M}(\mu,\varepsilon)$ we denote analogous partial sums of the series (2.1) and (2.6) correspondingly. it results from the items 2)–4) of the proved Theorem 8.1, that

Corollary 3. *The following equalities hold*

$$\left(H_0 - \widehat{\lambda}^s_{odd,n+2N}(\mu,\varepsilon)\right)\widehat{\psi}^{ex,s}_{odd,n+2N}(x,\mu,\varepsilon) = O\left(\mathcal{G}_n(r)\left(\left(\varepsilon r^{-1}\right)^2 + \varepsilon^2\mu^{-1}\right)^{N-1}\right)$$
$$\textit{when } r \to 0,\ \varepsilon r^{-1} \to 0,$$
$$\left(H_{\mu,\varepsilon} - \widehat{\lambda}^s_{odd,n+2N}(\mu,\varepsilon)\right)\widehat{\psi}^{in,s}_{odd,2(N+1)}(\xi,\mu,\varepsilon) = O\left((\varepsilon\rho)^{-1}\left((\varepsilon\rho)^2 + \varepsilon^2\mu^{-1}\right)^{N}\right)$$
$$\textit{when } \rho \to \infty,\ \varepsilon\rho \to 0,$$
$$\widehat{\psi}^{ex,s}_{odd,n+2N}(x,\mu,\varepsilon) - \widehat{\psi}^{in,s}_{odd,2(N+1)}(\xi,\mu,\varepsilon) = O\left(\left(r^2 + \varepsilon^2\mu^{-1} + \rho^{-2}\right)^{N}\right)$$
$$\textit{when } r \to 0, \quad \rho \to \infty,$$

meanwhile, the latter equality is differentiated by x_m (subject to $\xi = \varepsilon^{-1}x$).

9. Construction of complete formal asymptotic expansions in case of twofold characteristic constant λ_0 and the odd-dimensional domain

In the above sections there was considered the case of the simple characteristic constant λ_0 under the construction of the asymptotic expansion of the characteristic constant when the construction could be started not with the function $\psi_0(x)$, but for instance, with the function $\psi_0(x) + \varepsilon^q C\psi_0(x) = (1 + \varepsilon^q C)\psi_0(x)$ for any $q > 0$ and C, that, apparently, due to the linearity of the considered operators could result in the same asymptotics of the characteristic constant. Therefore it was useless to start construction of the asymptotics with the similar functions. In the case considered in the present section, when λ_0 is a twofold characteristic constant of the operator $\mathcal{H}_0$, the situation is different, as this characteristic constant is corresponded by the two eigenfunctions $\psi_0^{(1)}(x)$ and $\psi_0^{(2)}(x)$. Therefore while constructing the asymptotic expansions corresponding to the eigenfunctions of the operator $\mathcal{H}_{\mu,\varepsilon}$ which converge to the eigenfunctions $\psi_0^{(s)}(x)$, we start the construction with the following asymptotic series:

$$\psi_0^{(s)}(x) + \varepsilon\psi_0^{(s^*)}(x)\sum_{j=0}^{\infty}\sum_{i=2j}^{\infty}\alpha_{i+1,j}^{(s)}\varepsilon^i\mu^{-j}, \tag{9.1}$$

where $s^* = 2$, if $s = 1$ and, on the contrary, $s^* = 1$, if $s = 2$, and $\alpha_{i+1,j}^{(s)}$ are still arbitrary constants.

Remark 9.1. *The intuitive consideration of presence of the latter sums in (9.5) (their validity is seen from the further matching of the asymptotic expansions of the eigenfunctions) consists in the following observation: there is nothing restricting during the construction of the internal expansion of the eigenfunction, converging to $\psi_0^{(1)}(x)$ (to $\psi_0^{(2)}(x)$), to supplement a function which is proportional to $\psi_0^{(2)}(x)$ (proportional to $\psi_0^{(1)}(x)$) at every other stage of construction.*

While starting the construction of the asymptotic expansions with (9.1) and following the method of matching of the asymptotic expansions (repeating the algorithm described in section 5), we sequentially obtain first the functions $v_{p,0}^{(s)}$ and the dominant terms (at an increasing rate of negative degrees μ) of the internal expansions:

$$v_{0,0}^{(1)} \equiv \psi_0^{(1)}(0), \qquad v_{q,0}^{(1)}(\xi) = P_q^{(1)}(\xi) + \alpha_{1,0}^{(1)}P_q^{(2)}(\xi), \quad q \geqslant 1,$$

$$v_{1,0}^{(2)}(\xi) = \sum_{m=1}^{n}\frac{\partial\psi_0^{(2)}}{\partial x_m}(0)\xi_m + \alpha_{1,0}^{(2)}\psi_0^{(1)}(0), \quad v_{k,0}^{(2)}(\xi) = P_k^{(2)}(\xi) + \alpha_{1,0}^{(2)}P_k^{(1)}(\xi), \quad k \geqslant 2,$$

$$v_{2j+2,j+1}^{(1)}(\xi) = \psi_0^{(1)}(0)z_0^{(j+1)}(\xi), \qquad j \geqslant 0, \tag{9.2}$$

$$\begin{aligned} v_{2j+3,j+1}^{(2)}(\xi) = &\sum_{m=1}^{n}\frac{\partial\psi_0^{(2)}}{\partial x_m}(0)z_m^{(j+1)}(\xi) \\ &+ \psi_0^{(1)}(0)\left(\sum_{k=0}^{j}\alpha_{2k+1,k}^{(2)}z_0^{(j+1-k)}(\xi) + \alpha_{2j+3,j+1}^{(2)}\right), \quad j \geqslant 0; \end{aligned} \tag{9.3}$$

then of the internal asymptotic expansions:

$$
\begin{aligned}
\psi^{in,1}_{odd}(\xi,\mu,\varepsilon) &= \sum_{i=0}^{\infty}\varepsilon^i v^{(1)}_{i,0}(\xi) + \varepsilon^2\mu^{-1}\sum_{j=0}^{\infty}\sum_{i=2j}^{\infty}\varepsilon^i\mu^{-j}v^{(1)}_{2+i,j+1}(\xi),\\
\psi^{in,2}_{odd}(\xi,\mu,\varepsilon) &= \sum_{i=1}^{\infty}\varepsilon^i v^{(2)}_{i,0}(\xi) + \varepsilon^3\mu^{-1}\sum_{j=0}^{\infty}\sum_{i=2j}^{\infty}\varepsilon^i\mu^{-j}v^{(2)}_{3+i,j+1}(\xi)
\end{aligned}
\tag{9.4}
$$

(the analogue (5.15), (5.20)); then the supposed structures of the external asymptotic expansions:

$$
\begin{aligned}
\psi^{ex,1}_{odd}(x,\mu,\psi^{(1)}_0(x) &+ \varepsilon^n\mu^{-1}\sum_{j=0}^{\infty}\sum_{i=2j}^{\infty}\varepsilon^i\mu^{-j}\psi^{(1)}_{n+i,j+1}(x)\\
&+ \varepsilon\psi^{(2)}_0(x)\sum_{j=0}^{\infty}\sum_{i=2j}^{\infty}\alpha^{(1)}_{i+1,j}\varepsilon^i\mu^{-j},\\
\psi^{ex,2}_{odd}(x,\mu,\varepsilon) = \psi^{(2)}_0(x) &+ \varepsilon^{n+1}\mu^{-1}\sum_{j=0}^{\infty}\sum_{i=2j}^{\infty}\varepsilon^i\mu^{-j}\psi^{(2)}_{n+i+1,j+1}(x)\\
&+ \varepsilon\psi^{(1)}_0(x)\sum_{j=0}^{\infty}\sum_{i=2j}^{\infty}\alpha^{(2)}_{i+1,j}\varepsilon^i\mu^{-j},
\end{aligned}
\tag{9.5}
$$

(the analogue (6.5)) and the expected structures (2.10), (2.11) of the asymptotic expansions of the characteristic constants (the analogue (2.1), (2.6)).

