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BOUNDARY PROBLEM FOR THE GENERALIZED CAUCHY–RIEMANN EQUATION IN SPACES, DESCRIBED BY THE MODULUS OF CONTINUITY

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Abstract. The article is devoted to the Dirichlet problem in the unit disk G for $\partial_{\bar{z}}w + b(z)\overline{w} = 0$, $\Re w = g$ on ∂G, $\Im w = h$ at the point $z_0 = 1$, where g is a given Lipschitz continuous function. The coefficient b belongs to a subspace of $L_2(G)$ which is not contained in $L_q(G)$, $q > 2$ in the general case. Thus, I. Vekua's theory is not applicable in this case. The article shows that, as well as in the case of Dirichlet's problem for holomorphic functions, there appears a "logarithmic effect". The solution outside the point $z = 0$ satisfies the Lipschits conditions with logarithmic factors. The existence of a continuous solution of the problem in $\overline{G}$ is proved

Keywords: generalized Cauchy–Riemann equation; Dirichlet problem; modulus of continuity; Tikhonov's fixed point theorem

1. Introduction

Theory of generalized analytic functions is a theory of complex-valued functions $w = w(z)$, being solution of the equation

$$\partial_{\bar{z}}w(z) + A(z)w(z) + B(z)\overline{w}(z) = 0, z \in G, \tag{1.1}$$

where $\partial_{\bar{z}} := \frac{1}{2}\left(\frac{\partial}{\partial x} + i \cdot \frac{\partial}{\partial y}\right)$, and $A(z)$, $B(z)$ are given in the bounded domain G of the complex plane functions. In case, when $A(z) \equiv B(z) \equiv 0$, (1.1) pass to the condition of the function analyticity $w(z)$.

Theory of such functions was constructed by I. Vekua in the supposition, that $A(z)$, $B(z)$ belongs to the space $L_p(G)$, where $p > 2$ ([1]). In this case (1.1) is called a regular generalized Cauchy–Riemann system, and its solution is called generalized analytic functions. Coefficients of such functions can assume ¡¡weak¿¿ peculiarities, bounded by the demand of p-integrability. In particular, if $A(z)$, $B(z)$ reduce to infinity in some isolated special point, than the order of this peculiarity should be strongly less than 1. The study of problems for generalized equations with coefficients, possessing peculiarities in some isolated point, was made in papers of L.G. Mikhailov, Z.D. Usmanov, A. Tungatarov, N. Bliev, M. Otelbaev, M. Reissig and A.Y. Timofeev, R. Sacks, G.T. Makatsaria and others (for instance, [2]–[7]). Special attention in these papers is paid to research of existence of continuous solutions for boundary equations problems (1.1).

In paper [7] Dirichlet problem for generalized Cauchy–Riemann equation (1.1) is studied, where $G = \{z \in C : |z| < 1\}$, $A(z) \equiv 0$.

The novelty of the research consists in assumption of peculiarity in the point $z = 0$ where coefficients $B(z)$ belong to the power space of functions $S_p(G)$, which is a generalization of

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spaces

$$s_p(G) = \left\{ B(z) : \sup_{G\setminus\{0\}} (|B(z)| \cdot p(|z|)) < +\infty \right\}.$$

The set of functions $p(t)$, possessing sufficiently general properties, is denoted by P (see section 2). The space $S_p(G)$ consists of only those given in G functions $f(z)$, for each of which there is such a function $p(t) \in P$, that $f(z) \in s_p(G)$.

In [7] there was the following theorem proved

Theorem 1. *We consider the following Dirichlet problem:*

$$\partial_{\bar{z}} w + B(z)\overline{w} = 0, z \in G = \{z \in \mathbb{C} : |z| < 1\}, \tag{1.2}$$

$$\Re w = g(z), z \in \partial G, \Im w\,|_{z_0=1} = h, \tag{1.3}$$

where $B \in S_p(G)$, $g \in C^{\lambda_0}(\partial G)(0 < \lambda_0 < 1)$, $h \in \mathbb{R}$. *Then there is a unique solution of the problem* (1.2)–(1.3) $w = w(z)$, *and* $w \in C(\overline{G}) \bigcap C^{\lambda_0}(\overline{G} \setminus \{0\})$.

