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ON SOME FAMILIES OF COMPLEX LINES WHICH ARE SUFFICIENT FOR A HOLOMORPHIC EXTENSION OF FUNCTIONS

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Abstract. The present article is based on the result related to the holomorphic extension of functions. Functions with the one-dimensional holomorphic extension property along families of complex lines are discussed. Real analytic functions given on the boundary of a bounded domain D in $\mathbb{C}^n$, $n > 1$ with the one - dimensional holomorphic extension property along families of complex lines are considered. The existence of holomorphic extensions of these functions to D is studied depending on the form of the domain and location of the families of complex lines.

Keywords: real analytic function, holomorphic extension, functions with the one - dimensional holomorphic extension property

1. Preliminary results

The article contains some results, connected with the holomorphic extension of functions f, given on the boundary of a bounded domain $D \subset \mathbb{C}^n$, $n > 1$, in this domain. We will consider functions with one-dimensional holomorphic extension property along complex lines.

Results of functions with one-dimensional holomorphic extension property are trivial on the complex plane $\mathbb{C}$. Therefore our results are significantly multidimensional.

The first result, relating to our topic, was obtained by M.L. Agranovsky and R.E. Valsky in [1], who studied functions with the one-dimensional holomorphic extension property in a ball. The proof was based on properties of group of the ball automorphisms.

Stout in [2], applying complex Radon transform, projected Agranovsky and Valsky theorem on arbitrary limited domains with smooth boundary. An alternative proof of the Stout theorem was obtained by A.M. Kytmanov (see [3]), who applied Bochner-Martinelli integral. The idea of integral representations application (Bochner-Martinelli, Cauchy – Fantappie) proved to be useful in study of functions with one-dimensional holomorphic extension property.

Let D be a bounded domain in $\mathbb{C}^n$, $n > 1$, with a connected smooth boundary ∂D of the class C^2. Let us formulate the result of E.L. Stout [2].

We will consider complex lines of the form

$$l = \{\zeta \in \mathbb{C}^n : \ \zeta_j = z_j + b_j t, \ j = 1, \dots, n, \ t \in \mathbb{C}\}, \tag{1}$$

passing through the point $z \in \mathbb{C}^n$ in the direction of the vector $b \in \mathbb{CP}^{n-1}$ (direction b is defined with the precision to multiplication by the complex number $\lambda \neq 0$).

According to the Sard's theorem, for almost all $z \in \mathbb{C}^n$ and almost all $b \in \mathbb{CP}^{n-1}$ the meet $l \cap \partial D$ corresponds to a set of the finite number of piecewise-smooth curves (excluding the case of degeneracy, when $\partial D \cap l = \emptyset$).

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We will say, that the function $f \in C(\partial D)$ *possesses one-dimensional holomorphic extension property along a complex line* l $(l \cap \partial D \neq \emptyset)$, if there is the function f_l with the following properties:
1) $f_l \in C\left(\overline{D} \cap l\right)$;
2) $f_l = f$ in the set $\partial D \cap l$;
3) function f_l is holomorphic in internal (with respect to topology l) points of the set $\overline{D} \cap l$.

Theorem 1 ([2]). *If the function $f \in C(\partial D)$ possesses one-dimensional holomorphic extension property along complex lines of the form* (1)*, then f is holomorphically extended in D.*

Some more restricted family of complex lines, sufficient for holomorphic extension, was considered by M.L. Agranovsky and A.M. Semenov [4].

Let us consider the open set $V \subset D$ and the family $\mathfrak{L}_V$ of complex lines, meeting this set.

Theorem 2 ([4]). *If the function $f \in C(\partial D)$ possesses one-dimensional holomorphic extension property along lines from the family $\mathfrak{L}_V$ for some open set $V \subset D$, then the function f is holomorphically extended in D.*

Later some authors (see, for instance, papers [5] – [8]) considered different properties of complex lines (for instance, the family of complex lines, meeting the germ of the generating variety, passing through the germ of complex hypersurfaces and etc.), sufficient for holomorphic extension of functions from different classes. Let us demonstrate the result from paper [7], where it is stated, that the family of complex lines, passing through the boundary point of a complex ball, is sufficient for holomorphic extension of real analytic functions, given on the ball boundary.

Let $\mathbb{B}^n$ be a ball in $\mathbb{C}^n$, $\partial\mathbb{B}^n$ be a sphere, $z_0 \in \partial\mathbb{B}^n$ and C^w denotes a class of real analytic functions.