Substituting the series (2.10), (2.11), (9.5) into the equation (7.1) we obtain a fortiori holding equations

$$
H_0\psi^{(s)}_0 = \lambda_0\psi^{(s)}_0 \quad \text{in } \Omega
$$

and recurrent systems of the equations in $\Omega\backslash\{0\}$ for the remaining coefficients of the external expansions (9.5):

$$
\begin{aligned}
(H_0-\lambda_0)\,\psi^{(s)}_{n+2j,j+1} =& \lambda^{(s)}_{n+2j,j+1}\psi^{(s)}_0, \qquad j\geqslant 0\\
(H_0-\lambda_0)\,\psi^{(s)}_{n+i,1} =& \lambda^{(s)}_{n+i,1}\psi^{(s)}_0 + \psi^{(s^*)}_0\sum_{p=1}^{i}\alpha^{(s)}_{p,0}\lambda^{(s)}_{n+i-p,1} \quad i\geqslant 1,\\
(H_0-\lambda_0)\,\psi^{(s)}_{n+i+2j,j+1} =& \lambda^{(s)}_{n+i+2j,j+1}\psi^{(s)}_0\\
&+ \psi^{(s^*)}_0\sum_{p=1}^{i}\sum_{q=0}^{j}\alpha^{(s)}_{2q+p,q}\lambda^{(s)}_{n+i-p+2(j-q),j-q+1},\\
&1\leqslant i\leqslant n-3, \quad j\geqslant 1,\\
(H_0-\lambda_0)\,\psi^{(s)}_{n+i+2j,j+1} =& \lambda^{(s)}_{n+i+2j,j+1}\psi^{(s)}_0\\
&+ \sum_{k=0}^{i-n+2}\sum_{q=0}^{j-1}\lambda^{(s)}_{n+k+2q,q+1}\psi^{(s)}_{i-k+2(j-q),j-q}\\
&+ \psi^{(s^*)}_0\sum_{p=1}^{i}\sum_{q=0}^{j}\alpha^{(s)}_{2q+p,q}\lambda^{(s)}_{n+i-p+2(j-q),j-q+1},\\
&i\geqslant n-2, \quad j\geqslant 1,
\end{aligned}
\tag{9.6}
$$

(the analogue (7.2)), where

$$
\psi^{(2)}_{n+2j,j+1}(x) = \lambda^{(2)}_{n+2j,j+1} = \lambda^{(2)}_{n+1+2j,j+1} = 0 \tag{9.7}
$$

(the analogue (7.3), (8.13)).

Substituting the series (2.10), (2.11), (9.4) into the equation (7.1) we obtain for the coefficients of the internal expansions (9.4) the equations (7.8), in which the coefficients $v_{p,q}$, $\lambda_{k,l}$ are substituted for $v^{(s)}_{p,q}$, $\lambda^{(s)}_{k,l}$, and the equality (7.9) is substituted for the following:

$$v^{(2)}_{0,0}(\xi) = v^{(2)}_{2j+2,j+1}(\xi) = \lambda^{(2)}_{n+2j+2,j+1} = \lambda^{(2)}_{n+1+2j+2,j+1} = 0. \tag{9.8}$$

Therefore below for the coefficients of the internal expansion we refer to the equations (7.8), implying, that the indexes mentioned above are supplemented into them.

By analogy with the previous section the coefficients of the asymptotic expansions of the characteristic constants and external expansions of the eigenfunctions we construct in the following form

$$\lambda^{(s)}_{n+i+2j,j+1} = \sum_{t=0}^{i} \Lambda^{(t,s)}_{n+i+2j,j+1}, \quad j,i \geqslant 0, \tag{9.9}$$

$$\psi^{(s)}_{n+i+2j,j+1}(x) = \sum_{t=0}^{i} \Psi^{(t,s)}_{n+i+2j,j+1}(x), \quad j,i \geqslant 0, \tag{9.10}$$

$$\alpha^{(s)}_{2j+i,j} = \sum_{t=0}^{i} \alpha^{(t,s)}_{2j+t,j}, \quad j,i \geqslant 0, \tag{9.11}$$

and denote by $\Phi^{ex,s}_{odd,N}(x,\mu,\varepsilon)$ the series of the form (9.5), where $\psi^{(s)}_{n+i+2j,j+1}(x)$ and $\alpha^{(s)}_{2j+i,j}$ are substituted for

$$\Phi^{(N,s)}_{n+i+2j,j+1}(x) = \sum_{t=0}^{\min\{i,N\}} \Psi^{(t,s)}_{n+i+2j,j+1}(x), \quad j \geqslant 0,$$

$$\Theta^{(N,s)}_{2j+i,j+1} = \sum_{t=0}^{\min\{i,N\}} \alpha^{(t,s)}_{2j+i,j+1}, \quad j \geqslant 0,$$

correspondingly.

For the further matching of the series $\psi^{ex,s}_{odd}(x,\mu,\varepsilon)$ and $\psi^{in,s}_{odd}(x,\mu,\varepsilon)$ from (9.5) and (9.4) we need the following analogue of Lemma 8.1, which validity also results from the definition $\mathcal{A}^m$, $\mathcal{A}_m$, $\widetilde{\mathcal{B}}_m$, $\mathcal{K}_{m,l}$, $\mathcal{K}$, $\Phi^{ex,s}_{odd,N}(x,\mu,\varepsilon)$ and (9.9), (9.10), (9.11).

Lemma 9.1. *If the coefficients $\psi^{(s)}_{n+i+2j,j+1}(x)$ of the series (9.5) belong to $\mathcal{A}^i$, then*

$$\mathcal{K}(\psi^{ex,s}_{odd}(x,\mu,\varepsilon)) = \Psi^{in,s}_{odd}(\xi,\mu,\varepsilon),$$

where $\Psi^{in,s}_{odd}(\xi,\mu,\varepsilon)$ are the series of the form (9.4), in which the coefficients $v^{(s)}_{2+i,j+1}(\xi)$ are substituted for the series $V^{(s)}_{2+i,j+1}(\xi) \in \widetilde{\mathcal{B}}_{i-2j}$.

If $\Psi^{(t,s)}_{n+i+2j,j+1}(x) \in \mathcal{A}_{i-t}$, then the function $\psi^{(s)}_{n+i+2j,j+1}(x)$, determined by the equality (9.10), belongs to $\mathcal{A}^i$ and the following equalities take place:

$$V^{(s)}_{2j+2+t,j+1}(\xi) = \widetilde{V}^{(s)}_{2j+2+t,j+1}(\xi) + \sum_{k=0}^{\infty} Z^{(2j+2+t,j+1,s)}_k(\xi)\rho^{-n+2-2k},$$

where $\widetilde{V}^{(s)}_{2j+2,j+1}(\xi) \equiv 0$,

$$\widetilde{V}^{(s)}_{2j+2+t,j+1}(\xi) = \varepsilon^{-2j-2-t}\mu^{j+1}\mathcal{K}_{2j+2+t,j+1}\left(\Phi^{ex,s}_{odd,t-1}(x,\mu,\varepsilon)\right) \in \widetilde{\mathcal{B}}_t, \quad t \geqslant 1,$$

(i.e. $\widetilde{V}^{(s)}_{2j+2+t,j+1}$ does not depend on $\Psi^{(m,s)}_{p,q}$ when $m \geqslant t-1$), and $Z^{(2j+2+t,j+1,s)}_k r^{-n+2-2k}$ is the dominant term of the asymptotics $\Psi^{(t,s)}_{n+2j-t+k,j+1}(x)$ in the zero.

