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# ON A SPACE OF ENTIRE FUNCTIONS FAST DECREASING ON A REAL LINE

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**Abstract.** A space of entire functions decreasing fast on a real line is introduced. It contains the space of the Fourier-Laplace transforms of infinitely differentiable functions on a real line with a compact support as a proper subspace. The Fourier-Laplace transform of functions of this space is studied. Equivalent description in terms of estimates of derivatives of functions on a real line is obtained for the considered space.

**Keywords:** The Fourier-Laplace transform, entire functions, Paley-Wiener type theorem.

## 1. Introduction

**1.1. Problem setting**. In theory of generalized functions, theory of differential equations there is a significant interest given to infinitely differential functions and functions, fast decreasing on a real line. To solve different problems of analysis in such spaces, it is possible to use great opportunities, given by Fourier or Laplace transform. For some spaces of infinitely differential functions (including entire functions), functions, fast decreasing on a real line, these opportunities are demonstrated in papers of I.M. Gelfand and G.E. Shilov [1], B.L. Gurevich [2], G.E . Shilov [3], L. Hormander [4], K.I. Babenko [5], [6], R.S. Yulmukhametov [7], [8], A.M. Sedletsky [9], in books of I.M. Gelfand and G.E. Shilov [10], M.A. Evgrafov [11].

In the given paper we introduce a new class of entire functions spaces, fast decreasing on a real line, and the Fourier-Laplace transform of functions from this space is also studied here. These spaces are defined the following way. Everywhere further $\varphi$ will be a nonnegative continuous function on $[0,\infty)$, satisfying the conditions:

1) $\varphi(x)=0$ for $x\in[0,e]$;

2) $\lim\limits_{x\to+\infty}\dfrac{\varphi(x)}{x}=+\infty$;

3) function $\psi(x)=\varphi(e^x)$ is convex on $[0,\infty)$;

4) there are numbers $h>1$ and $K>0$ such that

$$2\varphi(x)\le\varphi(hx)+K,\ x\in[0,\infty).$$

Suppose $H(\mathbb{C})$ be a space of entire functions of a complex variable.

For arbitrary $\varepsilon>0$ and $k\in\mathbb{Z}_+$ assume

$$S_{\varepsilon,k}(\varphi)=\{f\in H(\mathbb{C}):p_{\varepsilon,k}(f)=\sup_{z\in\mathbb{C}}\frac{|f(z)|(1+|z|)^k}{e^{\varphi(\varepsilon|Im\ z|)}}<\infty\}.$$

Suppose $S(\varphi)=\bigcap\limits_{\varepsilon>0,k\in\mathbb{Z}_+}S_{\varepsilon,k}(\varphi)$. With ordinary operations of addition and multiplication by complex numbers $S(\varphi)$ is a linear space. Let us give $S(\varphi)$ topology, defined by the family of norms $p_{\varepsilon,k}$ $(\varepsilon>0,\ k\in\mathbb{Z}_+)$.

Note, that the function $f\in H(\mathbb{C})$ belongs to $S(\varphi)$ only when for any $\varepsilon>0,k\in\mathbb{Z}_+$ the following value is finite

$$q_{\varepsilon,k}(f)=\sup_{z\in\mathbb{C}}\frac{|f(z)z^k|}{e^{\varphi(\varepsilon|Im\ z|)}}\ .$$



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Indeed, for any $\varepsilon > 0$, $k \in \mathbb{Z}_+$ and $f \in H(\mathbb{C})$ the following inequalities occur:

$$q_{\varepsilon,k}(f) \le p_{\varepsilon,k}(f),$$
$$p_{\varepsilon,k}(f) \le 2^k \max_{0\le m\le k} q_{\varepsilon,m}(f).$$

Obviously, $S(\varphi)$ contains Fourier-Laplace transforms of infinitely differentiable functions on the real line with a compact support as its own space. It is easy to show, that operators of differentiation, shift and multiplication by polynomials are continuous in $S(\varphi)$.

The space $S(\varphi)$ represents a new class of entire functions, fast decreasing on a real line. It differs from the space of the form $W^\Omega$, studied in [10] and initially introduced by B.L. Gurevich [2]. Indeed, the spaces of the form $W^\Omega$ are introduced the following way. On the growing continuous unlimited function $w$ on $[0,\infty)$ such that $w(0) = 0$, we can define the function $\Omega$ on $[0,\infty)$: $\Omega(y) = \int\limits_0^y w(\xi)\ d\xi,\ y \ge 0$. Note, that $\Omega$ is a convex continuous function on $[0,\infty)$ and $\lim\limits_{y\to+\infty} \dfrac{\Omega(y)}{y} = +\infty$. The space $W^\Omega$ consists of functions $f \in H(\mathbb{C})$, for which there is a number $b > 0$ such that for all $k \in \mathbb{Z}_+$ with some $C_k > 0$

$$|z^k f(z)| \le C_k e^{\Omega(b|y|)},\ z \in \mathbb{C}.$$

The aim of the paper is to give an equivalent description of the space $S(\varphi)$ in terms of estimates of derivative functions on the real line and to study the Fourier-Laplace transform from $S(\varphi)$.

