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ON EXTREMAL TYPE OF AN ENTIRE FUNCTION OF ORDER LESS THAN UNITY WITH ZEROS OF PRESCRIBED DENSITIES AND STEP

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Abstract. Sharp lower estimate for the type of an entire function of the order $\rho \in (0;1)$ with respect to densities and step of its zeros located on the ray is proved.

Keywords: type of an entire function, the upper, lower densities and step of zeros.

Let us introduce necessary characteristics and give a precisely set problem. Let $\Lambda = (\lambda_n)_{n=1}^{\infty}$ be a written in a nondecreasing order of modules and going to infinity sequence of complex numbers; $n_\Lambda(R)$ be a number of elements from Λ, from the circle $|z| \leqslant R$. For the index $\rho > 0$ upper and lower ρ-densities are defined Λ :

$$\overline{\Delta}_\rho(\Lambda) := \varlimsup_{R\to+\infty} \frac{n_\Lambda(R)}{R^\rho}, \ \ \underline{\Delta}_\rho(\Lambda) := \varliminf_{R\to+\infty} \frac{n_\Lambda(R)}{R^\rho},$$

and also ρ-step of this sequence: $h_\rho(\Lambda) := \varliminf\limits_{n\to\infty} \left(|\lambda_{n+1}|^\rho - |\lambda_n|^\rho\right)$. The type of an entire function $f(z)$ with the order $\rho > 0$ is one of the basic characteristics of its increase and is specified by the formula

$$\sigma_\rho(f) := \varlimsup_{R\to+\infty} R^{-\rho} \ln \max_{|z|=R} |f(z)| .$$

We will be interested in the lowest possible increase of entire functions f of the order $\rho \in (0;1)$ with a number of positive zeros Λ_f. To be more precise, let us prescribe four numbers $\rho \in (0;1)$, $\beta > 0$, $\alpha \in [0;\beta]$, $h \in [0;\beta^{-1}]$ and specify the following extremal problem. To find a precise lower bound

$$s(\alpha,\beta,h;\rho) := \inf\left\{\sigma_\rho(f) : \Lambda = \Lambda_f \subset \mathbb{R}_+,\ \overline{\Delta}_\rho(\Lambda) = \beta,\ \underline{\Delta}_\rho(\Lambda) \geq \alpha,\ h_\rho(\Lambda) \geq h\right\}. \qquad (1)$$

Note, that the demand $0 \leqslant h \leqslant \beta^{-1}$ in the extremal problem specifying is not artificial. It results from the fact, that the inequality $\overline{\Delta}_\rho(\Lambda) h_\rho(\Lambda) \leqslant 1$, is easily checked by analogy to $\Lambda \subset \mathbb{R}_+$ and $\rho = 1$ in [1] constantly holds true.

Problem (1) is important in subjects of complex analysis, connected with completeness of functional systems, analytic extension and etc. A.U. Popov was the first who started to study such extremal problems [2].

The basic result of paper [2] consists in the fact, that the value

$$s(\beta;\rho) := \inf\left\{\sigma_\rho(f) :\ \Lambda = \Lambda_f \subset \mathbb{R}_+,\ \overline{\Delta}_\rho(\Lambda) = \beta\right\}$$

is equal to $\beta\, C(\rho)$, where $C(\rho) = \max\limits_{a>0} a^{-\rho} \ln(1+a)$. It gives an answer in problem (1) when $h = 0$ and $\alpha = 0$, whereas, it is obvious, that $s(0,\beta,0;\rho) = s(\beta;\rho)$.

The next step in solution of problem (1) can be considered in the recent result of V.B.Sherstyukov [3], who studied the case $\alpha > 0$ and obtained the value

$$s(\alpha\,,\beta\,;\rho) := \inf\left\{\sigma_\rho(f) :\ \Lambda = \Lambda_f \subset \mathbb{R}_+,\ \overline{\Delta}_\rho(\Lambda) = \beta,\ \underline{\Delta}_\rho(\Lambda) \geq \alpha\right\}.$$

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It turned out, that

$$s(\alpha\,,\beta\,;\rho)=\frac{\pi\alpha}{\sin\pi\rho}+\max_{a>0}\int\limits_{a(\alpha/\beta)^{1/\rho}}^{a}\frac{\beta a^{-\rho}-\alpha\tau^{-\rho}}{1+\tau}d\tau.$$

Hence, problem (1) is solved in case $h=0$ with arbitrary $\alpha\in[0;\beta]$.