The boundary function $g(z)$ under condition (1.3) belongs to Helder's space, described by the modulus of continuity $\mu(t) = t^{\lambda_0}$. It is known, that in general case the modulus of continuity satisfies the inequality

$$\mu(t) \geq c \cdot t$$

with some constant c.

In this connection, it is interesting to study problems (1.2)–(1.3) for the case, when $g(z)$ belongs to another space of functions, described by the module of continuity $\mu(t)$. What condition outside the point $z = 0$ continuous solutions $w(z)$ of the system (1.2)–(1.3) will satisfy?

In the given paper we study the case of ¡¡minimal¿¿ space, described by the modulus of continuity of the Lipschits space. In this case $\mu(t) = t$. Let us introduce symbols: $\mu_{1,0}(t) := t$; $\mu_{1,k}(t) := t \cdot (\ln \frac{1}{t})^k, k \geq 1 (0 < t < \frac{1}{e})$. It holds

Theorem 2. *Assume* $B(z) \in S_p(G)$, $g(z) \in C_{\mu_{1,0}}(\partial G)$, $h \in \mathbb{R}$. *Then there is a unique solution of the problem* (1.2)–(1.3) $w = w(z) \in C(\overline{G}) \bigcap C_{\mu_{1,5}}(\overline{G} \setminus \{0\})$.

In section 2 there is data on power functions, modulus of continuity and corresponding functional spaces. In section 3 there are auxiliary statements formulated. In the conclusion we give the algorithm of proof of Theorem 2.

2. Power functions, modulus of continuity. Basic functions spaces

In [7] there were introduced power functions $p(t)$, as functions, satisfying the following conditions.

1. Given and positive on some gap $(0, t_p]$, where the number t_p depends on the function $p(t)$, $t_p < 1$.
2. Do not decrease on $(0, t_p]$.
3. $\lim\limits_{t\to+0} p(t) = 0$.
4. $\displaystyle\int_0^{t_p} \frac{dt}{p(t)} < +\infty$.

Further we will consider functions $p(t)$ given on all the gap $(0, 1]$, expanding in case of necessity $p(t)$ on the gap $[t_p, 1]$ by the constant, which os equal to $p(t_p)$. In this case conditions 1, 2 and 4 will hold true already on all the gap $(0, 1]$. Let us denote the set of functions $p(t)$, satisfying conditions 1–4 by P.

It is easy to show, that for functions $p(t) \in P$ there is such a number $c_p > 0$, that

$$\frac{t}{p(t)} \leqslant c_p, t \in (0, 1]. \tag{2.1}$$

Let us give examples of power functions.

1. $p(t) = t^\alpha$, $0 < \alpha < 1$.
2. $p(t) = t \cdot \ln^\beta \frac{1}{t}$, $\beta > 1$.
3. $p(t) = t \cdot \ln \frac{1}{t} \cdot \ln \ln \frac{1}{t} \cdot \cdots \cdot \underbrace{\ln \ldots \ln}_{k-1} \frac{1}{t} \cdot (\underbrace{\ln \ldots \ln}_{k} \frac{1}{t})^\beta$, $\beta > 1$.

In the set of power functions P it is possible to introduce a partial order. Assume $p_1(t), p_2(t) \in P$. We will write $p_1 \prec p_2$, if $p_1(t) \leqslant p_2(t)$, $t \in (0, 1]$, and $\frac{p_1(t)}{p_2(t)} \to 0$ with $t \to +0$.

It is possible to show (see [7]), that for every function $p \in P$ there is $p_1 \in P$ with the property, that $p_1 \prec p$.

On the other hand, the correlation $\prec$ in the set of power functions P is not an order: not for any $p_1, p_2 \in P$ it is possible to say, that $p_1 \prec p_2$ or $p_2 \prec p_1$ (see [8]).

In paper [7] there are assumed peculiarities of the coefficients $B(z)$ $(A(z) \equiv 0)$, which belong to power space of functions $S_p(G)$,, which is in its turn a generalization of the spaces

$$s_p(G) = \left\{ B(z) : \sup_{G \setminus \{0\}} (|B(z)| \cdot p(|z|)) < +\infty \right\}.$$

Note, that for such functions the following condition holds true $B(z) \in L_{\infty,loc}(G \setminus \{0\})$. The space $S_p(G)$ consists of only those given in G functions $B(z)$, for every of which there is such a function $p(t) \in P$, that $B(z) \in s_p(G)$. It is easy to show, that $S_p(G) \subset L_2(G)$.