**1.2. Basic results**. For the arbitrary real-valued continuous function $g$ on $[0,\infty)$ such that $\lim\limits_{x\to+\infty} \dfrac{g(x)}{x} = +\infty$, assume $g^*(x) = \sup\limits_{y>0}(xy - g(y))$ be the function, conjugate by Jung with $g$ [11], $g[e](x) = g(e^x),\ x \ge 0$.

The following two theorems (proved in section 3) allow to give another description of the space $S(\varphi)$.

**Theorem 1.** *Suppose $f \in S(\varphi)$. Then $f \in C^\infty(\mathbb{R})$ and $\forall\varepsilon > 0$ $\forall m \in \mathbb{Z}_+$ $\exists c_{\varepsilon,m} > 0$ $\forall n \in \mathbb{Z}_+$ $\forall x \in \mathbb{R}$*

$$|x^m f^{(n)}(x)| \le c_{\varepsilon,m} n! \varepsilon^n e^{-\psi^*(n)}\ .$$

**Theorem 2.** *Suppose $f \in C^\infty(\mathbb{R})$ and for any $\varepsilon > 0, m \in \mathbb{Z}_+$ there is a number $d > 0$ such that for any $n \in \mathbb{Z}_+$*

$$(1 + |x|)^m |f^{(n)}(x)| \le d\varepsilon^n n! e^{-\psi^*(n)},\ x \in \mathbb{R}.$$

*Then $f$ (uniquely) extends up to the entire function from $S(\varphi)$.*

For $\varepsilon > 0, m \in \mathbb{Z}_+$ assume

$$G_{\varepsilon,m}(\psi^*) = \{f \in C^m(\mathbb{R}):$$
$$\|f\|_{\varepsilon,m} = \max_{0\le n\le m} \max\left(\sup_{x\in\mathbb{R}} |f^{(n)}(x)|, \sup_{x\in\mathbb{R},k\in\mathbb{N}} \frac{|x^k f^{(n)}(x)|}{k!\varepsilon^k e^{-\psi^*(k)}}\right) < \infty\}.$$

Assume $G(\psi^*) = \bigcap\limits_{\varepsilon>0,m\in\mathbb{Z}_+} G_{\varepsilon,m}(\psi^*)$. With ordinary operations of addition and multiplication by complex numbers $G(\psi^*)$ is a linear space. Let us give $G(\psi^*)$ topology, defined by the family of norms $\|f\|_{\varepsilon,m}$ ($\varepsilon >\ 0,\ m \in \mathbb{Z}_+$).

Let us define the Fourier transform of the function $f \in S(\varphi)$ by formula

$$\tilde{f}(x) = \int_{\mathbb{R}} f(\xi) e^{-ix\xi}\ d\xi,\ x \in \mathbb{R}.$$

It was proved in Section 4

**Theorem 3.** *The Fourier transform sets an isomorphism of the spaces $S(\varphi)$ and $G(\psi^*)$.*

For the case, when the function $\varphi$ is convex on $[0,\infty)$, it is shown in Section 5, that the space $G(\psi^*)$ admits a more simple description.

**Theorem 4.** *Let the function $\varphi$ be convex on $[0,\infty)$. Then the space $G(\psi^*)$ consists of functions $f \in C^\infty(\mathbb{R})$ such that for any $\varepsilon > 0$, $n \in \mathbb{Z}_+$ there is a constant $C_{\varepsilon,n} > 0$ such that*

$$|f^{(n)}(x)| \le C_{\varepsilon,n} e^{-\varphi^*(\frac{|x|}{\varepsilon})}, \ x \in \mathbb{R}.$$

## 2. Auxiliary results

Assume $a = \ln h$ and note, that condition 4) for the function $\varphi$ is equivalent to the following condition on $\psi$:

$$2\psi(x) \le \psi(x+a) + K, \ x \ge 0.$$

**Lemma 1.** *For any $M > 0$ there is a constant $C_M > 0$ such that*

$$\psi^*(x) \le x \ln \frac{x}{M} - x + C_M, \ x > 0.$$

**Proof**. It results from the definition of the function $\psi$ and condition 2) on $\varphi$, that for any $M > 0$ there is a constant $C_M > 0$ such that for all $y \ge 0$ $\psi(y) \ge Me^y - C_M$. Consequently,

$$\psi^*(x) = \sup_{y>0}(xy - \psi(y)) \le \sup_{y>0}(xy - Me^y) + C_M \le$$

$$\le \sup_{y\in\mathbb{R}}(xy - Me^y) + C_M = x \ln \frac{x}{M} - x + C_M.$$

From Lemma 1 we obtain the following

**Corollary 1.** *With any $\varepsilon > 0$ the series $\sum_{j=0}^{\infty} \frac{e^{\psi^*(j)}}{\varepsilon^j j!}$ conjugates.*

