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ON THE ASYMPTOTIC BEHAVIOR OF CAUCHY-STIELTJES INTEGRAL IN THE POLYDISC

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Abstract. In the paper the asymptotic behavior of Cauchy-Stieltjes integral of a complex-valued Borel measure on the skeleton in the polydisc is described. The main result holds outside a set of zero ω-capacity. It generalizes the theorem for the one-dimensional case.

Keywords: modulus of continuity, Cauchy-Stieltjes integral, polydisc, set of zero ω-capacity, non-tangential limit.

For $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$, $n \in \mathbb{N}$, let $|z| = \max\{|z_j| : 1 \leqslant j \leqslant n\}$ be the polydisc norm. Denote by $U^n = \{z \in \mathbb{C}^n : |z| < 1\}$ the unit polydisc with the distinguished boundary $T^n = \{z \in \mathbb{C}^n : |z_j| = 1, 1 \leqslant j \leqslant n\}$, and $\tau = [-\pi; \pi)$. For $z \in U^n, z_j = r_j e^{i\varphi_j}, w = (w_1, \ldots, w_n) \in T^n, w_j = e^{i\theta_j}, 1 \leqslant j \leqslant n$ we write $C_\alpha(z, w) = \prod_{j=1}^n \frac{1}{(1-z_j\bar{w}_j)^{\alpha_j}}$, where $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\alpha_j > 0$, $1 \le j \le n$, $C_{\alpha_j}(z_j, w_j) = \frac{1}{(1-z_j\bar{w}_j)^{\alpha_j}}$ is the generalized Cauchy kernel for the unit disc, $C_{\alpha_j}(0, w_j) = 1$. The symbol K will denote a constant not necessary the same in each occurrence.

The function in U^n defined by the equality

$$f(z_1, \ldots, z_n) = \int_{T^n} C_\alpha(z, w)\, d\mu(w), \quad z \in U^n \tag{1}$$

with $|\mu|(T^n) < +\infty$, where $|\mu|$ is the total variation of μ, is called the Cauchy-Stieltjes integral of a complex-valued Borel measure μ. The function $f(z_1, \ldots, z_n)$ is analytic in U^n.

For $\psi = (\psi_1, \ldots, \psi_n) \in \tau^n$, $\gamma = (\gamma_1, \ldots, \gamma_n) \in [0; \pi)^n$ we define the Stolz angle $S(\psi, \gamma) = S(\psi_1, \gamma_1) \times \ldots \times S(\psi_n, \gamma_n)$ in the polydisc, where $S(\psi_j, \gamma_j)$ is the Stolz angle for the unit disc with the vertex $e^{i\psi_j}$,

$$S(\psi_j, \gamma_j) = \{\left|z_j - e^{i\psi_j}\right| \le A(\gamma_j)(1 - r_j)\}, \quad 1 \le j \le n,$$

$A(\gamma_j) = \sqrt{1 + 4\mathrm{tg}^2\frac{\gamma_j}{2}}$.

In the case of the unit disc ($n = 1$), there is a strong dependence between local smoothness of the measure μ and the growth of f in the direction of $e^{i\psi}$ (see [1]–[3]). In particular, differentiability of μ implies boundedness of the Poisson-Stieltjes integral [4]. The idea of such results goes back to P. Fatou [4], and G. Hardy and J. Littlewood [5].

However, in the case $n > 1$ *local* differentiability of μ need not imply boundedness of Poisson-Stieltjes integral (see [6, Section 2.3]). In [7], [8] an interplay between smoothness and the growth of the Poisson-Stieltjes integral was considered. In particular, the growth of such integrals was characterized in terms of smoothness of the corresponding (positive) measure μ.

Let $\omega\colon \mathbb{R}_+^n \to \mathbb{R}_+$ be a semi-additive continuous increasing function in each variable vanishing if at least one of the arguments equals zero. We call ω a modulus of continuity.