According to the definition from [9, p. 41], the function $\omega(t)$, satisfying conditions

1. $\omega(t) \geq 0$ and does not decrease on $[0, 1]$;
2. $\omega(0) = 0$;
3. $\omega(t_1 + t_2) \leqslant \omega(t_1) + \omega(t_2)$;
4. $\omega(t)$ continuous on $[0, 1]$,

is called a modulus of continuity.

We will not demand satisfying of condition 4, and instead of condition 3 we will assume a stronger condition, that $\frac{\omega(t)}{t}$ does not increase with $t > 0$. It is obvious, that then $\omega(t)$ is half-additive. The set of all such functions will be denoted by Ω. Note, that for power functions from P,, generally speaking, the lack of growth condition does not hold true $\frac{p(t)}{t}$ when $t > 0$ (see [8]), and condition 4 of the functions of the class P, does not hold true for the modulus of continuity $\omega(t) = \begin{cases} t \cdot \ln \frac{1}{t}, & 0 < t \leqslant \frac{1}{e} \\ 0, & t = 0 \end{cases}$.

Let us now denote subsets $K \Subset \mathbb{C}$ and $\omega \in \Omega$ for the class of continuous functions $C_\omega(K)$, for the closed bounded subset , satisfying the condition

$$||f||_\omega := \max \left\{ \sup_K |f(t)|, \sup_{z_1 \neq z_2} \frac{|f(z_1) - f(z_2)|}{\omega(|z_1 - z_2|)} \right\} < \infty. \tag{2.2}$$

It is obvious, that the value (2.2) satisfies all the axioms of the norm. Moreover, the space $(C_\omega(K), || \cdot ||_\omega)$ is a Banach space (see, for instance, [10]). In case $\omega(t) = t^\lambda$ $(0 < \lambda < 1)$ we obtain Helder's space, and when $\omega(t) = t$ we deal with Lipschitz space.

3. Auxiliary statements

The following theorem is rather important in the proof of Theorem 1 in [7]:

Theorem 3. *Assume $b(z) \in S_p(G)$, then the function $T_G(b)(z)$ is continuous in the point $z = 0$.*

Here by $T_G(\cdot)$ we consider the basic operator of I. Vekua's theory, namely, the following:

$$T_G(b)(z) := -\frac{1}{\pi}\iint_G \frac{b(\zeta)}{\zeta - z}\, d\xi\, d\eta, \zeta = \xi + i\cdot\eta.$$

Moreover, the following property was applied.

Theorem 4. *Let $a(z)$ be a prescribed function from $L_\infty(G)$, $w(z) \in S_p(G)$. Then $T_G(a\cdot w) \in C^\lambda(\overline{G}\setminus\{0\})$ for any $\lambda \in (0,1)$, and*

$$|T_G(a\cdot w)(z)| \leqslant A_1(l,p)\cdot ||w||_p \cdot ||a||_{L_\infty(G)}, z \in \overline{G\setminus U_l}, \tag{3.1}$$

$$|T_G(a\cdot w)(z_1) - T_G(a\cdot w)(z_2)| \leqslant A_2(l,p,\lambda)\cdot ||w||_p\cdot ||a||_{L_\infty(G)}\cdot |z_1 - z_2|^\lambda, \tag{3.2}$$

where $U_l = \{z : |z| \leqslant \frac{1}{2^l}\}(l = 1,2,\ldots)$.

It results from (3.1) and (3.2), that for any $\lambda \in (0,l)$ and any $l \in \mathbb{N}$

$$||T_G(a\cdot w)||_{C^\lambda(\overline{G\setminus U_l})} \leqslant A(l,p,\lambda)\cdot ||w||_p\cdot ||a||_{L_\infty(G)}. \tag{3.3}$$

Remark. Whereas outside the circle U_l the function $a(z)\cdot w(z)$ is bounded, that instead of (3.2) we can state a more precise inequality (see [1, p. 39])

$$|T_G(a\cdot w)(z_1) - T_G(a\cdot w)(z_2)| \leqslant A_3(l,p)\cdot ||w||_p\cdot ||a||_{L_\infty(G)}\cdot |z_1 - z_2|\cdot \ln\frac{1}{|z_1 - z_2|}, \tag{3.4}$$

where $|z_1 - z_2| < \frac{1}{e}$, therefore, when the condition of the Theorem 4 holds true, the following estimate is valid:

$$||T_G(a\cdot w)||_{\omega_{1,1}} \leqslant A_4(l,p)\cdot ||w||_p\cdot ||a||_{L_\infty(G)} \tag{3.5}$$

in the sense of the space $C_{\omega_{1,1}}(\overline{G\setminus U_l})$.