We are implied to solve problem (1) in general case $h\in[0;\beta^{-1}]$ and $\alpha\in[0;\beta]$. In the article we prove a lower estimate for the type of an entire function of the order $\rho\in(0;1)$ with zeros on the ray and consider the construction scheme of the example, confirming precision of this estimate.

Thus, assume

$$\Lambda=(\lambda_n)_{n=1}^{\infty}\,,\ 0<\lambda_n\nearrow+\infty,\ \overline{\Delta}_{\rho}(\Lambda)=\varlimsup_{R\to+\infty}\frac{n_\Lambda(R)}{R^\rho}=\varlimsup_{n\to\infty}\frac{n}{\lambda_n^\rho}=\beta,$$

$$\underline{\Delta}_{\rho}(\Lambda)=\varliminf_{R\to+\infty}\frac{n_\Lambda(R)}{R^\rho}=\varliminf_{n\to\infty}\frac{n}{\lambda_n^\rho}\geq\alpha,\ h_{\rho}(\Lambda)=\varliminf_{n\to\infty}\left(\lambda_{n+1}^\rho-\lambda_n^\rho\right)\geq h>0.$$

The case $\alpha=0$, $h\in[0;\beta^{-1}]$ was studied in paper [4], therefore, further we consider, that $\alpha>0$.

Let $f(z)=\prod\limits_{n=1}^{\infty}\left(1-\dfrac{z}{\lambda_n}\right)$ be an entire function of the order $\rho\in(0;1)$ with zero set $\Lambda_f=\Lambda$.

Likewise in paper [3], we will consider function $\varphi_R(t)=\dfrac{n_\Lambda(Rt)}{(Rt)^\rho}$, $t>0$. with prescribed $R>0$. It is obvious, that within the frames of the problem conditions we obtain

$$\varlimsup_{t\to+\infty}\varphi_R(t)=\beta,\quad \varliminf_{t\to+\infty}\varphi_R(t)\geq\alpha.$$

In particular, arbitrarily prescribing $\alpha_1\in(0;\alpha)$ and $a>0$, we are going to find $c>0$ so, that with any $R\geq ac$ and $t\geq c/R$ the following inequality will hold true

$$\varphi_R(t)\geq\alpha_1. \tag{1}$$

For more convenience, we will introduce an auxiliary sequence $\Omega:=(\lambda_n^\rho)_{n=1}^{\infty}=(\mu_n)_{n=1}^{\infty}$. Taking into account the condition on ρ-step $h_{\rho}(\Lambda)$, we will write $\varliminf\limits_{n\to\infty}(\mu_{n+1}-\mu_n)\geq h$. Hence, for any $h_1\in(0;h)$ we obtain $\mu_{n+1}-\mu_n>h_1$ with $n\geq n_0=n_0(h_1)$ (we can consider, that $n_0\geq n_\Lambda(c^\rho)$). Estimating the standard way when $c/R\leqslant t\leqslant 1/a$ the number of points of the sequence Ω on the gap $[(Rt)^\rho\,;(R/a)^\rho]$ subject to its step, we obtain

$$n_\Omega\left((R/a)^\rho\right)-n_\Omega\left((Rt)^\rho\right)\leqslant\frac{1}{h_1}\left((R/a)^\rho-(Rt)^\rho\right).$$

Whereas $n_\Omega\left(x^\rho\right)=n_\Lambda(x)$, $x>0$, then, going back from Ω to Λ, we find

$$n_\Lambda\left(R/a\right)-n_\Lambda\left(Rt\right)\leqslant\frac{R^\rho}{h_1}\left(\frac{1}{a^\rho}-t^\rho\right).$$

Having divided this inequality on $(Rt)^\rho$ and having defined $\eta=\eta(R):=\varphi_R(1/a)$, we obtain the estimate

$$\eta\,(at)^{-\rho}-\varphi_R(t)\leqslant h_1^{-1}\left((at)^{-\rho}-1\right).$$