**Lemma 2.** *Suppose $\tau > 0$, and $g$ be a convex continuous function on $[0,\infty)$ such that $\lim_{x\to+\infty} \frac{g(x)}{x} = +\infty$. Then with some $C > 0$*

$$2g(x) \le g(x+\tau) + C, \ x \ge 0, \tag{1}$$

*only when there is a constant $A > 0$ such that*

$$g^*(x+y) \le g^*(x) + g^*(y) + \tau(x+y) + A, \ x, y \ge 0. \tag{2}$$

**Proof**. **Necessity**. Let us first note, that

$$g^*(x) \ge -\inf_{\xi\ge 0} g(\xi), \ x \ge 0. \tag{3}$$

Further, for arbitrary $x, y, t \in [0,\infty)$

$$g^*(x) + g^*(y) \ge (x+y)t - 2g(t).$$

According to (1), with any $x, y, t \ge 0$

$$g^*(x) + g^*(y) \ge (x+y)(t+\tau) - g(t+\tau) - C - \tau(x+y).$$

Consequently, with any $x, y \in [0,\infty)$

$$g^*(x) + g^*(y) \ge \sup_{\xi\ge\tau}((x+y)\xi - g(\xi)) - C - \tau(x+y). \tag{4}$$

Further, with any $x, y \in [0,\infty)$

$$\sup_{0\le\xi<\tau}((x+y)\xi - g(\xi)) \le (x+y)\tau - \inf_{0\le\xi<\tau} g(\xi) \le (x+y)\tau - \inf_{\xi\ge 0} g(\xi).$$

Subject to (3) we obtain

$$\sup_{0\le\xi<\tau}((x+y)\xi - g(\xi)) \le (x+y)\tau + g^*(x) \le$$

$$\leq (x+y)\tau + g^*(x) + g^*(y) + \inf_{\xi \geq 0} g(\xi).$$

Hereof and from inequality (4), assuming $A = \max(C, \inf_{\xi \geq 0} g(\xi))$, we obtain

$$g^*(x+y) \leq g^*(x) + g^*(y) + \tau(x+y) + A, \ \ x, y \geq 0.$$

**Sufficiency**. According to formula of inversion of the Jung transform [11] $g = (g^*)^*$. Applying this and (2), we obtain

$$2g(x) = \sup_{u \geq 0}(2xu - 2g^*(u)) \leq \sup_{u \geq 0}(2xu - g^*(2u) + 2\tau u + A) =$$

$$= \sup_{u \geq 0}((x+\tau)t - g^*(t)) + A = g(x+\tau) + A.$$

It remains to suppose $C = A$. the proof is complete.

The space $G(\psi^*)$ can be described the following way. For $\varepsilon > 0$, $m \in \mathbb{Z}_+$ assume

$$Q_{\varepsilon,m}(\psi^*) = \{f \in C^m(\mathbb{R}) : s_{\varepsilon,m}(f) = \max_{0 \leq n \leq m} \sup_{x \in \mathbb{R}, k \in \mathbb{Z}_+} \frac{(1+|x|)^k |f^{(n)}(x)|}{k! \varepsilon^k e^{-\psi^*(k)}} < \infty\}.$$

Assume $Q(\psi^*) = \bigcap_{\varepsilon > 0, m \in \mathbb{Z}_+} Q_{\varepsilon,m}(\psi^*)$. Let us give $Q(\psi^*)$ topology, defined by the family of norms $s_{\varepsilon,m}$ ($\varepsilon > 0$, $m \in \mathbb{Z}_+$).

It is valid

**Lemma 3.** *$Q(\psi^*) = G(\psi^*)$.*

**Proof**. Assume $f \in Q(\psi^*)$. Then for any $\varepsilon > 0, m \in \mathbb{Z}_+$ $\|f\|_{\varepsilon,m} \leq s_{\varepsilon,m}(f)$. It implies, that $f \in G(\psi^*)$. Moreover, the mapping of the insertion $I : Q(\psi^*) \to G(\psi^*)$ is continuous.

Let us now $f \in G(\psi^*)$. Then $\forall \varepsilon > 0 \ \forall m \in \mathbb{Z}_+ \ \|f\|_{\frac{\varepsilon}{2},m} < \infty$. Consequently, whatever $m \in \mathbb{Z}_+$ could be for $n \in \mathbb{Z}_+$ such that $0 \leq n \leq m$

$$|f^{(n)}(x)| \leq \|f\|_{\frac{\varepsilon}{2},m}, \ \ x \in \mathbb{R}. \tag{5}$$

Note, that for $n \in \mathbb{Z}_+$ such that $0 \leq n \leq m$

$$\sup_{|x| \leq 1, k \in \mathbb{Z}_+} \frac{(1+|x|)^k |f^{(n)}(x)|}{k! \varepsilon^k e^{-\psi^*(k)}} \leq \sup_{|x| \leq 1, k \in \mathbb{Z}_+} \frac{2^k |f^{(n)}(x)|}{k! \varepsilon^k e^{-\psi^*(k)}} \ .$$