If, meanwhile, the functions $\Psi^{(t,s)}_{n+i+2j,j+1}(x)$ *are in* $\Omega\setminus\{0\}$ *the solutions of the equations*

$$\begin{aligned}
(H_0-\lambda_0)\,\Psi^{(t,s)}_{n+2j,j+1} &=\Lambda^{(t,s)}_{n+2j,j+1}\psi^{(s)}_0, \qquad j\geqslant 0\\
(H_0-\lambda_0)\,\Psi^{(t,s)}_{n+i,1} &=\Lambda^{(t,s)}_{n+i,1}\psi^{(s)}_0+\psi^{(s^*)}_0\sum_{p=1}^{i}\alpha^{(s)}_{p,0}\Lambda^{(t,s)}_{n+i-p,1} \quad i\geqslant 1,\\
(H_0-\lambda_0)\,\Psi^{(t,s)}_{n+i,1} &=\Lambda^{(t,s)}_{n+i,1}\psi_0, \quad i\geqslant 0,\\
(H_0-\lambda_0)\,\Psi^{(t,s)}_{n+i+2j,j+1} &=\Lambda^{(t,s)}_{n+i+2j,j+1}\psi_0\\
&\quad+\sum_{k=0}^{i-n+2}\sum_{q=0}^{j-1}\sum_{p=0}^{t}\Lambda^{(p,s)}_{n+k+2q,q+1}\Psi^{(t-p,s)}_{i-k+2(j-q),j-q}\\
&\quad+\psi^{(s^*)}_0\sum_{p=1}^{i}\sum_{q=0}^{j}\alpha^{(s)}_{2q+p,q}\Lambda^{(t,s)}_{n+i-p+2(j-q),j-q+1},\\
&\qquad i\geqslant 1 \quad j\geqslant 1,
\end{aligned} \tag{9.12}$$

or the equations

$$\begin{aligned}
(H_0-\lambda_0)\,\Psi^{(t,s)}_{n+2j,j+1} &=\Lambda^{(t,s)}_{n+2j,j+1}\psi^{(s)}_0, \qquad j\geqslant 0\\
(H_0-\lambda_0)\,\Psi^{(t,s)}_{n+i,1} &=\Lambda^{(t,s)}_{n+i,1}\psi^{(s)}_0+\psi^{(s^*)}_0\sum_{p=1}^{i}\sum_{l=0}^{t}\alpha^{(l,s)}_{p,0}\Lambda^{(t-l,s)}_{n+i-p,1} \quad i\geqslant 1,\\
(H_0-\lambda_0)\,\Psi^{(t,s)}_{n+i,1} &=\Lambda^{(t,s)}_{n+i,1}\psi_0, \quad i\geqslant 0,\\
(H_0-\lambda_0)\,\Psi^{(t,s)}_{n+i+2j,j+1} &=\Lambda^{(t,s)}_{n+i+2j,j+1}\psi_0\\
&\quad+\sum_{k=0}^{i-n+2}\sum_{q=0}^{j-1}\sum_{p=0}^{t}\Lambda^{(p,s)}_{n+k+2q,q+1}\Psi^{(t-p,s)}_{i-k+2(j-q),j-q}\\
&\quad+\psi^{(s^*)}_0\sum_{p=1}^{i}\sum_{q=0}^{j}\sum_{l=0}^{t}\alpha^{(l,s)}_{2q+p,q}\Lambda^{(t-l,s)}_{n+i-p+2(j-q),j-q+1},\\
&\qquad i\geqslant 1 \quad j\geqslant 1,
\end{aligned} \tag{9.13}$$

then the functions $\psi^{(s)}_{n+i+2j,j+1}(x)$*, determined by the equalities (9.10) are the solutions of the equations (9.6), (9.7), when* $\lambda^{(s)}_{n+i+2j,j+1}$*, determined by the equalities (9.9), and the series* $\widetilde{V}^{(s)}_{2j+2+t,j+1}$ *are formal asymptotic solutions of the equations (7.8) when* $\rho\to\infty$*, where in the right side* $v_{p,q}$ *and* $\lambda_{p,q}$ *are substituted for* $V^{(s)}_{p,q}$ *and* $\lambda^{(s)}_{p,q}$ *when* $q>0$*.*

First we consider the matching of the series $\psi^{ex,1}_{odd}(x,\mu,\varepsilon)$ and $\psi^{in,1}_{odd}(\xi,\mu,\varepsilon)$. In this case we search for the equations (9.13). Following the algorithm of the proof of Theorem 8.1, we see, that the functions $v^{(1)}_{2j+2,j+1}(\xi)$, $j\geqslant 0$, determined by the equalities (9.2), belong to $\mathcal{B}_0$, are the solutions of the equations (7.8) (in the first line), and on the strength of Lemmas 4.6, 4.7 possess with $\rho\to\infty$ the following asymptotics

$$V^{(1)}_{2j+2,j+1}(\xi)=\psi^{(1)}_0(0)c^{(j+1)}_{0,0}\rho^{2-n}+\sum_{k=1}^{\infty}Y_k(\xi)\rho^{-2k-n+2}, \quad j\geqslant 0.$$

Whence due to Lemma 9.1 we obtain the dominant terms of the asymptotics in the zero for the functions $\Psi^{(0,1)}_{n+2j+k,j+1}(x)$:

$$\begin{aligned} \Psi^{(0,1)}_{n+2j,j+1}(x) \sim& \psi^{(1)}_0(0)c^{(j+1)}_{0,0}r^{2-n}, \\ \Psi^{(0,1)}_{n+2j+k,j+1}(x) \sim& Y_k(x)r^{-2k-n+2}, \qquad k \geqslant 1. \end{aligned} \tag{9.14}$$

The functions

$$\Psi^{(0,1)}_{n+2j,j+1}(x) = \psi^{(1)}_0(0)c^{(j+1)}_{0,0}E_0(x) \tag{9.15}$$

possess the required asymptotics in the zero and due to Lemma 4.4 they satisfy the corresponding equations

$$(H_0 - \lambda_0)\Psi^{(0,1)}_{n+2j,j+1} = \Lambda^{(0,1)}_{n+2j,j+1}\psi^{(1)}_0, \quad j \geqslant 0$$

from (9.13) when

$$\Lambda^{(0,1)}_{n+2j,j+1} = \psi^{(1)}_0(0)c^{(j+1)}_{0,0}\Lambda_0. \tag{9.16}$$

Remark 9.2 (derivation of the formula (2.12))**.** *On the strength of (9.9), (9.10), (9.16) and (9.15), in particular, we obtain, that*

$$\lambda^{(1)}_{n+2j,j+1} = \psi^{(1)}_0(0)c^{(j+1)}_{0,0}\Lambda_0, \quad \psi_{n+2j,j+1}(x) = \psi^{(1)}_0(0)c^{(j+1)}_{0,0}E_0(x) \in \mathcal{A}^0.$$

Substituting into these equalities the values Λ_0, $c^{(1)}_{0,0}$ from Lemmas 4.4, 4.6, we obtain the equality (2.12) for $\lambda^{(1)}_{n,1}$.