Hereof, assuming $b:=h_1^{-1}$, we finally obtain

$$\varphi_R(t)\geq b+(\eta-b)\,(at)^{-\rho}\,,\quad t\in[c/R;1/a]\,. \tag{2}$$

For $t\geq 1/a$ due to obvious inequality $n_\Lambda\left(Rt\right)\geq n_\Lambda\left(R/a\right)$ the following estimate holds true

$$\varphi_R(t) = \frac{n_\Lambda(Rt)}{(Rt)^\rho} \geq \frac{n_\Lambda(R/a)}{(Rt)^\rho} = \frac{n_\Lambda(R/a)}{(R/a)^\rho}\,(at)^{-\rho} = \eta\,(at)^{-\rho}\,, \text{ i.e.}$$

$$\varphi_R(t) \geq \eta\,(at)^{-\rho}\,, \quad t \geq 1/a\,. \tag{3}$$

Estimation of an entire function type $f(z)$ is based on integral representation of maximal logarithm of its module (see [3]):

$$F(R) := R^{-\rho} \ln \max_{|z|=R} |f(z)| = \int\limits_0^{+\infty} \varphi_R(t)\,\frac{t^{\rho-1}}{1+t}\,dt\,.$$

To simplify further results it is convenient to make a substitution of the variable $t = 1/\tau$, in the latter integral, that gives

$$F(R) = \int\limits_0^{+\infty} \varphi_R(1/\tau)\,\frac{\tau^{-\rho}}{1+\tau}\,d\tau\,. \tag{4}$$

Let us rewrite estimates (2) — (4), obtained with any $R \geq ac$, in gaps of the variable change τ :

$$\varphi_R(1/\tau) > \alpha_1\,, \quad \tau \in (0;\, R/c]\ ; \tag{5}$$

$$\varphi_R(1/\tau) \geq b + (\eta - b)a^{-\rho}\tau^\rho\,, \quad \tau \in [a;\, R/c]\ ; \tag{6}$$

$$\varphi_R(1/\tau) \geq \eta a^{-\rho}\tau^\rho\,, \quad \tau \in (0; a]. \tag{7}$$

To obtain a precise lower estimate of the integral in (5) we will need to divide the gap $(0;\, R/c]$ into parts, in which we choose the largest of right-side parts of the corresponding inequalities (6) — (8). Denoting the result of such a procedure of the best choice by $\psi_R(\tau)$ (comparing in advance in turn estimate (6) with estimates (7) and (8) and finding points $\tau_1 = a\,(\alpha_1/\eta)^{1/\rho}$, $\tau_2 = a\,((b-\alpha_1)/(b-\eta))^{1/\rho}$), we obtain the inequality

$$\varphi_R(1/\tau) \geq \psi_R(\tau),\ \tau \in (0;\, R/c]\,,\ R \geq ac, \tag{8}$$

where

$$\psi_R(\tau) = \begin{cases} \alpha_1,\ \tau \in E := (0;\tau_1] \cup (\tau_2; R/c]\,, \\ \eta a^{-\rho}\tau^\rho,\ \tau \in (\tau_1; a), \\ b + (\eta - b)a^{-\rho}\tau^\rho,\ \tau \in [a; \tau_2]. \end{cases} \tag{9}$$

Gaps in definition (10) of the function $\psi_R(\tau)$ are not empty with sufficiently large R. Indeed, the following $\alpha_1 < \eta = \varphi_R(1/a) < b$, holds true for such R, that results from (6) with $\tau = a$, and also from the inequality $h\beta \leqslant 1$ and the choice of h_1. From (5) and (9) with $R \geq ac$ we obtain

$$F(R) \geq \int\limits_0^{R/c} \varphi_R(1/\tau)\,\frac{\tau^{-\rho}}{1+\tau}\,d\tau \geq \int\limits_0^{R/c} \psi_R(\tau)\,\frac{\tau^{-\rho}}{1+\tau}\,d\tau\,.$$