Whereas $\lim_{k \to \infty} \frac{2^k}{k! \varepsilon^k e^{-\psi^*(k)}} = 0$, then there is a number $C(\varepsilon) > 1$ such that for $n \in \mathbb{Z}_+$ such that $0 \leq n \leq m$

$$\sup_{|x| \leq 1, k \in \mathbb{Z}_+} \frac{(1+|x|)^k |f^{(n)}(x)|}{k! \varepsilon^k e^{-\psi^*(k)}} \leq C(\varepsilon) \sup_{|x| \leq 1} |f^{(n)}(x)|.$$

Consequently, for any $f \in G(\psi^*)$ with any $m \in \mathbb{Z}_+$

$$\max_{0 \leq n \leq m} \sup_{|x| \leq 1, k \in \mathbb{Z}_+} \frac{(1+|x|)^k |f^{(n)}(x)|}{k! \varepsilon^k e^{-\psi^*(k)}} \leq C(\varepsilon) \|f\|_{\frac{\varepsilon}{2},m}. \tag{6}$$

Further, for $n \in \mathbb{Z}_+$ such that $0 \leq n \leq m$

$$\sup_{|x| > 1, k \in \mathbb{Z}_+} \frac{(1+|x|)^k |f^{(n)}(x)|}{k! \varepsilon^k e^{-\psi^*(k)}} \leq \sup_{|x| > 1, k \in \mathbb{Z}_+} \frac{(2|x|)^k |f^{(n)}(x)|}{k! \varepsilon^k e^{-\psi^*(k)}} \leq \|f\|_{\frac{\varepsilon}{2},m}. \tag{7}$$

It results from estimates (5) – (7), that for any $f \in G(\psi^*)$ with any $\varepsilon > 0, m \in \mathbb{Z}_+$

$$s_{\varepsilon,m}(f) \leq C(\varepsilon) \|f\|_{\frac{\varepsilon}{2},m}.$$

Thereby we state a topological equality $Q(\psi^*) = G(\psi^*)$.

## 3. Equivalent description of the space $S(\varphi)$

**Proof of Theorem 1**. Assume $f \in S(\varphi)$. Applying Cauchy integral formula, we obtain with any $m, n \in \mathbb{Z}_+$

$$(1+|x|)^m f^{(n)}(x) = \frac{n!}{2\pi i}\int_{L_R(x)} \frac{(1+|x|)^m f(\zeta)}{(\zeta - x)^{n+1}}\,d\zeta,\ x \in \mathbb{R},$$

where for $R > 0$ $L_R(x) = \{\zeta \in \mathbb{C} : |\zeta - x| = R\}$. Hereof with any $R > 0$ and $\varepsilon > 0$

$$(1+|x|)^m |f^{(n)}(x)| \le n! \max_{\zeta \in L_R} \frac{(1+|\zeta - x|)^m (1+|\zeta|)^m |f(\zeta)|}{R^n} \le$$

$$\le n! p_{\varepsilon,m}(f) \frac{(1+R)^m e^{\varphi(\varepsilon R)}}{R^n}\ .$$

Applying conditions 2) and 4) for the function $\varphi$, we obtain with some $c_{\varepsilon,m} >\ 0$ for any $R > 0$

$$(1+|x|)^m |f^{(n)}(x)| \le c_{\varepsilon,m} n! p_{\varepsilon,m}(f) \frac{e^{\varphi(\varepsilon h R)}}{R^n} = c_{\varepsilon,m} n! p_{\varepsilon,m}(f)(\varepsilon h)^n \frac{e^{\varphi(\varepsilon h R)}}{(\varepsilon h R)^n}\ .$$

Consequently, for any $x \in \mathbb{R}$

$$(1+|x|)^m |f^{(n)}(x)| \le c_{\varepsilon,m} n! p_{\varepsilon,m}(f)(\varepsilon h)^n \inf_{R \ge 1} \frac{e^{\varphi(R)}}{R^n} =$$

$$= c_{\varepsilon,m} n! p_{\varepsilon,m}(f)(\varepsilon h)^n \exp(-\sup_{R\ge 1}(n \ln R - \varphi(R))) =$$

$$= c_{\varepsilon,m} n! p_{\varepsilon,m}(f)(\varepsilon h)^n \exp(-\sup_{r \ge 0}(nr - \psi(r))) = c_{\varepsilon,m} n! p_{\varepsilon,m}(f)(\varepsilon h)^n e^{-\psi^*(n)}.$$

Theorem 1 has been proved.