When $k \geqslant 1$ the equations (9.13) for $\Psi^{(0,1)}_{n+2j+k,j+1}(x)$ take the form

$$\begin{aligned} (H_0 - \lambda_0)\Psi^{(0,1)}_{n+2j+k,j+1} =& \Lambda^{(0,1)}_{n+2j+k,j+1}\psi^{(1)}_0 \\ &+ \psi^{(2)}_0 \sum_{m=1}^{k}\sum_{q=0}^{j-1} \alpha^{(0,1)}_{2q+m,q}\Lambda^{(0,1)}_{n+2(j-q)+k-m,(j-q)+1}. \end{aligned}$$

On the strength of Lemma 4.2 from the condition of solubility of these equations with the given in (9.14) features in the zero solutions, firstly, we determine $\Lambda^{(0,1)}_{n+2j+k,j+1}$ and $\Psi^{(0,1)}_{n+2j+k,j+1}(x) \in \mathcal{A}_k$, and secondly, considering, that $\Lambda^{(0,1)}_{n,1} = \lambda^{(1)}_{n,1} \neq 0$ on the strength of (2.12) and the condition $\langle V\rangle \neq 0$, we obtain $\alpha^{(0,1)}_{2j+k,j+1}$. Let us note, that $\alpha^{(1)}_{2j+1,j} = \alpha^{(0,1)}_{2j+1,j}$ due to (9.11).

At the next stage we similarly determine $\Psi^{(1,1)}_{n+2j+1+k,j+1}$, $\Lambda^{(1,1)}_{n+2j+1+k,j+1}$ and $\alpha^{(1,1)}_{2j+k+2,j}$ when $k \geqslant 0$, and consequently due to (9.9), (9.10) and (9.11) we finally obtain $\psi^{(1)}_{n+2j+1,j+1}$, $\lambda^{(1)}_{n+2j+1,j+1}$ and $\alpha^{(1)}_{2j+2,j}$. And so on.

As a result we obtain the validity of the following analogue of Theorem 8.1 and its corollary 3.

Theorem 9.1. *Let n be odd, $\langle V\rangle \neq 0$, λ_0 be the twofold characteristic constant of the operator $\mathcal{H}_0$, $\psi^{(1)}_0$ and $\psi^{(2)}_0$ be the corresponding orthonormalized in $L_2(\Omega)$ eigenfunctions, satisfying the condition (2.8) and chosen in compliance with (2.9).*

Then there exist the series $\psi^{ex,1}_{odd}(x,\mu,\varepsilon)$ of the form (9.5), the series $\psi^{in,1}_{odd}(\xi,\mu,\varepsilon)$ of the form (9.4) and the series (2.10) such that:

1) the equality (2.12) holds;

2) $\psi^{(1)}_{n+2j+i,j+1} \in \mathcal{A}^i$, $v^{(1)}_{2j+2+i,j+1} \in \mathcal{B}_i$;

3) for their partial sums the statements of the corollary 3 hold.

Let us proceed to the matching of the series $\psi^{ex,2}_{odd}(x,\mu,\varepsilon)$ and $\psi^{in,2}_{odd}(\xi,\mu,\varepsilon)$. In this case it is sufficient to apply the equations (9.12). Following the described above algorithm we see, that the functions $v^{(2)}_{2j+3,j+1}(\xi)$, $j \geqslant 0$, determined by the equalities (9.3), belong to $\mathcal{B}_0 \subset \mathcal{B}_1$, are the solutions of the equations (7.8) (subject to the equalities (9.8)) and on the strength of Lemmas 4.6, 4.7 possess when $\rho \to \infty$ the following asymptotics

$$V^{(2)}_{2j+3,j+1}(\xi) = \left(\sum_{m=1}^{n} \frac{\partial \psi_0^{(2)}}{\partial x_m}(0) c^{(j+1)}_{m,0} + \alpha^{(2)}_{2j+1,j} \psi_0^{(1)}(0) c^{(j+1)}_{0,0} \right) \rho^{2-n} + \sum_{i=1}^{n} \left(\sum_{m=1}^{n} \frac{\partial \psi_0^{(2)}}{\partial x_i}(0) c^{(j+1)}_{m,i} + \alpha^{(2)}_{2j+1,j} \psi_0^{(1)}(0) c^{(j+1)}_{0,i} \right) \xi_i \rho^{-n} + \sum_{k=2}^{\infty} \left(Y_k^{(j+1,0,2)}(\xi) + \alpha^{(2)}_{2j+1,j} Y_k^{(j+1,0,1)}(\xi) \right) \rho^{-2k-n+2}.$$

Whence, firstly, on the strength of (9.9), (9.10) sequentially results, that

$$\begin{aligned} \Psi^{(0,2)}_{n+2j+k,j+1}(x) =& \Lambda^{(0,2)}_{n+2j+k,j+1} = 0, \quad k \geqslant 0, \\ \psi^{(2)}_{n+2j,j+1}(x) =& \lambda^{(2)}_{n+2j,j+1} = 0, \end{aligned} \tag{9.17}$$

and secondly, on the strength of Lemma 9.1 we obtain, that $\Psi^{(1,2)}_{n+2j+k+1,j+1}(x)$:

$$\Psi^{(1,2)}_{n+2j+1,j+1}(x) \sim \left(\sum_{m=1}^{n} \frac{\partial \psi_0^{(2)}}{\partial x_m}(0) c^{(j+1)}_{m,0} + \alpha^{(2)}_{2j+1,j} \psi_0^{(1)}(0) c^{(j+1)}_{0,0} \right) \rho^{2-n}, \tag{9.18}$$

$$\Psi^{(1,2)}_{n+2j+2,j+1}(x) \sim \sum_{i=1}^{n} \left(\sum_{m=1}^{n} \frac{\partial \psi_0^{(2)}}{\partial x_m}(0) c^{(j+1)}_{m,i} + \alpha^{(2)}_{2j+1,j} \psi_0^{(1)}(0) c^{(j+1)}_{0,i} \right) \xi_i \rho^{-n}, \tag{9.19}$$

$$\Psi^{(1,2)}_{n+2j+k+1,j+1}(x) \sim \left(Y_k^{(j+1,0,2)}(\xi) + \alpha^{(2)}_{2j+1,j} Y_k^{(j+1,0,1)}(\xi) \right) \rho^{-2k-n+2}, \qquad k \geqslant 2, \tag{9.20}$$

when $r \to 0$. The equations (9.12) for these functions subject to the equalities (9.17) take the following form:

$$(H_0 - \lambda_0) \Psi^{(1,2)}_{n+2j+1,j+1} = \Lambda^{(1,2)}_{n+2j+1,j+1} \psi_0^{(2)}, \tag{9.21}$$

$$(H_0 - \lambda_0) \Psi^{(1,2)}_{n+2j+2,j+1} = \Lambda^{(1,2)}_{n+2j+2,j+1} \psi_0^{(2)}, \tag{9.22}$$

$$(H_0 - \lambda_0) \Psi^{(1,2)}_{n+2j+k+1,j+1} = \Lambda^{(1,2)}_{n+2j+k+1,j+1} \psi_0^{(2)}, \quad k \geqslant 2. \tag{9.23}$$

On the strength of Lemma 4.4 the equations (9.21) possess the solutions with the asymptotics (9.18) in the zero only if the multiplier in (9.18) is equal to zero, i.e.

$$\alpha^{(2)}_{2j+1,j} = -\frac{1}{\psi_0^{(1)}(0) c^{(j+1)}_{0,0}} \sum_{m=1}^{n} \frac{\partial \psi_0^{(2)}}{\partial x_m}(0) c^{(j+1)}_{m,0}, \tag{9.24}$$

that in its turn results in the equalities

$$\Lambda^{(1,2)}_{n+2j+1,j+1} = \Psi^{(1,2)}_{n+2j+1,j+1} = 0. \tag{9.25}$$

Remark 9.3 (on the structure of the external expansion)**.** *The equalities result from (9.25), (9.17) and (9.9), (9.10):*

$$\psi^{(2)}_{n+2j,j+1}(x) = \psi^{(2)}_{n+2j+1,j+1}(x) = \lambda^{(2)}_{n+2j,j+1} = \lambda^{(2)}_{n+1+2j,j+1} = 0,$$

which are more detailed than the equalities (9.7).