A Borel set $E \subset T^n$ is called a *set of positive ω-capacity* if there exists a nonnegative measure ν on T^n such that

$$\int_E d\nu = \int_{T^n} d\nu = 1 \tag{2}$$

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and

$$\sup_{x\in\mathbb{R}^n}\int\limits_{T^n}\frac{d\nu\left(e^{it_1},\ldots,e^{it_n}\right)}{\omega\left(|t_1-x_1|,\ldots,|t_n-x_n|\right)}<+\infty. \tag{3}$$

Otherwise, E is called a *set of zero ω-capacity.*

The following properties of sets of zero ω-capacity are easy to check:

1) If E_1 and E_2 are Borel subsets of T^n, $E_1\subset E_2$, and E_2 has ω-capacity zero, then E_1 has ω-capacity zero.

2) If Borel sets $E_i, i=1,2,\ldots$ have ω-capacity zero, then the set $E=\bigcup\limits_{i=1}^{\infty}E_i$ has ω-capacity zero too.

3) If ω_1 and ω_2 are moduli of continuity, $\omega_1(t)\le\omega_2(t)$, $t\in\mathbb{R}^n_+$, and a Borel subset E of T^n has positive ω_1-capacity, then E has positive ω_2-capacity.

4) If $\int\limits_0^1\ldots\int\limits_0^1\frac{dt_1\ldots dt_n}{\omega(t_1,\ldots,t_n)}<\infty$ and a set $E\subset T^n$ has zero ω-capacity, then E has zero n-dimensional Lebesgue measure.

Let us prove the last property. Indeed, if m is the Lebesgue on T^n and $m(E)>0$, let

$$d\nu\left(e^{ix_1},\ldots,e^{ix_n}\right)=\left(\mathcal{X}_E\left(e^{ix_1},\ldots,e^{ix_n}\right)/m(E)\right)dm\left(e^{ix_1},\ldots,e^{ix_n}\right),$$

where $\mathcal{X}_E\left(e^{ix_1},\ldots,e^{ix_n}\right)$ is the characteristic function of E. Thus,

$$\int\limits_{T^n}\frac{d\nu\left(e^{ix_1},\ldots,e^{ix_n}\right)}{\omega\left(|x_1-t_1|,\ldots,|x_n-t_n|\right)}=\frac{1}{m\left(E\right)}\int\limits_{T^n}\frac{\mathcal{X}_E\left(e^{it_1},\ldots,e^{it_n}\right)dm\left(e^{it_1},\ldots,e^{it_n}\right)}{\omega\left(|x_1-t_1|,\ldots,|x_n-t_n|\right)}\le$$

$$\le\frac{1}{m\left(E\right)}\int\limits_{T^n}\frac{dt_1\ldots dt_n}{\omega\left(|x_1-t_1|,\ldots,|x_n-t_n|\right)}\le\frac{2}{m\left(E\right)}\int\limits_0^{\pi}\ldots\int\limits_0^{\pi}\frac{dt_1\ldots dt_n}{\omega\left(t_1,\ldots,t_n\right)}<\infty.$$

Therefore, ω-capacity of E is positive.

The notion of zero ω-capacity for Borel subsets of T provides a useful measure of finiteness of exceptional sets for the radial (non-tangential) growth of functions of the form (1) (see Theorem B ([3]) below, [1], and [2]). In the case $n=1$, $\omega(t)=t^{\beta}$, $\beta\in(0,1)$, the definition and properties of ω-capacity are given in [9, Chapter 3].

Theorem A ([3]). *Let $\alpha>0$, $\psi\in[-\pi;\pi]$, g be a function of bounded variation on $[-\pi;\pi]$, and a modulus of continuity ω satisfies the condition*

$$\int\limits_0^1 t^{-\alpha-1}\omega\left(t\right)dt=\infty. \tag{4}$$

If

$$|g(t)-g(\psi)|=o(\omega(|t-\psi|)),\quad t\to\psi$$

then

$$\left|\int\limits_{-\pi}^{\pi}C_\alpha\left(z,e^{-it}\right)dg\left(t\right)\right|\Bigg/\int\limits_{\left|1-ze^{-i\psi}\right|}^{1}t^{-\alpha-1}\omega(t)dt,\quad z\in U \tag{5}$$

has the non-tangential limit zero at $e^{i\psi}$.