**Proof of Theorem 2**. Suppose $f \in C^\infty(\mathbb{R})$ and for any $\varepsilon > 0, m \in \mathbb{Z}_+$ there be a number $d > 0$ such that for any $n \in \mathbb{Z}_+$

$$(1+|x|)^m |f^{(n)}(x)| \le d\varepsilon^n n! e^{-\psi^*(n)},\ x \in \mathbb{R}. \tag{8}$$

In particular, $|f^{(n)}(x)| \le d\varepsilon^n n!,\ x \in \mathbb{R}$. Obviously, the sequence $\left(\sum_{n=0}^{k} \frac{f^{(n)}(0)}{n!} x^n\right)_{k=0}^{\infty}$ conjugates to $f$ uniformly on the compacts of a number scale, and the series $\sum\limits_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n$ conjugates uniformly on the compacts in $\mathbb{C}$ and, consequently, the sum $F_f(z)$ of this series is an entire function in $\mathbb{C}$. Note, that $F_f(x) = f(x),\ x \in \mathbb{R}$. Thus, we have obtained an analytic extension of the function $f$ up to the entire function in $\mathbb{C}$ of the function $F_f$.

Applying the equality

$$F_f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x)}{n!}(iy)^n,\ z = x + iy\ (x, y \in \mathbb{R}),$$

and the inequality (8), we will estimate the growth of $F_f$. For any $\varepsilon > 0, m \in \mathbb{Z}_+$

$$(1+|z|)^m |F_f(z)| \le \sum_{n=0}^{\infty} \frac{(1+|x|)^m (1+|y|)^{m+n} |f^{(n)}(x)|}{n!} \le$$

$$\le \sum_{n=0}^{\infty} \frac{d\varepsilon^n}{e^{\psi^*(n)}}(1+|y|)^{n+m} \le d(1+|y|)^m \sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n e^{\sup\limits_{n\ge 0}(n\ln(2\varepsilon(1+|y|)) - \psi^*(n))} \le$$

$$\le 2d(1+|y|)^m e^{\sup\limits_{x \ge 0}(x\ln(2\varepsilon(1+|y|)) - \psi^*(x))} = 2d(1+|y|)^m e^{\psi(\ln(2\varepsilon(1+|y|)))}.$$

In the end of this inequality there was applied a convexity $\psi$. Finally, we obtain with all $\varepsilon > 0, m \in \mathbb{Z}_+$,

$$(1+|z|)^m |F_f(z)| \le 2d(1+|y|)^m e^{\varphi(2\varepsilon(1+|y|))},\ z \in \mathbb{C}. \tag{9}$$

Whereas a never-decreasing function $\varphi$ satisfies conditions 2) and 4), then it is possible to find a constant $C_{\varepsilon,m,\varphi} > 0$ (depending on $\varepsilon$, $m$ and $\varphi$) such that everywhere in $\mathbb{C}$

$$(1+|z|)^m|F(z)| \le C_{\varepsilon,m,\varphi} d e^{\varphi(4\varepsilon h|y|)}.$$

Due to arbitrariness of numbers $b > 0, m \in \mathbb{Z}_+$ we make a conclusion, that $F_f \in S(\varphi)$. The uniqueness of the continuation results from the theorem of uniqueness for analytical functions.

Theorem 2 has been proved.

## 4. On Fourier transforms from the space $S(\varphi)$

**Proof of Theorem 3**. Assume $f \in S(\varphi)$. Then $\forall \varepsilon > 0\ \forall x \in \mathbb{R}$

$$|\tilde{f}^{(n)}(x)| \le \int_{\mathbb{R}} |f(\xi)||\xi|^n \, d\xi \le \int_{\mathbb{R}} \frac{|f(\xi)|(1+|\xi|)^{n+2}}{1+\xi^2} d\xi \le \pi p_{\varepsilon,n+2}(f) \ . \tag{10}$$

Whereas with any $m \in \mathbb{N}$, $n \in \mathbb{Z}_+$, $x, \eta \in \mathbb{R}$

$$x^m \tilde{f}^{(n)}(x) = x^m \int_{\mathbb{R}} f(\zeta)(-i\zeta)^n e^{-ix\zeta} \, d\xi, \ \zeta = \xi + i\eta,$$

then

$$|x^m \tilde{f}^{(n)}(x)| \le \int_{\mathbb{R}} |f(\zeta)||\zeta|^n e^{x\eta}|x|^m \, d\xi \le \int_{\mathbb{R}} |f(\zeta)|(1+|\zeta|)^{n+2} e^{x\eta}|x|^m \ \frac{d\xi}{1+\xi^2} \ .$$

Let us consider the case $x \ne 0$. Assume $\eta = -\frac{x}{|x|}t, \ t > 0$. Then with any $t > 0$, $\varepsilon > 0$

$$|x^m \tilde{f}^{(n)}(x)| \le \pi p_{\varepsilon,n+2}(f) e^{-t|x|} e^{\varphi(\varepsilon t)}|x|^m \le$$

$$\le \pi p_{\varepsilon,n+2}(f) e^{\sup\limits_{r>0}(-tr+m\ln r)} e^{\varphi(\varepsilon t)} \le \pi p_{\varepsilon,n+2}(f) e^{m\ln m - m - m\ln t} e^{\varphi(\varepsilon t)}.$$