Proving Theorem 1 we applied the Dirichlet problem solution for holomorphic functions, namely, the following (see [11, p. 131])

Theorem 5. *If the function g is given on ∂G and is continuous, according to Helder with the index λ ($0 < \lambda < 1$), then there is a unique holomorphic in G function f, continuous in the closed circle G and satisfying conditions*

$$\Re f = g(z), z \in \partial G, \Im f|_{z=z_0} = c, \tag{3.6}$$

where $z_0 \in \partial G$ is a prescribed point, and f is continuous, according to Helder in $\overline{G}$ with the same index λ, i.e. $f \in C^\lambda(\overline{G})$.

Remark. As it results from [11, p. 131], the following estimate is valid

$$||f||_{C^\lambda(\overline{G})} \leqslant A(\lambda)||f||_{C^\lambda(\partial G)}. \tag{3.7}$$

As it is shown in [10] and [12], analogues of Theorem 5 hold true for more general, than Helder, spaces of functions, of described modulus of continuity. In particular, it holds, that

Theorem 6. *If $g \in C_{\mu_{1,k}}(\partial G)$ ($k \geq 0$), then there is a unique holomorphic in G function f, satisfying (3.6), and $f \in C_{\mu_{1,k+2}}(\overline{G})$.*

The analogue of the inequality (3.7) in this case

$$||f||_{C_{\mu_{1,k+2}}(\overline{G})} \leqslant A\cdot ||f||_{C_{\mu_{1,k+2}}(\partial G)} \tag{3.8}$$

is used in section 4 on step 2 and step 3 with the application of a point immobility.

Remark. Theorems 5 and 6 hold true in case of real and imaginary parts change in the condition (3.6).

4. Algorithm of Theorem 2 proof

The proof of Theorem 2 is made according to algorithm of Theorem 1 proof (see [7, p. 661–662]).

1-st step. We search for the solution $w = w(z)$ (1.2)–(1.3) in the form

$$w(z) = \Phi(z) \cdot \exp \omega(z), \tag{4.1}$$

where $\Phi(z)$ is a holomorphic in G function, which is continuous in $\overline{G}$, and $\exp \omega(z) \in L_\infty(G)$. Substituting (4.1) into (1.2), we obtain the equation for $\omega = \omega(z)$:

$$\frac{\partial \omega}{\partial \bar{z}} + B(z) \cdot \frac{\overline{\Phi}(z)}{\Phi(z)} \cdot \frac{\overline{\exp \omega(z)}}{\exp \omega(z)} = 0, z \in G. \tag{4.2}$$

We select the solution (4.2) so, that the following conditions could hold true

$$\Im \omega|_{\partial G} = 0, \Re \omega|_{z_0=1} = 0. \tag{4.3}$$

From (4.2) we obtain ([1]) for the solution (4.2) representation

$$\omega(z) = \tilde{\Phi}(\omega, \Phi)(z) - T_G\left(B \cdot \frac{\overline{\Phi}}{\Phi} \cdot \frac{\overline{\exp \omega}}{\exp \omega}\right)(z), z \in G, \tag{4.4}$$

where $\tilde{\Phi}$ is an arbitrary holomorphic function, depending on ω and Φ. It results from Theorem 3, 4 and Remark, that $T_G\left(B \cdot \frac{\overline{\Phi}}{\Phi} \cdot \frac{\overline{\exp \omega}}{\exp \omega}\right) \in C(\overline{G}) \bigcap C_{\mu_{1,1}}(\overline{G} \setminus \{0\})$, and (see (3.5))

$$\left|\left|T_G\left(B \cdot \frac{\overline{\Phi}}{\Phi} \cdot \frac{\overline{\exp \omega}}{\exp \omega}\right)\right|\right|_{C(\overline{G})} \leqslant C_B, \left|\left|T_G\left(B \cdot \frac{\overline{\Phi}}{\Phi} \cdot \frac{\overline{\exp \omega}}{\exp \omega}\right)\right|\right|_{C_{\mu_{1,1}}(\overline{G \setminus U_l})} \leqslant C_{B,l}, \tag{4.5}$$

where constants C_B and $C_{B,l}$ do not depend on ω and Φ.