Theorem 3 ([7]). *Let the function $f \in C^w(\partial\mathbb{B}^n)$ possess one-dimensional holomorphic extension property along all complex lines, passing trough z_0. Then the function f holomorphically extends in $\mathbb{B}^n$.*

2. Two-dimensional case

Let us consider a two-dimensional complex space $\mathbb{C}^2$, points of which we will denote as $w = (w_1, w_2)$, $z = (z_1, z_2)$ and etc. Let D be a bounded strictly convex domain in $\mathbb{C}^2$ with a real analytic function boundary ∂D, i.e. $D = \{w \,|\, \rho(w) < 0\}$, where the function $\rho(w_1, w_2)$ is real analytic in some surrounding of the domain closure $\overline{D}$. With all this $grad\,\rho = \left(\frac{\partial \rho}{\partial w_1}, \frac{\partial \rho}{\partial w_2}\right) \neq 0$ on ∂D. Let the following condition hold true for all the points of the boundary

$$\left(\frac{\partial \rho}{\partial w_2}(w)\right)^2 \frac{\partial^2 \rho}{\partial w_1^2}(w) - 2\frac{\partial \rho}{\partial w_1}(w)\frac{\partial \rho}{\partial w_2}(w)\frac{\partial^2 \rho}{\partial w_1 \partial w_2}(w) + \left(\frac{\partial \rho}{\partial w_1}(w)\right)^2 \frac{\partial^2 \rho}{\partial w_2^2}(w) = 0. \quad (2)$$

We will also denote the family of complex lines, passing through the point w_0, $w_0 \in \partial D$ by $\mathfrak{L}_{w_0}$.

Theorem 4. *Let the function $f \in C^w(\partial D)$ possess one-dimensional holomorphic extension property along all complex lines from $\mathfrak{L}_{w_0}$, meeting D, then the function f holomorphically extends in D.*

Remark 1. *If the point w_0 is fixed in advance, then the condition* (2) *can be satisfied only in the point w_0.*

Proof. Let us make a shift, for the point $w_0 \in \partial D$ to pass to 0 and make an orthogonal transform

$$w = Bz,$$

given by the matrix

$$B = \begin{pmatrix} \frac{\partial \rho}{\partial w_2}(0) & i\frac{\partial \rho}{\partial \bar{w}_1}(0) \\ -\frac{\partial \rho}{\partial w_1}(0) & i\frac{\partial \rho}{\partial \bar{w}_2}(0) \end{pmatrix}.$$

This transform is nonsingular, whereas $|B| \neq 0$. In case of such a transform real analytic property of the function $\rho(Bz) = \tilde{\rho}(z)$ remains. The component-like variant of the transform will be as follows.

$$\begin{cases} \frac{\partial \rho}{\partial w_2}(0)\, z_1 + i\frac{\partial \rho}{\partial \bar{w}_1}(0)\, z_2 & = w_1 \\ -\frac{\partial \rho}{\partial w_1}(0)\, z_1 + i\frac{\partial \rho}{\partial \bar{w}_2}(0)\, z_2 & = w_2. \end{cases}$$

Assume $z_1 = x_1 + ix_2$, $z_2 = x_3 + ix_4$.

Lemma 1. *In case of complex-linear transform of coordinates $w = Bz$ the condition (2) for the function $\rho(w_1, w_2)$, considered in the boundary point $w_0 = 0$, takes the following form*

$$\frac{\partial^2 \varphi}{\partial x_1 \partial x_2}(0) = 0, \qquad \frac{\partial^2 \varphi}{\partial x_1^2}(0) = \frac{\partial^2 \varphi}{\partial x_2^2}(0), \tag{3}$$

where the implicit function $x_4 = \varphi(x_1, x_2, x_3)$, is defined by the equation $\rho(x_1, x_2, x_3, x_4) = 0$, and satisfies the conditions $\varphi(0) = 0$, $\frac{\partial \varphi}{\partial x_k}(0) = 0$, $k = \overline{1,3}$.

Proof. Let us find connection between partial derivatives of functions $\tilde{\rho}(z)$ and $\rho(w)$, and also conditions for the function $\tilde{\rho}(z)$. We will obtain

$$\frac{\partial \tilde{\rho}}{\partial z_1} = \frac{\partial \rho}{\partial w_1}\frac{\partial w_1}{\partial z_1} + \frac{\partial \rho}{\partial \bar{w}_1}\frac{\partial \bar{w}_1}{\partial z_1} + \frac{\partial \rho}{\partial w_2}\frac{\partial w_2}{\partial z_1} + \frac{\partial \rho}{\partial \bar{w}_2}\frac{\partial \bar{w}_2}{\partial z_1} = \frac{\partial \rho}{\partial w_2}(0)\frac{\partial \rho}{\partial w_1} - \frac{\partial \rho}{\partial w_1}(0)\frac{\partial \rho}{\partial w_2}.$$