Theorem B ([3]). *Let $\alpha>0$, g be a function of bounded variation on $[-\pi;\pi]$, and a modulus of continuity ω satisfies condition (4). Then (5) has the non-tangential limit zero at all ψ in $[-\pi;\pi]$, except, possibly, a set of zero ω-capacity.*

We give an example of a set of zero ω-capacity that will be used later. For $a \in \mathbb{R}$, $1 \leqslant j \leqslant n$ we denote

$$T_a^{(j)} = \left\{ \left(e^{i\theta_1}, \ldots, e^{i\theta_n}\right) \in T^n : \quad \theta_j = a \right\},$$
$$\tau_a^{(j)} = \left\{ (\theta_1, \ldots, \theta_n) \in \tau^n : \ \theta_j = a (\mathrm{mod}\ 2\pi) \right\}.$$

Let $e_j = \left\{a_j^1, \ldots, a_j^s, \ldots\right\}$, $s \in \mathbb{N}$. We are to prove that the set $E_j = \bigcup_{a \in e_j} T_a^{(j)}$ is of positive ω-capacity. Suppose the contrary. Then the conditions (2) and (3) hold. It follows from (2) that

$$\exists a_j^s : \ \nu \left(T_{a_j^s}^{(j)} \right) > 0.$$

Consequently, using (3) and the definition of ω, we get

$$\infty > \sup_{x \in \mathbb{R}^n} \int\limits_{\tau^n} \frac{d\nu\left(e^{i\theta_1}, \ldots, e^{i\theta_n}\right)}{\omega\left(\left|\theta_1 - x_1\right|, \ldots, \left|\theta_n - x_n\right|\right)} >$$

$$> \sup_{x_j = a_j^s} \int\limits_{\tau^n} \frac{d\nu\left(e^{i\theta_1}, \ldots, e^{i\theta_n}\right)}{\omega\left(\left|\theta_1 - x_1\right|, \ldots, \left|\theta_j - x_j\right|, \ldots, \left|\theta_n - x_n\right|\right)} \geq$$

$$\geq \int\limits_{\tau_a^{(j)}} \frac{d\nu\left(e^{i\theta_1}, \ldots, e^{i\theta_n}\right)}{\omega\left(\left|\theta_1 - x_1\right|, \ldots, 0, \ldots, \left|\theta_n - x_n\right|\right)} = \infty.$$

Hence, the set E_j has ω-capacity zero. In particular, $T_a^{(j)}$ has ω-capacity zero.

Theorem 1. *Let $\alpha_j > 0, \beta_j > 0,\ 1 \leq j \leq n,\ n \in \mathbb{N}$, ω be a modulus of continuity satisfying*

$$\int\limits_0^1 \cdots \int\limits_0^1 \frac{\omega\left(t_1, \ldots, t_n\right)}{t_1^{\alpha_1+1} \cdot \ldots \cdot t_n^{\alpha_n+1}} dt_1 \ldots dt_n = +\infty.$$

Let μ be a complex-valued Borel measure on T^n with $|\mu|\left(T^n\right) < +\infty$, and $e^{i\psi} \in T^n$.
If

$$|\mu|\left(\left\{e^{i\theta} \in T^n : |\theta_j - \psi_j| \leq t_j, 1 \leq j \leq n\right\}\right) = o\left(\omega\left(t_1, \ldots, t_n\right)\right), \min_j t_j \to 0+,$$

then

$$\left| \int\limits_{T^n} C_\alpha\left(z, w\right) d\mu\left(w\right) \right| = o\left(\log^n \frac{1}{\delta} \cdot \int\limits_{\left|z_1 - e^{i\psi_1}\right|}^1 \cdots \int\limits_{\left|z_n - e^{i\psi_n}\right|}^1 \frac{\omega\left(t_1, \ldots, t_n\right) dt_1 \ldots dt_n}{t_1^{\alpha_1+1} \cdot \ldots \cdot t_n^{\alpha_n+1}} \right),$$

where $\delta \to 0$, $|z_j| = 1 - \delta^{\frac{1}{\beta_j}}$, $z \in S\left(\psi, \gamma\right)$.