Similarly on the strength of Lemma 4.4 the functions

$$\Psi^{(1,2)}_{n+2j+2,j+1}(x) = \sum_{i=1}^{n}\Bigg(\sum_{m=1}^{n}\frac{\partial\psi_0^{(2)}}{\partial x_m}(0)c^{(j+1)}_{m,i} + \alpha^{(2)}_{2j+1,j}\psi_0^{(1)}(0)c^{(j+1)}_{0,i}\Bigg)E_i(x)$$

possess the asymptotics (9.19) and are the solutions of the equations (9.22) when

$$\Lambda^{(1,2)}_{n+2j+2,j+1} = \sum_{i=1}^{n}\left(\sum_{m=1}^{n}\frac{\partial\psi_0^{(2)}}{\partial x_m}(0)c^{(j+1)}_{m,i} + \alpha^{(2)}_{2j+1,j}\psi_0^{(1)}(0)c^{(j+1)}_{0,i}\right)\Lambda_i^{(2)}. \tag{9.26}$$

And, finally, on the strength of the corollary 2 there exist functions $\Psi^{(1,2)}_{n+2j+k+1,j+1} \in \mathcal{A}^k$, possessing the asymptotics (9.20) and being the solutions of the equations (9.23) with some $\Lambda^{(1,2)}_{n+2j+k+1,j+1}$.

At the next stage similarly from the condition of solubility of the equations (9.12) for $\Psi^{(2,2)}_{n+2j+2,j+1}(x)$ we obtain $\alpha^{(2)}_{2j+2,j}$ and obtain, that

$$\Lambda^{(2,2)}_{n+2j+2,j+1} = \Psi^{(2,2)}_{n+2j+2,j+1} = 0. \tag{9.27}$$

Further, on the strength of the corollary 2 there are the functions $\Psi^{(2,2)}_{n+2j+k+2,j+1} \in \mathcal{A}^k$, $k \geqslant 1$, possessing the asymptotics, requiring the asymptotics and being the solutions of the equations (9.12) with some $\Lambda^{(2,2)}_{n+2j+k+2,j+1}$. And so on.

Remark 9.4 (derivation of the formula (2.13)). *Since $\Lambda^{(2)}_{n+2,1} = \Lambda^{(1,2)}_{n+2,1}$ on the strength of (9.9) and (9.17), (9.27), then, substituting into (9.26) the values $\alpha^{(2)}_{1,0}$ from (9.24)and Λ_k, $c^{(1)}_{m,k}$ from Lemmas 4.4, 4.6, we derive the equality (2.13).*

As a result we obtain the validity of the following analogue of Theorem 8.1 and its corollary 3.

Theorem 9.2. *Let the conditions of Theorem 9.1 hold.*

Then there exist the series $\psi^{ex,2}_{odd}(x,\mu,\varepsilon)$ of the form (9.5), the series $\psi^{in,2}_{odd}(\xi,\mu,\varepsilon)$ of the form (9.4) and the series (2.11) such that:

1) the equality (2.13) holds;

2) $\psi^{(2)}_{n+2j+i,j+1} \in \mathcal{A}^i$, $v^{(2)}_{2j+2+i,j+1} \in \mathcal{B}_i$;

3) for their partial sums the statements of the corollary 3 hold.

10. Construction of the complete formal asymptotic expansions in case of the even-dimensional domains

For the case of the even domains the asymptotic expansions are more bulky and contain the degrees $\ln\varepsilon$. It is connected with the fact, that the asymptotics in the zero of the coefficients of the external expansion contain logarithmic terms, which during rewriting in the internal variables generate the summands, containing $\ln\varepsilon$. therefore in the internal and the external expansions of the eigenfunctions and expansions of the characteristic constant sequentially occur summands of the form $\varepsilon^i\mu^{-j}\ln\varepsilon\, v_{i,j,1}(\xi)$, $\varepsilon^i\mu^{-j}\ln\varepsilon\,\psi_{i,j,1}(x)$ and $\varepsilon^i\mu^{-j}\ln\varepsilon\,\lambda_{i,j,1}$. In its turn rewriting the asymptotics in the zero of the coefficients of the external expansion $\psi_{i,j,1}(x)$ in the internal variables sequentially generates the summands, containing $\ln^2\varepsilon$, in the internal, external expansions of the eigenfunctions and in expansion of the characteristic constant. Applying the used in the previous sections algorithm of matching of the asymptotic expansions, it is easy to follow, that for the even n in case of the simple characteristic constant λ_0 the chain of the origin of the first terms containing increasing degrees $\ln\varepsilon$, looks as follows:

$$\begin{aligned}
&v_{0,0}=\psi_0(0),\quad v_{k,0}=P_k,\quad k\geqslant 1\\
\Rightarrow\ &\varepsilon^{2+2j}\mu^{-j-1}:\ v_{2+2j,j+1}=\psi_0(0)z_0^{(j+1)}j\geqslant 0\\
\Rightarrow\ &\varepsilon^{n+2j}\mu^{-j-1}:\ \psi_{n+2j,j+1}=\psi_0(0)c_{0,0}^{(j+1)}E_0;\quad \lambda_{n+2j,j+1}=\psi_0(0)c_{0,0}^{(j+1)}\Lambda_0\\
\Rightarrow\ &\varepsilon^{n+2j}\mu^{-j-1}\ln\varepsilon:\ v_{n+2j,j+1,1}=\psi_0(0)A_j^{(1)},\quad v_{n+2j+k,j+1,1}(\xi)=\psi_0(0)R_k^{(1)}(\xi)\\
\Rightarrow\ &\dots\\
\Rightarrow\ &\varepsilon^{qn+2j}\mu^{-j-q}\ln^q\varepsilon:\ v_{qn+2j,j+q,q}=\psi_0(0)A_j^{(q)},\\
&v_{qn+2j+k,j+q,q}=\psi_0(0)R_k^{(q)}\\
\Rightarrow\ &\varepsilon^{qn+2j+2}\mu^{-j-q-1}\ln^q\varepsilon:\ v_{qn+2j+2,j+q+1,q}=\psi_0(0)A_j^{(q)}z_0^{(j+1)}\\
\Rightarrow\ &\varepsilon^{3n+2j}\mu^{-j-3}\ln^q\varepsilon:\ \psi_{(q+1)n+2j,j+q+1,q}=\psi_0(0)c_{0,0}^{(j+1)}A_j^{(q)}E_0;\\
&\qquad\qquad\qquad\qquad \lambda_{(q+1)n+j,j+q+1,q}=\psi_0(0)c_{0,0}^{(j+1)}A_j^{(q)}\Lambda_0\\
\Rightarrow\ &\varepsilon^{(q+1)n+2j}\mu^{-j-q-1}\ln^{q+1}\varepsilon:\ v_{(q+1)n+2j,j+q+1,q+1}=\psi_0(0)A_j^{(q+1)},\\
&v_{(q+1)n+2j+k,j+q+1,q+1}=\psi_0(0)R_k^{(q+1)}\ \Rightarrow\dots
\end{aligned}$$