$$\frac{\partial \tilde{\rho}}{\partial z_2} = \frac{\partial \rho}{\partial w_1}\frac{\partial w_1}{\partial z_2} + \frac{\partial \rho}{\partial \bar{w}_1}\frac{\partial \bar{w}_1}{\partial z_2} + \frac{\partial \rho}{\partial w_2}\frac{\partial w_2}{\partial z_2} + \frac{\partial \rho}{\partial \bar{w}_2}\frac{\partial \bar{w}_2}{\partial z_2} = i\frac{\partial \rho}{\partial \bar{w}_1}(0)\frac{\partial \rho}{\partial w_1} + i\frac{\partial \rho}{\partial \bar{w}_2}(0)\frac{\partial \rho}{\partial w_2}.$$

It is clear from the above calculations, that

$$\frac{\partial \tilde{\rho}}{\partial z_1}(0) = 0,$$

and the value

$$\frac{\partial \tilde{\rho}}{\partial z_2}(0) = i\left(\frac{\partial \rho}{\partial \bar{w}_1}(0)\frac{\partial \rho}{\partial w_1}(0) + \frac{\partial \rho}{\partial \bar{w}_2}(0)\frac{\partial \rho}{\partial w_2}(0)\right) = i\left(\left|\frac{\partial \rho}{\partial w_1}(0)\right|^2 + \left|\frac{\partial \rho}{\partial w_2}(0)\right|^2\right) \neq 0$$

is simply imaginary.

Let us consider the partial derivative of the function $\tilde{\rho}(z)$ of the second order.

$$\begin{aligned}\frac{\partial^2\tilde{\rho}}{\partial z_1^2} &= \frac{\partial\rho}{\partial w_2}(0)\left(\frac{\partial^2\rho}{\partial w_1\partial w_1}\frac{\partial w_1}{\partial z_1}+\frac{\partial^2\rho}{\partial w_1\partial w_2}\frac{\partial w_2}{\partial z_1}\right)-\\ &-\frac{\partial\rho}{\partial w_1}(0)\left(\frac{\partial^2\rho}{\partial w_2\partial w_1}\frac{\partial w_1}{\partial z_1}+\frac{\partial^2\rho}{\partial w_2\partial w_2}\frac{\partial w_2}{\partial z_1}\right)=\\ &=\frac{\partial\rho}{\partial w_2}(0)\left(\frac{\partial\rho}{\partial w_2}(0)\frac{\partial^2\rho}{\partial w_1^2}-\frac{\partial\rho}{\partial w_1}(0)\frac{\partial^2\rho}{\partial w_1\partial w_2}\right)-\\ &-\frac{\partial\rho}{\partial w_1}(0)\left(\frac{\partial\rho}{\partial w_2}(0)\frac{\partial^2\rho}{\partial w_2\partial w_1}-\frac{\partial\rho}{\partial w_1}(0)\frac{\partial^2\rho}{\partial w_2^2}\right)=\\ &=\left(\frac{\partial\rho}{\partial w_2}(0)\right)^2\frac{\partial^2\rho}{\partial w_1^2}-2\frac{\partial\rho}{\partial w_1}(0)\frac{\partial\rho}{\partial w_2}(0)\frac{\partial^2\rho}{\partial w_1\partial w_2}+\left(\frac{\partial\rho}{\partial w_1}(0)\right)^2\frac{\partial^2\rho}{\partial w_2^2}.\end{aligned}$$

Considering the shift of coordinates made, when the boundary point w_0 transformed to zero, and the condition (2) to the boundary of the domain D, the latter equality means, that

$$\frac{\partial^2\tilde{\rho}}{\partial z_1^2}(0)=0.$$

Further for more convenience instead of function $\tilde{\rho}(z)$, giving the boundary of the domain D, we will write $\rho(z)$. In other words,

$$\rho(z_1,z_2)=0 \tag{4}$$

with the condition

$$\begin{cases}\dfrac{\partial\rho}{\partial z_1}(0) &= 0\\ \dfrac{\partial^2\rho}{\partial z_1^2}(0) &= 0,\end{cases} \tag{5}$$

and also with the condition, that the value $\dfrac{\partial\rho}{\partial z_2}(0)\neq 0$ is simply imaginary.