Substituting into the latter integral the expression from definition (10), we will write the following transforms:

$$\int\limits_0^{R/c} \psi_R(\tau)\,\frac{\tau^{-\rho}}{1+\tau}\,d\tau = \alpha_1 \int\limits_E \frac{\tau^{-\rho}}{1+\tau}\,d\tau + \int\limits_{\tau_1}^{a} \frac{\eta a^{-\rho}}{1+\tau}\,d\tau$$

$$+\int\limits_a^{\tau_2} \left(b + (\eta - b)a^{-\rho}\tau^{\rho}\right)\,\frac{\tau^{-\rho}}{1+\tau}\,d\tau = \alpha_1 \left(\int\limits_0^{+\infty} - \int\limits_{\tau_1}^{\tau_2} - \int\limits_{R/c}^{+\infty}\right) \frac{\tau^{-\rho}}{1+\tau}\,d\tau + \int\limits_{\tau_1}^{a} \frac{\eta a^{-\rho}}{1+\tau}\,d\tau$$

$$+\int\limits_a^{\tau_2} \frac{b\tau^{-\rho} + (\eta - b)a^{-\rho}}{1+\tau}\,d\tau = \frac{\pi\alpha_1}{\sin\pi\rho} - \alpha_1 \int\limits_{\tau_1}^{a} \frac{\tau^{-\rho}}{1+\tau}\,d\tau - \alpha_1 \int\limits_a^{\tau_2} \frac{\tau^{-\rho}}{1+\tau}\,d\tau + o(1)$$

$$+\int\limits_{\tau_1}^{a} \frac{\eta a^{-\rho}}{1+\tau}\,d\tau + \int\limits_a^{\tau_2} \frac{b\tau^{-\rho} + (\eta - b)a^{-\rho}}{1+\tau}\,d\tau, \quad R \to +\infty.$$

Grouping the first integral with the third, and the second with the fourth, we obtain the estimate

$$F(R) \geq \frac{\pi\alpha_1}{\sin\pi\rho} + \int\limits_{\tau_1}^{a} \frac{\eta a^{-\rho} - \alpha_1\tau^{-\rho}}{1+\tau}\,d\tau + \int\limits_a^{\tau_2} \frac{(b-\alpha_1)\tau^{-\rho} + (\eta - b)a^{-\rho}}{1+\tau}\,d\tau + o(1),\ R \to +\infty.$$

Now we proceed to the upper bound of the sequence $R_j \to +\infty$, for which $\eta = \eta(R_j) \to \beta$. Then we will direct α_1 to α, h_1 to h and take into account continuous dependence of values τ_1, τ_2 and b from their arguments. We will obtain in the result, that $b \to \dfrac{1}{h}$, $\tau_1 \to a\,(\alpha/\beta)^{1/\rho}$, $\tau_2 \to a\,((1-\alpha h)/(1-\beta h))^{1/\rho}$, and with any $a > 0$ the following holds true

$$\sigma_\rho(f) \geq \varlimsup_{R_j \to +\infty} F(R_j) \geq \frac{\pi\alpha}{\sin\pi\rho} + \int\limits_{a(\alpha/\beta)^{1/\rho}}^{a} \frac{\beta a^{-\rho} - \alpha\tau^{-\rho}}{1+\tau}d\tau + \frac{s}{h}\int\limits_a^{a\nu^{1/\rho}} \frac{\nu\tau^{-\rho} - a^{-\rho}}{1+\tau}d\tau$$

(for short we assumed $s := 1 - \beta h$ and $\nu := (1-\alpha h)/(1-\beta h)$). Taking into account the parameter arbitrariness $a > 0$, we will finally obtain

$$\sigma_\rho(f) \geq \frac{\pi\alpha}{\sin\pi\rho} + \sup_{a>0}\left\{\int\limits_{a(\alpha/\beta)^{1/\rho}}^{a} \frac{\beta a^{-\rho} - \alpha\tau^{-\rho}}{1+\tau}d\tau + \frac{s}{h}\int\limits_a^{a\nu^{1/\rho}} \frac{\nu\tau^{-\rho} - a^{-\rho}}{1+\tau}d\tau\right\}. \qquad (10)$$

To complete the proof, it is necessary to give an example of sequence Λ, on which estimate (11) is obtained. The general principle of such examples construction is considered in detail in [3]. Points of extremal sequence are set on gaps of different types, each of which is responsible for a definite characteristic of sequence expansion. The example construction in our case does not differ from consideration from paper [3], but technically it is somehow more complicated due to embedding of gaps into the construction, which enable a rear step. A detailed foundation of such a construction is extremely bulky and it would occupy more place, then proof of the inequality itself (11), and, therefore, it is considered not expedient to be given here.

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