Let us proceed to precise lower bound in all $t > 0$ in the right part of this inequality (the left part depends on $t$). Whereas

$$\inf_{t>0}(-m\ln t + \varphi(\varepsilon t)) = m\ln\varepsilon + \inf_{u>0}(-m\ln u + \varphi(u)) = m\ln\varepsilon - \sup_{u>0}(m\ln u - \varphi(u)) =$$

$$= m\ln\varepsilon - \sup_{u\ge1}(m\ln u - \varphi(u)) = m\ln\varepsilon - \psi^*(m),$$

then

$$|x^m \tilde{f}^{(n)}(x)| \le \pi p_{\varepsilon,n+2}(f)\varepsilon^m e^{m\ln m - m} e^{-\psi^*(m)}. \tag{11}$$

If $x = 0$, then for $m \in \mathbb{N}$ and for any $n \in \mathbb{Z}_+$ $x^m \tilde{f}^{(n)}(x) = 0$, Hereof, from estimates (10) and (11), and taking into account, that $m^m \le e^m m!$ for all $m \in \mathbb{N}$, we obtain with any $\varepsilon > 0, k \in \mathbb{Z}_+$ $\|\tilde{f}\|_{\varepsilon,k} \le \pi p_{\varepsilon,k+2}(f), \ f \in S(\varphi)$. It implies, that the linear mapping $\mathcal{F} : S(\varphi) \to G(\psi^*)$, acting by the rule: $f \in S(\varphi) \to \tilde{f}$, is continuous.

Let us show, that $\mathcal{F}$ is surjective. Assume $g \in G(\psi^*)$. Then (applying Lemma 3) with any $\varepsilon > 0$, $k, m \in \mathbb{Z}_+$ and $n \in \mathbb{Z}_+$ such that $n \le m$

$$(1+|x|)^k|g^{(n)}(x)| \le s_{\varepsilon,m}(g)\varepsilon^k k! e^{-\psi^*(k)}, \ x \in \mathbb{R}.$$

Suppose $f(\xi) = \frac{1}{2\pi}\int_{\mathbb{R}} g(x)e^{ix\xi} \, dx, \ \xi \in \mathbb{R}$. For any $n \in \mathbb{Z}_+$

$$f^{(n)}(\xi) = \frac{1}{2\pi}\int_{\mathbb{R}} g(x)(ix)^n e^{ix\xi} \, dx, \ \xi \in \mathbb{R}.$$

Hereof, (integrated in parts) for any $m \in \mathbb{Z}_+$

$$(i\xi)^m f^{(n)}(\xi) = \frac{1}{2\pi}(-1)^m \int_{\mathbb{R}} (g(x)(ix)^n)^{(m)} e^{ix\xi} \, dx, \ \xi \in \mathbb{R}.$$

Assume $r = \min(m, n)$. Then

$$(i\xi)^m f^{(n)}(\xi) = \frac{1}{2\pi}(-1)^m \int_{\mathbb{R}} \sum_{j=0}^{r} C_m^j g^{(m-j)}(x)((ix)^n)^{(j)} e^{ix\xi} \, dx, \ \xi \in \mathbb{R}.$$

Hereof

$$|\xi^m f^{(n)}(\xi)| \le \frac{1}{2\pi}\sum_{j=0}^{r} C_m^j \int_{\mathbb{R}} |g^{(m-j)}(x)| \frac{n!}{(n-j)!}|x|^{n-j}\, dx \le$$

$$\le \frac{1}{2\pi}\sum_{j=0}^{r} C_m^j \frac{n!}{(n-j)!}\int_{\mathbb{R}} |g^{(m-j)}(x)|(1+|x|)^{n-j+2}\, \frac{dx}{1+x^2} \le$$

$$\le \frac{1}{2}\sum_{j=0}^{r} C_m^j \frac{n!}{(n-j)!} s_{\varepsilon,m}(g)(n-j+2)!\varepsilon^{n-j+2}e^{-\psi^*(n-j+2)} \le$$

$$\le \frac{1}{2}\sum_{j=0}^{r} C_m^j \frac{n!}{(n-j)!} s_{\varepsilon,m}(g)(n-j+2)!\varepsilon^{n-j+2}e^{-\psi^*(n-j)} \le$$

$$\le \frac{1}{2}n!\varepsilon^{n+2}s_{\varepsilon,m}(g)\sum_{j=0}^{r} C_m^j (n-j+1)(n-j+2)\varepsilon^{-j}e^{-\psi^*(n-j)}.$$

Applying the conditions on $\psi$ and Lemma 2, we obtain with some $K_\psi > 0$

$$\psi^*(x+y) \le \psi^*(x) + \psi^*(y) + a(x+y) + K_\psi,\ x, y \ge 0. \tag{12}$$