2-nd step. Further we select an arbitrary function $\tilde{\Phi}$ so, that the condition holds true (4.3), i.e.

$$\begin{cases} \Im \tilde{\Phi}\Big|_{\partial G} = \Im T_G\left(B \cdot \frac{\overline{\Phi}}{\Phi} \cdot \frac{\overline{\exp \omega}}{\exp \omega}\right)\Big|_{\partial G} \\ \Re \tilde{\Phi}\Big|_{z_0=1} = \Re T_G\left(B \cdot \frac{\overline{\Phi}}{\Phi} \cdot \frac{\overline{\exp \omega}}{\exp \omega}\right)\Big|_{z_0=1}. \end{cases} \tag{4.6}$$

The right-side part of the first correlation (4.6) is a function of class $C_{\mu_{1,1}}(\partial G)$. According to the Remark and Theorem 6, there is a unique holomorphic in G function $\tilde{\Phi}(z)$, which satisfies conditions (4.6), and

$$||\tilde{\Phi}||_{C_{\mu_{1,3}}(\overline{G})} \leqslant C \cdot ||\tilde{\Phi}||_{C_{\mu_{1,3}}(\partial G)}.$$

Therefore, the right-side part (4.4), and the left-side part, i.e. the function $\omega(z)$, for any $\Phi \in H(G) \bigcap C(\overline{G})$ and $\omega \in L_\infty(G)$ is a function of the class $C(\overline{G}) \bigcap C_{\mu_{1,3}}(\overline{G} \setminus \{0\})$. Further, applying Tikhonov theorem on a point immobility to the mapping

$$\omega \to F_1(\Phi, \omega) := \tilde{\Phi}(\omega, \Phi) - T_G\left(B \cdot \frac{\overline{\Phi}}{\Phi} \cdot \frac{\overline{\exp \omega}}{\exp \omega}\right),$$

we obtain, that for any $\Phi \in H(G) \bigcap C(\overline{G})$ there is a unique function $\omega(z) \in C(\overline{G}) \bigcap C_{\mu_{1,3}}(\overline{G} \setminus \{0\})$ with properties (4.2)–(4.3). As a result, we obtain, that for any $\Phi \in H(G) \bigcap C(\overline{G})$ there is a function $w(z)$ of the form (4.1), which is in G solution of the equation (4.1).

3-rd step. We select such a holomorphic in G function $\hat{\Phi} \in C(\overline{G})$, that the function $w = \hat{\Phi} \cdot \exp \omega$, where ω would be the function of the 2-nd step, and the following boundary conditions could hold true (1.3). For this purpose we will consider the mapping

$$\omega \to \hat{\Phi},$$

where

$$\Re\hat{\Phi} = \Re(w \cdot \exp(-\omega)) = \exp(-\Re\omega) \cdot g(z) = g_1(z), z \in \partial G,$$
$$\Im\hat{\Phi} = \exp(-\Re\omega(z_0 = 1)) \cdot \Im w(z_0 = 1) = h.$$

Note, that $g_1(z) \in C_{\mu_{1,3}}(\partial G)$. Therefore, we select a holomorphic in G function $\hat{\Phi}$, so, that

$$\Re\hat{\Phi}\Big|_{\partial G} = g_1(z), \Im\hat{\Phi}\Big|_{z_0=1} = h.$$

According to Theorem 6, there is such a function and it is unique. Moreover, $\hat{\Phi}(z) \in C_{\mu_{1,5}}(\overline{G})$.

To prove this solution existence, we will study the mapping

$$K : \Phi \in H(G) \bigcap C(\overline{G}) \to \omega = K_1(\Phi) \to \hat{\Phi} = K_2(\omega), \hat{\Phi} = K(\Phi),$$

where $\omega = K_1(\Phi)$ is an immobile point for $\omega = F_1(\omega, \Phi)$, and $\hat{\Phi}$ is solution of the given above Dirichlet problem. Applying Schauder theorem on a point immobility to the mapping $\hat{\Phi} = K(\Phi)$, likewise in [7], we obtain the proof of the solution existence (1.2)–(1.3).

4-th step. Likewise in [7], we can prove the uniqueness of the solution (1.2)–(1.3) in the class of solutions in the sense of Sobolev functions from $C(\overline{G})$.

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