Partial derivatives in complex variables can be expressed via derivatives in real variables as follows:

$$\frac{\partial\rho}{\partial z_1}=\frac{1}{2}\left(\frac{\partial\rho}{\partial x_1}-i\frac{\partial\rho}{\partial x_2}\right),$$

$$\frac{\partial\rho}{\partial z_2}=\frac{1}{2}\left(\frac{\partial\rho}{\partial x_3}-i\frac{\partial\rho}{\partial x_4}\right).$$

Thus, it results from the correlations of derivatives connection of complex and real variables, and also from the system of conditions (5), that

$$\frac{\partial\rho}{\partial x_1}(0)=0,\quad \frac{\partial\rho}{\partial x_2}(0)=0,\quad \frac{\partial\rho}{\partial x_3}(0)=0. \tag{6}$$

Then we will write the second condition in the system (5) in real variables. We will obtain

$$\frac{\partial}{\partial x_1}\rho=\left(\frac{\partial}{\partial z_1}+\frac{\partial}{\partial \bar{z}_1}\right)\rho,$$
$$\frac{\partial}{\partial x_2}\rho=i\left(\frac{\partial}{\partial z_1}-\frac{\partial}{\partial \bar{z}_1}\right)\rho,$$

$$\frac{\partial^2\rho}{\partial x_1^2}=\frac{\partial}{\partial x_1}\left(\frac{\partial\rho}{\partial x_1}\right)=\left(\frac{\partial}{\partial z_1}+\frac{\partial}{\partial\bar z_1}\right)\left(\frac{\partial\rho}{\partial z_1}+\frac{\partial\rho}{\partial\bar z_1}\right)=\frac{\partial^2\rho}{\partial z_1^2}+\frac{\partial^2\rho}{\partial\bar z_1\partial z_1}+$$
$$+\frac{\partial^2\rho}{\partial z_1\partial\bar z_1}+\frac{\partial^2\rho}{\partial\bar z_1^2}=\frac{\partial^2\rho}{\partial z_1^2}+\frac{\partial^2\rho}{\partial\bar z_1^2}+2\frac{\partial^2\rho}{\partial z_1\partial\bar z_1}.$$
$$\frac{\partial^2\rho}{\partial x_2^2}=\frac{\partial}{\partial x_2}\left(\frac{\partial\rho}{\partial x_2}\right)=i\left(\frac{\partial}{\partial z_1}-\frac{\partial}{\partial\bar z_1}\right)i\left(\frac{\partial\rho}{\partial z_1}-\frac{\partial\rho}{\partial\bar z_1}\right)=$$
$$=-\left(\frac{\partial^2\rho}{\partial z_1^2}-\frac{\partial^2\rho}{\partial\bar z_1\partial z_1}-\frac{\partial^2\rho}{\partial z_1\partial\bar z_1}+\frac{\partial^2\rho}{\partial\bar z_1^2}\right)=2\frac{\partial^2\rho}{\partial z_1\partial\bar z_1}-\frac{\partial^2\rho}{\partial z_1^2}-\frac{\partial^2\rho}{\partial\bar z_1^2}.$$
$$\frac{\partial^2\rho}{\partial x_1\partial x_2}=\frac{\partial}{\partial x_1}\left(\frac{\partial\rho}{\partial x_2}\right)=\left(\frac{\partial}{\partial z_1}+\frac{\partial}{\partial\bar z_1}\right)i\left(\frac{\partial\rho}{\partial z_1}-\frac{\partial\rho}{\partial\bar z_1}\right)=$$
$$=i\left(\frac{\partial^2\rho}{\partial z_1^2}-\frac{\partial^2\rho}{\partial\bar z_1\partial z_1}+\frac{\partial^2\rho}{\partial z_1\partial\bar z_1}-\frac{\partial^2\rho}{\partial\bar z_1^2}\right)=i\left(\frac{\partial^2\rho}{\partial z_1^2}-\frac{\partial^2\rho}{\partial\bar z_1^2}\right).$$

Therefore, taking into account the second condition of the system (5) and also real-analytic property of the function ρ, it follows from the above results, that the conditions for function $\rho(x_1,x_2,x_3,x_4)$ will have the form

$$\frac{\partial^2\rho}{\partial x_1^2}(0)=\frac{\partial^2\rho}{\partial x_2^2}(0),\qquad\frac{\partial^2\rho}{\partial x_1\partial x_2}(0)=0. \tag{7}$$