It results from this chain and the given in the previous section matching of asymptotic expansions, that if $\psi_0(0)=0$, then the internal expansion possesses the form

$$\psi_{even}^{in,s}(\xi,\mu,\varepsilon)=\sum_{q=0}^{\infty}\mu^{-q}\varepsilon^{qn}\ln^q\varepsilon\,\psi_q^{in,s}(\xi,\mu,\varepsilon),\tag{10.1}$$

where $s=1$, the series $\psi_0^{in,1}(\xi,\mu,\varepsilon)$ coincides with the series $\psi_{odd}^{in,1}(\xi,\mu,\varepsilon)$ from (5.15), and the series $\psi_l^{in,1}(\xi,\mu,\varepsilon)$ when $l\geqslant 1$ possess the same structure, the external expansions takes the form

$$\psi_{even}^{ex,s}(\xi,\mu,\varepsilon)=\sum_{q=0}^{\infty}\mu^{-q}\varepsilon^{qn}\ln^q\varepsilon\,\psi_q^{ex,s}(x,\mu,\varepsilon),\tag{10.2}$$

where $s=1$, the series $\psi_0^{ex,1}(x,\mu,\varepsilon)$ coincides with the series $\psi_{odd}^{ex,1}(x,\mu,\varepsilon)$ from (6.5), and the series $\psi_l^{ex,1}(\xi,\mu,\varepsilon)+\psi_0(x)$ when $l\geqslant 1$ possess the same structure, and the asymptotic expansion of the characteristic constant takes the form

$$\lambda_{even}^{s}(\mu,\varepsilon)=\sum_{q=0}^{\infty}\mu^{-q}\varepsilon^{qn}\ln^q\varepsilon\,\lambda_q^{s}(\mu,\varepsilon),\tag{10.3}$$

where $s=1$, the series $\lambda_0^1(\mu,\varepsilon)$ coincides with the series $\lambda_{odd}^1(\mu,\varepsilon)$ from (6.7), and the series $\lambda_l^1(\mu,\varepsilon)+\lambda_0$ when $l\geqslant 1$ possess the same structure. Consequently, the series $\lambda_{even}^1(\mu,\varepsilon)$ takes the form (2.1).

If $\psi_0(0)=0$, then for the even $n\geqslant 4$ in case of the simple characteristic constant λ_0 the chain of occurrence of the first terms, containing increasing degrees $\ln\varepsilon$, possesses the following form:

$$v_{0,0} = \psi_0(0) = 0 \quad v_{1,0}(\xi) = \sum_{m=1}^{n} \frac{\partial \psi_0}{\partial x_m}(0)\xi_m, \quad v_{k,0} = P_k, \quad k \geqslant 2$$

$$\Rightarrow v_{3+2j,j+1} = \sum_{m=1}^{n} \frac{\partial \psi_0}{\partial x_m}(0) z_m^{(j+1)}, \quad j \geqslant 0$$

$$\Rightarrow \psi_{n+2j+1,j+1} = \sum_{m=1}^{n} \frac{\partial \psi_0}{\partial x_m}(0) c_{m,0}^{(j+1)} E_0;$$

$$\psi_{n+2j+2,j+1} = \sum_{m=1}^{n}\sum_{i=1}^{n} \frac{\partial \psi_0}{\partial x_m}(0) c_{m,i}^{(j+1)} E_i + B_j^{(1)} E_0;$$

$$\lambda_{n+2j+2,j+1} = \sum_{m=1}^{n}\sum_{i=1}^{n} \frac{\partial \psi_0}{\partial x_m}(0) c_{m,i}^{(j+1)} \Lambda_i$$

$$\Rightarrow v_{n+2j+1,j+1,1} = A_j^{(1)}, \quad v_{n+2j+1+l,j+1,1}(\xi) = R_l^{(1)}(\xi), \quad l \geqslant 1$$

$$\Rightarrow \dots$$

$$\Rightarrow v_{qn+2j+1,j+q,q} = A_j^{(q)}, \quad v_{qn+2j+1+l,j+q,q} = R_l^{(q)}$$

$$\Rightarrow v_{qn+2j+3,j+q+1,q} = A_j^{(q)} z_0^{(j+1)}$$

$$\Rightarrow \psi_{(q+1)n+2j+1,j+q+1,q} = A_j^{(q)} c_{0,0}^{(j+1)} E_0;$$

$$\psi_{(q+1)n+2j+2,j+q+1,q} = A_j^{(q)} \sum_{i=1}^{n} c_{0,i}^{(j+1)} E_i + B_j^{(q+1)} E_0;$$

$$\lambda_{(q+1)n+2j+2,j+q+1,q} = A_j^{(q)} \sum_{i=1}^{n} c_{0,i}^{(j+1)} \Lambda_i$$

$$\Rightarrow v_{2n+2j+1,j+2,2} = A_j^{(q+1)}, \quad v_{(q+1)n+2j+2,j+q+1,q+1} = R_l^{(k+1)} \Rightarrow \dots$$

Remark 10.1 (the case $\psi_0(0) = 0$, $n = 2$)**.** *Since in the considered case*

$$E_0(x) = -\ln r + c(\Omega) + O(r \ln r), \quad r \to 0,$$

on the strength of Lemma 4.5, the for the matching of the dominant terms of the external and the internal asymptotic expansions of the eigenfunction in the presented above chain it is sufficient to choose

$$v_{3+2j,j+1} = \sum_{m=1}^{n} \frac{\partial \psi_0}{\partial x_m}(0) \left(z_m^{(j+1)} + c_{m,0}^{(j+1)} c(\Omega) \right),$$

$$v_{qn+2j+3,j+q+1,q} = A_j^{(q)} \left(z_0^{(j+1)} + c_{0,0}^{(j+1)} c(\Omega) \right), \quad j \geqslant 0,\ q \geqslant 1.$$

It results from this chain and the considered in the previous section matching of the asymptotic expansions, that if $\psi_0(0) = 0$, then the internal expansion possesses the form (10.1), where $s = 2$, the series $\psi_0^{in,2}(\xi,\mu,\varepsilon)$ coincides with the series $\psi_{odd}^{in,2}(\xi,\mu,\varepsilon)$ from (5.20), and the series $\psi_l^{in,2}(\xi,\mu,\varepsilon)$ when $l \geqslant 1$ possess the same structure with precision to the constant summand, the internal expansion possesses the form (10.2), where $s = 2$, the series $\psi_0^{ex,2}(x,\mu,\varepsilon)$ coincides with the series $\psi_{odd}^{ex,2}(x,\mu,\varepsilon)$ from (6.5), and the series $\psi_l^{ex,2}(x,\mu,\varepsilon) + \psi_0(x)$ when $l \geqslant 1$ possess the same structure, and the asymptotic expansion of the characteristic constant possesses the form (10.3), where $s = 2$, the series $\lambda_0^2(\mu,\varepsilon)$ coincides with the series $\lambda_{odd}^2(\mu,\varepsilon)$ from (6.7), and the series $\lambda_l^2(\mu,\varepsilon)+\lambda_0$ when $l\geqslant 1$ possess the same structure. Consequently, the series $\lambda^2_{even}(\mu,\varepsilon)$ takes the form (2.6).