Theorem 2. *Let $\alpha_j > 0, \beta_j > 0,\ 1 \leq j \leq n,\ n \in \mathbb{N}$, ω be a modulus of continuity satisfying*

$$\int\limits_0^1 \cdots \int\limits_0^1 \frac{\omega\left(t_1, \ldots, t_n\right)}{t_1^{\alpha_1+1} \cdot \ldots \cdot t_n^{\alpha_n+1}} dt_1 \ldots dt_n = +\infty,$$

and μ be a complex-valued Borel measure on T^n with $|\mu|\left(T^n\right) < +\infty$. Then

$$\left| \int\limits_{T^n} C_\alpha\left(z, w\right) d\mu\left(w\right) \right| = o\left(\log^n \frac{1}{\delta} \cdot \int\limits_{\left|z_1 - e^{i\psi_1}\right|}^1 \cdots \int\limits_{\left|z_n - e^{i\psi_n}\right|}^1 \frac{\omega\left(t_1, \ldots, t_n\right) dt_1 \ldots dt_n}{t_1^{\alpha_1+1} \cdot \ldots \cdot t_n^{\alpha_n+1}} \right),$$

where $\delta \to 0$, $|z_j| = 1 - \delta^{\frac{1}{\beta_j}}$, $z \in S\left(\psi, \gamma\right)$, for $\left(e^{i\psi_1}, \ldots, e^{i\psi_n}\right) \in T^n$ except, possibly, a set of zero ω-capacity.

The proof of Theorem 1, as a matter of fact, is contained in that of Theorem 2, which generalizes Theorem B.

Corollary 1. *Let $\omega(t_1,\dots,t_n) = t_1^{\varkappa_1}\cdot\ldots\cdot t_n^{\varkappa_n}$ be a modulus of continuity, $\varkappa_j > 0$, $\alpha_j > 0, \beta_j > 0$, $1 \le j \le n, n \in \mathbb{N}$, and $\exists j_0 :$ $\alpha_{j_0} \ge \varkappa_{j_0}$, μ be a complex-valued Borel measure on T^n with $|\mu|(T^n) < +\infty$. Then*

$$\left|\int\limits_{T^n} C_\alpha(z,w)\,d\mu(w)\right| = o\left(\log^n\frac{1}{\delta}\cdot\int\limits_{|z_1-e^{i\psi_1}|}^{1}\cdots\int\limits_{|z_n-e^{i\psi_n}|}^{1}\frac{dt_1\ldots dt_n}{t_1^{\alpha_1+1-\varkappa_1}\cdot\ldots\cdot t_n^{\alpha_n+1-\varkappa_n}}\right),$$

as $\delta\to 0$, where $|z_j| = 1-\delta^{\frac{1}{\beta_j}}$, $z\in S(\psi,\gamma)$, for $\left(e^{i\psi_1},\dots,e^{i\psi_n}\right)\in T^n$ except, possibly, a set of zero ω-capacity.