Therefore, for any $\xi \in \mathbb{R}$

$$|\xi^m f^{(n)}(\xi)| \le \frac{1}{2}(n+2)!m!\varepsilon^{n+2}s_{\varepsilon,m}(g)e^{-\psi^*(n)}\sum_{j=0}^{r}\frac{e^{\psi^*(j)+an+K_\psi}}{\varepsilon^j j!}\ .$$

Assuming $c_{\varepsilon,m} = 2\varepsilon^2 m! e^{K_\psi}\sum_{j=0}^{\infty}\frac{e^{\psi^*(j)}}{\varepsilon^j j!}$, we obtain with all $n \in \mathbb{Z}_+$

$$|\xi^m f^{(n)}(\xi)| \le c_{\varepsilon,m}(2\varepsilon e^a)^n n! s_{\varepsilon,m}(g)e^{-\psi^*(n)},\ \xi \in \mathbb{R}.$$

Consequently, with all $n \in \mathbb{Z}_+$ and $\xi \in \mathbb{R}$

$$(1+|\xi|)^m|f^{(n)}(\xi)| \le 2^m(c_{\varepsilon,0} + c_{\varepsilon,m})s_{\varepsilon,m}(g)(2\varepsilon e^a)^n n! e^{-\psi^*(n)}.$$

According to Theorem 2 the function $f$ (uniquely) extends up to an entire function of the class $S(\varphi)$. Therefore, $f \in S(\varphi)$. It is obvious, that $g = \mathcal{F}(f)$. Taking into account the inequality (9), we obtain

$$(1+|z|)^m|f(z)| \le 2^{m+1}(c_{\varepsilon,0} + c_{\varepsilon,m})s_{\varepsilon,m}(g)(1+|y|)^m e^{\varphi(4\varepsilon e^a(1+|y|))},\ z \in \mathbb{C}.$$

Due to conditions 2) and 4) on $\varphi$ we can find a constant $K_{\varepsilon,m,\varphi} > 0$ such that

$$p_{8\varepsilon e^a h,m}(f) \le K_{\varepsilon,m,\varphi}s_{\varepsilon,m}(g).$$

Applying Lemma 3, we obtain, that the inverse mapping $I^{-1}$ is continuous.

Thus, it was proved, that the Fourier transform sets a topological isomorphism of the spaces $S(\varphi)$ and $G(\psi^*)$.

## 5. A special case of the function $\varphi$

**Proof of Theorem 4**. Let the function $\varphi$ satisfy conditions 1) – 4) and be convex on $[0, \infty)$. Let us show, that in this case the space $G(\psi^*)$ consists of functions $f \in C^\infty(\mathbb{R})$ such that for any $\varepsilon > 0$, $n \in \mathbb{Z}_+$ there is a constant $C_{\varepsilon,n} > 0$ such that

$$|f^{(n)}(x)| \le C_{\varepsilon,n}e^{-\varphi^*(\frac{|x|}{\varepsilon})},\ x \in \mathbb{R}. \tag{13}$$

Suppose $f \in G(\psi^*)$, $\varepsilon \in (0,1)$ be arbitrary, $b = \frac{\varepsilon}{2e^{a+1}}$. According to Lemma 3 $\forall n, k \in \mathbb{Z}_+$

$$|f^{(n)}(x)| \le s_{b,n}(f)\frac{k! b^k e^{-\psi^*(k)}}{(1+|x|)^k}\ ,\ x \ge 0. \tag{14}$$

Applying the inequality: $k! < 3\frac{k^{k+1}}{e^k}$ for any $k \in \mathbb{N}$, inequality (12) and monotony $\psi^*$, we obtain for $k \in \mathbb{N}$, $t \in [k, k+1)$, $b \in (0,1)$, $x \in \mathbb{R}$

$$\frac{k!b^k e^{-\psi^*(k)}}{(1+|x|)^k} < 3\frac{b^k k^{k+1} e^{-\psi^*(k)}}{e^k(1+|x|)^k} \leq \frac{3b^t t^{t+1} e^{-\psi^*(t)+\psi^*(1)+at+K_\psi+1}}{be^t(1+|x|)^t}(1+|x|) =$$

$$= \frac{3e^{K_\psi+1+\psi^*(1)}}{b} e^{t\ln b+(t+1)\ln t-\psi^*(t)+at-t-t\ln(1+|x|)}(1+|x|).$$

Let us apply the inequality [12]

$$(\varphi[e])^*(x) + (\varphi^*[e])^*(x) = x\ln x - x,\ x > 0, \tag{15}$$

and assume, that $C = \frac{3e^{K_\psi+1+\psi^*(1)}}{b}$. Then

$$\frac{k!b^k e^{-\psi^*(k)}}{(1+|x|)^k} < C(1+|x|)e^{t\ln\frac{be^a}{1+|x|}+\ln t+(\varphi^*[e])^*(t)} < C(1+|x|)e^{t\ln\frac{be^{a+1}}{1+|x|}+(\varphi^*[e])^*(t)}.$$