Remark 10.2. *The equations for the coefficients of the asymptotic expansions (10.1), (10.2) of the eigenfunctions are derived the same way as in the previous sections. The series (10.2) and (10.3) are substituted into the equation*

$$H_{\mu,\varepsilon}\psi^{\mu,\varepsilon}=\lambda^{\mu,\varepsilon}\psi^{\mu,\varepsilon}, \tag{10.4}$$

and we rewrite the equalities with similar degrees $\varepsilon, \ln\varepsilon$ and μ. In the result we obtain the equations for the coefficients of the external expansion (10.2). Analogously, by means of substituting the series (10.1) and (10.3) into the equation (10.4), by changing in it to the internal variable ξ and by rewriting the equalities with similar degrees $\varepsilon, \ln\varepsilon$ and μ we obtain the equations for the coefficients of the internal expansion (10.1). If the coefficients of the expansions satisfy the obtained such way equations, we consider, that the series (10.1), (10.2), (10.3) are the asymptotic solutions of the equations (10.4).

By analogy with the indexes, applied in the considered above chains, when $l\geqslant 1$ fort he coefficients of the series $\psi_l^{in,s}(\xi,\mu,\varepsilon)$, $\psi_l^{ex,s}(x,\mu,\varepsilon)$ and $\lambda_l^{(s)}(\mu,\varepsilon)$ when $\varepsilon^i\mu^k$ we apply the notations $v_{i,k,l}$, $\psi_{i,k,l}$ and $\lambda_{i,k,l}$ correspondingly.

Following the procedure of matching of the asymptotic expansions, presented in section 8, it is easy to obtain the validity of the following statement.

Theorem 10.1. *Let λ_0 be a simple characteristic constant of the operator $\mathcal{H}_0$, ψ_0 be the corresponding normalized in $L_2(\Omega)$ eigenfunction. Then with even n there exist the series (10.1), (10.2), (10.3) such that:*

1) they are the asymptotic solutions of the equation (10.4);

2) the series $\lambda^1_{even}(\mu,\varepsilon)$, $\lambda^2_{even}(\mu,\varepsilon)$ coincide with the series (2.1), (2.6), correspondingly, though, for them the equalities (2.2), (2.3), (2.4), (2.5) hold, (the latter subject to the statement of Lemma 4.6 for $n=2$);

3) the series $\psi_0^{ex,s}(x,\mu,\varepsilon)$ coincide with the series $\psi_{odd}^{ex,s}(x,\mu,\varepsilon)$ from (6.5);

4) $\psi_{n+2j+i,j+1}$, $\psi_{n+2j+nl+i,j+1,l}\in\mathcal{A}^i$, $v_{2j+2+i,j+1}$, $v_{2j+ln+2+i,j+1,l}\in\mathcal{B}_i$;

5) for the partial sums of the series (10.1), (10.2), (10.3) the statements of the corollary 3 hold,(with the substitution of the index "odd" for "even" in the formulation).

Let us formulate the analogue of this theorem for the case multiple to λ_0. Following the algorithm presented in the previous section 9, it is easy to rewrite the chains of occurrence of the first terms, containing increasing degrees $\ln\varepsilon$, and in case of the twofold characteristic constant λ_0, and to make sure, that the asymptotic expansions possess the form (10.1), (10.2), (10.3), where the series $\psi_0^{in,s}(\xi,\mu,\varepsilon)$ coincide with the series $\psi_{odd}^{in,s}(\xi,\mu,\varepsilon)$ from (9.4), and the series $\psi_l^{in,s}(\xi,\mu,\varepsilon)$ when $l\geqslant 1$ possess the same structure (the latter - with precision to the constant summand), the series $\psi_0^{ex,s}(x,\mu,\varepsilon)$ coincide with the series $\psi_{odd}^{ex,s}(x,\mu,\varepsilon)$from (9.5), and the series $\psi_l^{ex,s}(x,\mu,\varepsilon)+\psi_0^{(s)}(x)$ when $l\geqslant 1$ possess the same structure, and the asymptotic expansions of the characteristic constants possess the form (2.10), (2.11). Analogous to the previous section we prove the validity of the following statement.

Theorem 10.2. *Let λ_0 be a twofold characteristic constant of the operator $\mathcal{H}_0$, $\langle V\rangle\neq 0$, $\psi_0^{(1)}$ and $\psi_0^{(2)}$ be the corresponding orthonormalized in $L_2(\Omega)$ eigenfunctions, satisfying (2.8), chosen in compliance with (2.9). Then with the even n there are the series (10.1), (10.2), (10.3) such that:*

1) they are the asymptotic solutions of the equation (10.4);

2)the series $\lambda^1_{even}(\mu,\varepsilon)$, $\lambda^2_{even}(\mu,\varepsilon)$ coincide with the series (2.10) and (2.11) correspondingly, though, for them the equalities (2.12)and (2.13) hold;

3) the series $\psi_0^{ex,s}(x,\mu,\varepsilon)$ *coincide with the series* $\psi_{odd}^{ex,s}(x,\mu,\varepsilon)$ *from (9.5);*
4) $\psi^{(s)}_{n+2j+i,j+1}$, $\psi^{(s)}_{n+2j+nl+i,j+1,l}\in\mathcal{A}^i$, $v^{(s)}_{2j+2+i,j+1}$, $v^{(s)}_{2j+ln+2+i,j+1,l}\in\mathcal{B}_i$;
5) for the partial sums of the series (10.1), (10.2), (10.3) the statements of the corollary 3 hold (with the substitution of the index "odd" for "even" in the formulation).

The construction of the formal asymptotic expansions of the (2.1)–(2.13) characteristic constants, of the corresponding eigenfunctions by method of matching of the asymptotic expansions has been completed. Let us also note, that in the process of the asymptotics construction the condition (1.9) was not applied. It is apparent, that the series (2.1), (2.6), (2.10), (2.11) are asymptotic even under a weaker condition (1.6).

11. Justification of the asymptotic expansions

Everywhere below, firstly, the asymptotic expansions of the eigenfunctions and the characteristic constants are considered chosen in compliance with the statements of Theorems 8.1, 9.1, 9.2, 10.1, 10.2, and secondly, since the further description does not depend on evenness of n, then in the notations of these series and from the partial sums we omit the indexes "odd" and "even". Subject to the statements of the mentioned theorems the justification of the constructed asymptotic expansions is standard (see, for instance, [8]).

Let us denote

$$\widetilde{\psi}_N^{(s)}(x,\mu,\varepsilon):=\left(1-\chi\left(\frac{r}{\sqrt{\varepsilon}}\right)\right)\widehat{\psi}^{ex,s}_{n+2N}(x,\mu,\varepsilon)+\chi\left(\frac{r}{\sqrt{\varepsilon}}\right)\widehat{\psi}^{in,s}_{2(N+1)}\left(\frac{x}{\varepsilon},\mu,\varepsilon\right)$$

where $\chi(t)$ is an infinite differential patching function, which is identically equal to the unit when $t<1$ and to the zero when $t>2$. The validity of the the following Lemma results from the statement of Theorems 8.1, 9.1, 10.1, 10.2.