Corollary 2. *Let $\varkappa_j \in (0;1), \alpha_j > 0, \beta_j > 0$, $1 \le j \le n, n \in \mathbb{N}$, and $\exists j_0 :$ $\alpha_{j_0} \ge \varkappa_{j_0}$, $\omega(t_1,\dots,t_n) = t_1^{\varkappa_1}\cdot\ldots\cdot t_n^{\varkappa_n}$. Let μ be a complex-valued Borel measure on T^n with $|\mu|(T^n) < +\infty$. Then*

$$\left|\int\limits_{T^n} C_\alpha(z,w)\,d\mu(w)\right| = o\left(\log^n\frac{1}{\delta}\cdot\int\limits_{|z_1-e^{i\psi_1}|}^{1}\cdots\int\limits_{|z_n-e^{i\psi_n}|}^{1}\frac{dt_1\ldots dt_n}{t_1^{\alpha_1-\varkappa_1+1}\cdot\ldots\cdot t_n^{\alpha_n-\varkappa_n+1}}\right),$$

as $\delta\to 0$, where $|z_j| = 1-\delta^{\frac{1}{\beta_j}}$, $z\in S(\psi,\gamma)$, except, possibly, a set of zero Lebesgue measure.

Corollary 2 follows from property 4 of the sets of zero ω-capacity and Corollary 1.

Corollary 3. *Let $\beta_j > 0$, $\omega(t_1,\dots,t_n) = t_1^{\alpha_1}\cdot\ldots\cdot t_n^{\alpha_n}\cdot\log^{l_1}\frac{1}{t_1}\cdot\ldots\cdot\log^{l_n}\frac{1}{t_n}$, $\alpha_j > 0$, $1\le j\le n$, $n\in\mathbb{N}$, $l_j\in\mathbb{R}$, $\exists j_0 : l_{j_0}\ge -1$. Let μ be a complex-valued Borel measure on T^n with $|\mu|(T^n) < +\infty$. Then*

$$\left|\int\limits_{T^n} C_\alpha(z,w)\,d\mu(w)\right| = o\left(\log^n\frac{1}{\delta}\int\limits_{|z_1-e^{i\psi_1}|}^{1}\cdots\int\limits_{|z_n-e^{i\psi_n}|}^{1}\frac{\log^{l_1}\frac{1}{t_1}\cdot\ldots\cdot\log^{l_n}\frac{1}{t_n}dt_1\ldots dt_n}{t_1\cdot\ldots\cdot t_n}\right),$$

as $\delta\to 0$, where $|z_j| = 1-\delta^{\frac{1}{\beta_j}}$, $z\in S(\psi,\gamma)$, for $\left(e^{i\psi_1},\dots,e^{i\psi_n}\right)\in T^n$ except, possibly, a set of zero ω-capacity.

Unfortunately, the author does not know whether it is possible to omit the factor $\log^n\frac{1}{\delta}$ in the assertion of the theorem.

To prove Theorem 1 we need the following lemma due to Prof. I.Chyzhykov who has kindly allowed to use it.

Lemma 1. *Let $\omega(t_1,\dots,t_n)$ be a modulus of continuity and μ be a complex-valued Borel measure, $|\mu|(T^n) < +\infty$. Then*

$$|\mu|\left(\{e^{i\theta}\in T^n : |\theta_j-\varphi_j|\le t_j, 1\le j\le n\}\right) = o\left(\omega(t_1,\dots,t_n)\right), \tag{6}$$

as $\min\limits_j t_j\to 0+$, except, possibly, a set of zero ω-capacity of values $(e^{i\varphi_1},\dots,e^{i\varphi_n})$.

Proof of the lemma. Let $e_j = \left\{a\in\mathbb{R} :\ |\mu|\left(T_a^{(j)}\right) > 0\right\}$. For arbitrary j the set e_j is at most countable. Denote $E_j = \bigcup\limits_{a\in e_j} T_a^{(j)}$. As it was proved above, every set E_j has zero ω-capacity. Thus $E = \bigcup\limits_{j=1}^{n} E_j$ has ω-capacity zero as well.

Let now $e^{i\varphi} \in T^n \setminus E$. Then $|\mu|\left(T^{(j)}_{\varphi_j}\right) = 0,\ 1 \le j \le n$.

Let $\min\limits_j t_j \to 0+$. Passing, if necessary, to a subsequence, we may assume that $t_1 \to 0+$.