It implies, that

$$\inf_{k\in\mathbb{N}} \frac{k!b^k e^{-\psi^*(k)}}{(1+|x|)^k} \leq C(1+|x|)\inf_{t\geq 1} e^{t\ln\frac{be^{a+1}}{1+|x|}+(\varphi^*[e])^*(t)}.$$

Subject to $be^{a+1} < 1$, we obtain with some $C_1 = C_1(b,\varphi) > 0$

$$\inf_{k\in\mathbb{N}} \frac{k!b^k e^{-\psi^*(k)}}{(1+|x|)^k} \leq C_1(1+|x|)^2 \inf_{t\geq 0} e^{t\ln\frac{be^{a+1}}{1+|x|}+(\varphi^*[e])^*(t)}.$$

Let us rewrite the latter inequality in the following form

$$\inf_{k\in\mathbb{N}} \frac{k!b^k e^{-\psi^*(k)}}{(1+|x|)^k} \leq C_1(1+|x|)^2 e^{-\sup\limits_{t>0}(t\ln\frac{1+|x|}{be^{a+1}}-(\varphi^*[e])^*(t))} =$$

$$= C_1(1+|x|)^2 e^{-(\varphi^*([e])^{**}(\ln\frac{1+|x|}{be^{a+1}}))}.$$

Applying formula of inversion of the Jung transform, we obtain

$$\inf_{k\in\mathbb{N}} \frac{k!b^k e^{-\psi^*(k)}}{(1+|x|)^k} \leq C_1 e^{2\ln(1+|x|)-\varphi^*(\frac{1+|x|}{be^{a+1}})}.$$

Whereas $\lim\limits_{x\to+\infty}\dfrac{\varphi^*(x)}{x} = +\infty$, then there is a constant $C_2 = C_2(b,\varphi) > 0$ such that

$$\inf_{k\in\mathbb{N}} \frac{k!b^k e^{-\psi^*(k)}}{(1+|x|)^k} \leq C_2 e^{-\frac{1}{2}\varphi^*(\frac{1+|x|}{be^{a+1}})}.$$

Whereas $\varphi^*(2u) \geq 2\varphi^*(u)$ for any $u \geq 0$, then

$$\inf_{k\in\mathbb{N}} \frac{k!b^k e^{-\psi^*(k)}}{(1+|x|)^k} \leq C_2 e^{-\varphi^*(\frac{1+|x|}{2be^{a+1}})}.$$

Hereof and from (14) (assuming $C_{n,\varepsilon} = s_{b,n}(f)C_2$) we obtain

$$|f^{(n)}(x)| \leq s_{b,n}(f)C_2 e^{-\varphi^*(\frac{|x|}{2be^{a+1}})} = C_{n,\varepsilon}e^{-\varphi^*(\frac{|x|}{\varepsilon})}.$$

Thus, the estimate (13) has been obtained.

Let us now $f \in C^\infty(\mathbb{R})$ satisfy inequality (13). Let us show, that $f \in G(\psi^*)$. For $x \neq 0$ and any $n \in \mathbb{Z}_+$

$$|f^{(n)}(x)| \leq C_{\varepsilon,n} e^{-\varphi^*(e^{\ln\frac{|x|}{\varepsilon}})},$$

i.e.

$$|f^{(n)}(x)| \leq C_{\varepsilon,n} e^{-\varphi^*[e](\ln\frac{|x|}{\varepsilon})}.$$

Applying formula of inversion of the Jung transform, we obtain
$$|f^{(n)}(x)| \leq C_{\varepsilon,n} e^{-\sup\limits_{t\geq 0}(t\ln\frac{|x|}{\varepsilon} - (\varphi^*[e])^*(t))}, \ x \neq 0.$$
Applying equality (15), we deal with
$$|f^{(n)}(x)| \leq C_{\varepsilon,n} e^{-\sup\limits_{t\geq 0}(t\ln\frac{|ex|}{\varepsilon} - t\ln t + \psi^*(t))}, \ x \neq 0.$$
Consequently,
$$|f^{(n)}(x)| \leq C_{\varepsilon,n} e^{-\sup\limits_{k\in\mathbb{N}}(k\ln\frac{|ex|}{\varepsilon} - k\ln k + \psi^*(k))}, \ x \neq 0.$$
Therefore, with any $\varepsilon > 0, k \in \mathbb{N}$
$$|f^{(n)}(x)x^k| \leq C_{\varepsilon,n}\varepsilon^k \left(\frac{k}{e}\right)^k e^{-\psi^*(k)}, \ x \neq 0.$$
Taking into account, that $k^k \leq e^k k!$ for any $k \in \mathbb{N}$, we obtain
$$|f^{(n)}(x)x^k| \leq C_{\varepsilon,n}\varepsilon^k k! e^{-\psi^*(k)}, \ k \in \mathbb{N}, x \neq 0.$$
This inequality is also valid in the point $x = 0$ with any $k \in \mathbb{N}$ and for any $x \in \mathbb{R}$ with $k = 0$ (due to (13)). Hence, $f \in G(\psi^*)$.

Theorem 4 has been proved.

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