Lemma 11.1. *For* $\widetilde{\psi}_N^{(s)}$ *the following equalities hold:*

$$\|\widetilde{\psi}_N^{(s)}-\psi_0^{(s)}\|_{L_2(\Omega)}\underset{\varepsilon\to0}{\longrightarrow}0, \tag{11.1}$$

$$\mathcal{H}_{\mu,\varepsilon}\widetilde{\psi}_N^{(s)}=\widehat{\lambda}^s_{n+2N}\widetilde{\psi}_N^{(s)}+f_N^{(s)}, \tag{11.2}$$

though, if the condition (1.9) is satisfied, then

$$\|f_N^{(s)}\|_{L_2(\Omega)}=O\left(\varepsilon^{M(N)}\right),\qquad M(N)\underset{N\to\infty}{\longrightarrow}\infty. \tag{11.3}$$

Let us denote by $\sigma\left(\mathcal{H}_{\mu,\varepsilon}\right)$ the spectre of the operator $\mathcal{H}_{\mu,\varepsilon}$. On the strength of the well-known estimate of the resolvent (see, for instance, [1, Chapter 5, § 3]) we possess

$$\left\|\widetilde{\psi}_N^{(s)}\right\|_{L_2(\Omega)}\leqslant\frac{\|f_N^{(s)}\|_{L_2(\Omega)}}{\operatorname{dist}\left\{\sigma\left(\mathcal{H}_{\mu,\varepsilon}\right),\widehat{\lambda}^s_{n+2N}\right\}}.$$

It results from this estimate and (11.1), (11.3), that

$$\operatorname{dist}\left\{\sigma\left(\mathcal{H}_{\mu,\varepsilon}\right),\widehat{\lambda}^s_{n+2N}\right\}=O\left(\varepsilon^{M(N)}\right),\qquad M(N)\underset{N\to\infty}{\longrightarrow}\infty.$$

This equality on the strength of Theorem 2.1, its corollary 1 and the arbitrary choice of N justifies the asymptotic expansions (2.1)–(2.13) of the characteristic constant and, in particular, completes the proof of Theorems 2.2, 2.3.

Let us also note, that in case of the twofold characteristic constant λ_0

$$\left|\lambda^{\mu,\varepsilon,2}-\lambda^{\mu,\varepsilon,1}\right|\geqslant c\varepsilon^n\mu^{-1},\quad c>0, \tag{11.4}$$

on the strength of (2.10)–(2.12) and the inequality $\langle V\rangle \neq 0$. Consequently, the characteristic constants $\lambda^{\mu,\varepsilon,1}$ and $\lambda^{\mu,\varepsilon,2}$ are simple and for the finale proof of Theorem 2.4 it is left to show, that

$$\|\psi^{\mu,\varepsilon,s}-\psi_0^{(s)}\|_{L_2(\Omega)} \underset{\varepsilon\to 0}{\longrightarrow} 0. \tag{11.5}$$

Let us expand $\widetilde{\psi}_N^{(1)}$ into the direct sum:

$$\widetilde{\psi}_N^{(1)} = b_N(\mu,\varepsilon)\psi^{\mu,\varepsilon,1}+\psi^{\perp}_{\mu,\varepsilon}, \tag{11.6}$$

$$\text{where} \qquad b_N(\mu,\varepsilon)=\left(\widetilde{\psi}_N^{(1)},\psi^{\mu,\varepsilon,1}\right)_{L_2(\Omega)}, \qquad \left(\psi^{\perp}_{\mu,\varepsilon},\psi^{\mu,\varepsilon,1}\right)_{L_2(\Omega)}=0. \tag{11.7}$$

On the strength of (11.2), (11.6) we obtain, that

$$\mathcal{H}_{\mu,\varepsilon}\psi^{\perp}_{\mu,\varepsilon} =\lambda^{\mu,\varepsilon,1}\psi^{\perp}_{\mu,\varepsilon}+\widetilde{f}_N^{(1)}, \tag{11.8}$$

$$\text{where} \qquad \widetilde{f}_N^{(1)}=\left(\widehat{\lambda}^1_{n+2N}-\lambda^{\mu,\varepsilon,1}\right)\left(b_N(\mu,\varepsilon)\psi^{\mu,\varepsilon,1}+\psi^{\perp}_{\mu,\varepsilon}\right)+f_N^{(1)}.$$

It results from the latter equality and from (11.7), (11.1) and (11.3), that

$$\|\widetilde{f}_N^{(1)}\|_{L_2(\Omega)}=O\left(\varepsilon^{M(N)}\right), \qquad M(N)\underset{N\to\infty}{\longrightarrow}\infty. \tag{11.9}$$

Since two simple characteristic constants $\lambda^{\mu,\varepsilon,1}$ and $\lambda^{\mu,\varepsilon,2}$ converge to λ_0, then it results from (11.8) and the second equality in (11.7), that

$$\left\|\psi^{\perp}_{\mu,\varepsilon}\right\|_{L_2(\Omega)} \leqslant \frac{\|\widetilde{f}_N^{(1)}\|_{L_2(\Omega)}}{\left|\lambda^{\mu,\varepsilon,2}-\lambda^{\mu,\varepsilon,1}\right|}.$$

It results from this inequality (11.9) and (11.4), that

$$\left\|\psi^{\perp}_{\mu,\varepsilon}\right\|_{L_2(\Omega)} \underset{\varepsilon\to 0}{\longrightarrow} 0.$$

Whence and from (11.6)and (11.1) we obtain the convergence (11.5) when $s=1$. In its turn it results from this convergence, corollary 1 and the orthonormalization $\psi^{\mu,\varepsilon,1}$ and $\psi^{\mu,\varepsilon,2}$ in $L_2(\Omega)$, that the convergence (11.5) and when $s=2$. Theorem 2.4 has been completely proved.

The first author is thankful to the Kazakh National University in the name of Al-Faraby, where a part of the present work was carried out for their hospitality.

BIBLIOGRAPHY

1. Kato T. *The theory of perturbations of linear operators* Mir, M., 1972.
2. Bikmetov A.R. *Asymptotics of the characteristic constants of the elements of boundary-value problem of Schrödinger operator with greater potential, localised on the small set* // Journal of counting math. and math. phys. 2006. V. 79, No 4. P. 666–681.
3. Bikmetov A.R., Gadylshin R.R. *On the spectre of the Schrödinger operator with growing potential, localized on the squeezing set* // Math. notes. 2006. V. 79, No 5. P. 787–790.
4. Golovaly Yu.D., Manko S.S.*Exact models for Schrödinger operator with δ' like potensials*// Ukrainian Mathematical Bulletin. V.6, No 2. 2009. P. 173–207.
5. Husnullin I.H. *A perturbed boundary-value problem on the characteristic constants for the Schrödinger operator on the section* // Journal of counting math. and math. phys. V.50, No 4. 2010. P. 679–698.
6. Gadylshin R.R., Husnullin I.H. *Perturbation of the Schrödinger operator by a narrow potential* // Ufa math. journal. V.3, No3. 2011. P. 55–66.
7. Ilyin A.M. *Matching of the asymptotic expansions of the solutions of boundary-value problems* Nauka, M., 1989.

8. Gadylshin R.R. *The method of matching of the asymptotic expansions in the singularly perturbed boundary-value problem for the Laplace operator* // Modern mathematics and its applications. V.5. 2003. P. 3–32.
9. O.A. Oleinik, Yosifian G.A., Shamaev A. S., *Mathematical problems of the theory of strong nonhomogeneous elastic mediums* Izd-vo MSU, M., 1990.
10. O.A. Oleinik, J. Sanchez-Hubert, Yosifian G.A. *On vibrations of a membrane with concentrated masses* // Bull. Sc. math. Ser. 2 1991. V. 115. P. 1–27.
11. Ilyin A.M. *A boundary-value problem for the elliptic equation of the second order in the domain with a narrow slot. II. the domain with a small slot* // Math. issue. 1977. V. 103, No 2. P. 265–284.

Bikmetov Aidar Renatovich,
Bashkir State Pedagogical University by the name of M.Akmulla,
3a, Oktyabrskaya Revolution str.,
Ufa, Russia, 450000
E-mail: bikmetovar@yandex.ru

Gadyl'shin Rustem Rashitovich,
Bashkir State Pedagogical University by the name of M.Akmulla,
3a, Oktyabrskaya Revolution str.,
Ufa, Russia, 450000
E-mail: gadylshin@yandex.ru