Fixing φ_1, we consider the set

$$G^k_{\varphi_1} = \left\{\left(e^{i\theta_1}, \ldots, e^{i\theta_n}\right) \in T^n :\ |\varphi_1 - \theta_1| < \frac{1}{k}\right\}, \quad k \in \mathbb{N}.$$

Then $G^{k+1}_{\varphi_1} \subset G^k_{\varphi_1}$ and $\bigcap\limits_{k\in\mathbb{N}} G^k_{\varphi_1} = T^{(1)}_{\varphi_1}$. By countable additivity,

$$\lim_{k\to\infty} |\mu|\left(G^k_{\varphi_1}\right) = |\mu|\left(T^{(1)}_{\varphi_1}\right) = 0.$$

Assume that

$$\int\limits_{T^n} \frac{|d\mu|\left(e^{i\theta_1}, \ldots, e^{i\theta_n}\right)}{\omega\left(|\varphi_1 - \theta_1|, \ldots, |\varphi_n - \theta_n|\right)} < +\infty.$$

Denote $g^k_{\varphi_1} = \left\{(\theta_1, \ldots, \theta_n) :\ -\pi \le \theta_j < \pi,\ 2 \le j \le n,\ e^{i\theta} \in G^k_{\varphi_1}, |\varphi_j - \theta_j| < \frac{1}{k}\right\}$. Then, using countable additivity of the Lebesgue integral, we obtain

$$\lim_{k\to\infty} \int\limits_{g^k_{\varphi_1}} \frac{|d\mu|\left(e^{i\theta_1}, \ldots, e^{i\theta_n}\right)}{\omega\left(|\varphi_1 - \theta_1|, \ldots, |\varphi_n - \theta_n|\right)} =0.$$

Let $t_2, \ldots, t_n \in [0;\pi]$. Then $\forall\varepsilon > 0 \quad \exists k_0 \in \mathbb{N} \quad \forall k > k_0 \quad |t_1| < \frac{1}{k}$ we have

$$\varepsilon > \int\limits_{g^k_{\varphi_1}} \frac{|d\mu|\left(e^{i\theta_1}, \ldots, e^{i\theta_n}\right)}{\omega\left(|\varphi_1 - \theta_1|, \ldots, |\varphi_n - \theta_n|\right)} \ge$$

$$\ge \int\limits_{|\varphi_1-\theta_1|\le t_1} \cdots \int\limits_{|\varphi_n-\theta_n|\le t_n} \frac{|d\mu|\left(e^{i\theta_1}, \ldots, e^{i\theta_n}\right)}{\omega\left(|\varphi_1 - \theta_1|, \ldots, |\varphi_n - \theta_n|\right)} \ge$$

$$\ge \frac{|\mu|\left(\{e^{i\theta} : |\varphi_1 - \theta_1| \le t_1, \ldots, |\varphi_n - \theta_n| \le t_n\}\right)}{\omega\left(t_1, \ldots, t_n\right)}.$$

Hence,

$$|\mu|\left(\left\{\left(e^{i\theta_1}, \ldots, e^{i\theta_n}\right) \in T^n : |\theta_j - \varphi_j| \le t_j,\ 1 \le j \le n\right\}\right) = o\left(\omega\left(t_1, \ldots, t_n\right)\right),\ t_1 \to 0+.$$

The lemma is proved.

Now we can prove Theorem 1.

Proof of Theorem 1. By Lemma 1, there exists a set E of zero ω-capacity with the following property. Given $e^{i\psi} \in T^n \setminus E$, for arbitrary $\varepsilon > 0$ there exists η_ε such that

$$|\mu|\left(\left\{e^{i\theta} : |\theta_j - \psi_j| < t_j,\ 1 \le j \le n\right\}\right) \le \varepsilon\omega\left(t_1, \ldots, t_n\right), \quad 0 \le \min_{1\le j\le n} t_j < \eta_\varepsilon. \tag{7}$$

For fixed δ, $0 < \delta < \frac{1}{2}$, $m = (m_1, \ldots, m_n) \in \mathbb{Z}^n_+$ we denote

$$R_m = \left\{ \begin{array}{ll} e^{i\theta} \in T^n : 2^{-\frac{1}{\beta_j}} \le & \dfrac{|\theta_j - \psi_j|}{2A\left(\gamma_j\right) \cdot \left(2^{m_j}\delta\right)^{\frac{1}{\beta_j}}} \le 1, \quad if\ m_j > 0; \\ & \dfrac{|\theta_j - \psi_j|}{2A\left(\gamma_j\right) \cdot \left(2^{m_j}\delta\right)^{\frac{1}{\beta_j}}} \le 1, \quad if\ m_j = 0,\ 1 \le j \le n \end{array} \right\},$$

where $\beta_j > 0,\ \gamma_j \in [0;\pi)$.

For $z_j \in S(\psi_j, \gamma_j)$, $e^{i\theta} \in R_m$, $m_j > 0$, we obtain

$$|z_j - e^{i\theta_j}| \geq |e^{i\psi_j} - e^{i\theta_j}| - |e^{i\psi_j} - z_j| \geq \frac{2}{\pi}|\psi_j - \theta_j| - A(\gamma_j)(1 - r_j) \geq$$
$$\geq \frac{4}{\pi}A(\gamma_j)2^{\frac{m_j-1}{\beta_j}}\delta^{\frac{1}{\beta_j}} - A(\gamma_j)\delta^{\frac{1}{\beta_j}} \geq \frac{1}{4}A(\gamma_j)2^{\frac{m_j-1}{\beta_j}}\delta^{\frac{1}{\beta_j}}.$$

If $z_j \in S(\psi_j, \gamma_j)$, $e^{i\theta} \in R_m$, $m_j = 0$, we have $|z_j - e^{i\theta_j}| \geq 1 - r_j = \delta^{\frac{1}{\beta_j}}$. Therefore, we can write both inequality in the form

$$|z_j - e^{i\theta_j}| \geq K2^{\frac{m_j}{\beta_j}}\delta^{\frac{1}{\beta_j}}, \quad z_j \in S(\psi_j, \gamma_j), e^{i\theta} \in R_m, \tag{8}$$

where $K = K(\beta_j, \gamma_j)$ is a constant depending on β_j and γ_j only.

For $m = (m_1, \ldots, m_n) \in \mathbb{Z}_+^n$ we consider two cases:

1) $\forall j:\ 2A(\gamma_j) \cdot (2^{m_j}\delta)^{\frac{1}{\beta_j}} \geq \eta_\varepsilon$;

2) $\exists j:\ 2A(\gamma_j) \cdot (2^{m_j}\delta)^{\frac{1}{\beta_j}} < \eta_\varepsilon$.

In the first case we have $|C_{\alpha_j}(z_j, w_j)| \leq \frac{K(\beta_j,\alpha_j)}{\eta_\varepsilon^{\alpha_j}}$.

We denote $F = F(\beta, \gamma, \delta, \varepsilon) = \bigcup_{m:1)} R_m$. Then using the last estimate of the Cauchy kernel we deduce

$$\left|\int\limits_F C_\alpha(z, w)\, d\mu(w)\right| \leq |\mu|(T^n) \cdot \frac{K}{\eta_\varepsilon^{\alpha_1+\ldots+\alpha_n}}. \tag{9}$$

We now consider the second case. By (7) we have

$$|\mu|(R_{m_1\ldots m_n}) \leq \varepsilon \cdot \omega\left(2A(\gamma_1) \cdot (2^{m_1}\delta)^{\frac{1}{\beta_1}}, \ldots, 2A(\gamma_n) \cdot (2^{m_n}\delta)^{\frac{1}{\beta_n}}\right). \tag{10}$$

Then, using (8) we deduce

$$\left|\sum_m \int\limits_{R_m} C_\alpha(z, w)\, d\mu(w)\right| \leq$$
$$\leq \varepsilon \sum_m \omega\left(2A(\gamma_1)(2^{m_1}\delta)^{\frac{1}{\beta_1}}, \ldots, 2A(\gamma_n)(2^{m_n}\delta)^{\frac{1}{\beta_n}}\right) \prod_{j=1}^n \frac{1}{K(2^{m_j}\delta)^{\frac{\alpha_j}{\beta_j}}}, \tag{11}$$

where the sum is taken over $m = (m_1, \ldots, m_n)$ satisfying the condition from the second case.

On the other hand

$$\int\limits_{2A(\gamma_1)\cdot(2^{m_1}\delta)^{\frac{1}{\beta_1}}}^1 \cdots \int\limits_{2A(\gamma_n)\cdot(2^{m_n}\delta)^{\frac{1}{\beta_n}}}^1 \frac{\omega(t_1, \ldots, t_n)\, dt_1 \ldots dt_n}{{t_1}^{\alpha_1+1} \cdot \ldots \cdot {t_n}^{\alpha_n+1}} \geq$$
$$\geq \frac{K\omega\left(2A(\gamma_1) \cdot (2^{m_1}\delta)^{\frac{1}{\beta_1}}, \ldots, 2A(\gamma_n) \cdot (2^{m_n}\delta)^{\frac{1}{\beta_n}}\right)}{\prod_{j=1}^n \left(2^{m_j}\delta\right)^{\frac{\alpha_j}{\beta_j}}}, \tag{12}$$

where $K = K(\alpha, \beta, \gamma)$. From (11) and (12), we obtain

$$\left|\sum_m \int\limits_{R_{m_1\ldots m_n}} C_\alpha(z, w)\, d\mu(w)\right| \leq \varepsilon \cdot N^* \cdot \int\limits_{A(\gamma_1)\cdot\delta^{\frac{1}{\beta_1}}}^1 \cdots \int\limits_{A(\gamma_n)\cdot\delta^{\frac{1}{\beta_n}}}^1 \frac{\omega(t_1, \ldots, t_n)\, dt_1 \ldots dt_n}{{t_1}^{\alpha_1+1} \cdot \ldots \cdot {t_n}^{\alpha_n+1}},$$

where $|z_j| = 1 - \delta^{\frac{1}{\beta_j}}$, $1 \le j \le n$, N^* is the number of m satisfying the condition from the second case. It is easy to see that

$$N^* \le \left(\max_j \log_2 \frac{\pi^{\beta_j}}{(2A(\gamma_j))^{\beta_j}\delta}\right)^n \le K \cdot \log^n \frac{1}{\delta}.$$

Then

$$\left|\sum_m \int\limits_{R_m} C_\alpha(z,w)\, d\mu(w)\right| \le K\varepsilon \log^n \frac{1}{\delta} \cdot \int\limits_{A(\gamma_1)\delta^{\frac{1}{\beta_1}}}^{1} \dots \int\limits_{A(\gamma_n)\cdot\delta^{\frac{1}{\beta_n}}}^{1} \frac{\omega(t_1,\dots,t_n)\, dt_1 \dots dt_n}{{t_1}^{\alpha_1+1}\cdot \ldots \cdot {t_n}^{\alpha_n+1}}.$$

Finally, using the definition of the Stolz angle, we get

$$\left|\sum_m \int\limits_{R_{m_1\dots m_n}} C_\alpha(z,w)\, d\mu(w)\right| \le$$

$$\le 2K\varepsilon \log^n \frac{1}{\delta} \cdot \int\limits_{\left|z_1 - e^{i\psi_1}\right|}^{1} \dots \int\limits_{\left|z_n - e^{i\psi_n}\right|}^{1} \frac{\omega(t_1,\dots,t_n)\, dt_1 \dots dt_n}{{t_1}^{\alpha_1+1}\cdot \ldots \cdot {t_n}^{\alpha_n+1}}. \qquad (13)$$

The assertion of Theorem 2 follows from (9) and (13).

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