Due to its pass to real coordinates, the function, giving the boundary of the domain D, takes the form

$$\rho(x_1,x_2,x_3,x_4)=0.$$

Whereas the gradient of the function $\rho(x_1,\dots,x_4)$ differs from zero, due to correlations (6) $\dfrac{\partial\rho}{\partial x_4}(0)\neq0$. Then, according to the theorem of an implicit function (chapter 2, p. 26.1 from [9]), the function, giving the boundary of the domain in some surrounding of the boundary point 0, takes the form

$$x_4=\varphi(x_1,x_2,x_3), \tag{8}$$

where

$$\frac{\partial\varphi}{\partial x_k}=-\frac{\partial\rho}{\partial x_k}\Big(x_1,x_2,x_3,\varphi(x_1,x_2,x_3)\Big)\Big/\frac{\partial\rho}{\partial x_4}\Big(x_1,x_2,x_3,\varphi(x_1,x_2,x_3)\Big),\qquad k=\overline{1,3}.$$

The function φ satisfies the conditions $\varphi(0)=0$, $\dfrac{\partial\varphi}{\partial x_k}(0)=0$, $k=\overline{1,3}$. Further, applying the correlations (6) and (7), we will find conditions for the function $\varphi(x_1,x_2,x_3)$. For this purpose we will consider the derivative $\dfrac{\partial^2\varphi}{\partial x_k\partial x_j}$, $j=\overline{1,3}$. We will obtain

$$\frac{\partial}{\partial x_j}\frac{\partial\rho}{\partial x_k}\Big(x_1,x_2,x_3,\varphi(x_1,x_2,x_3)\Big)=\frac{\partial^2\rho}{\partial x_k\partial x_j}+\frac{\partial^2\rho}{\partial x_k\partial x_4}\frac{\partial\varphi}{\partial x_j}=\frac{\partial^2\rho}{\partial x_k\partial x_j}-\frac{\dfrac{\partial^2\rho}{\partial x_k\partial x_4}}{\dfrac{\partial\rho}{\partial x_4}}\frac{\partial\rho}{\partial x_j},$$

$$\frac{\partial}{\partial x_j}\frac{\partial\rho}{\partial x_4}\Big(x_1,x_2,x_3,\varphi(x_1,x_2,x_3)\Big)=\frac{\partial^2\rho}{\partial x_4\partial x_j}+\frac{\partial^2\rho}{\partial x_4\partial x_4}\frac{\partial\varphi}{\partial x_j}=\frac{\partial^2\rho}{\partial x_4\partial x_j}-\frac{\dfrac{\partial^2\rho}{\partial x_4^2}}{\dfrac{\partial\rho}{\partial x_4}}\frac{\partial\rho}{\partial x_j}.$$

Therefore,

$$\frac{\partial^2\varphi}{\partial x_k\partial x_j}=-\frac{\left(\frac{\partial^2\rho}{\partial x_k\partial x_j}\frac{\partial\rho}{\partial x_4}-\frac{\partial^2\rho}{\partial x_k\partial x_4}\frac{\partial\rho}{\partial x_j}\right)\frac{\partial\rho}{\partial x_4}-\left(\frac{\partial^2\rho}{\partial x_4\partial x_j}\frac{\partial\rho}{\partial x_4}-\frac{\partial^2\rho}{\partial x_4^2}\frac{\partial\rho}{\partial x_j}\right)\frac{\partial\rho}{\partial x_k}}{\left(\frac{\partial\rho}{\partial x_4}\right)^3},$$

and

$$\frac{\partial^2\varphi}{\partial x_k^2}=-\frac{\left(\frac{\partial^2\rho}{\partial x_k^2}\frac{\partial\rho}{\partial x_4}-\frac{\partial^2\rho}{\partial x_k\partial x_4}\frac{\partial\rho}{\partial x_k}\right)\frac{\partial\rho}{\partial x_4}-\left(\frac{\partial^2\rho}{\partial x_4\partial x_k}\frac{\partial\rho}{\partial x_4}-\frac{\partial^2\rho}{\partial x_4^2}\frac{\partial\rho}{\partial x_k}\right)\frac{\partial\rho}{\partial x_k}}{\left(\frac{\partial\rho}{\partial x_4}\right)^3}.$$

Taking into account conditions (6) and (7), it is easy to see, that

$$\frac{\partial^2\varphi}{\partial x_1\partial x_2}(0)=0,\qquad \frac{\partial^2\varphi}{\partial x_1^2}(0)=\frac{\partial^2\varphi}{\partial x_2^2}(0).\quad\square$$

We will continue to prove the theorem. Further we will consider sections $D_a(\tau)$ of the domain D

$$D_a(\tau)=\left(\frac{\tau}{1+|a|^2}a,\frac{\tau}{1+|a|^2}\right)\qquad \forall\tau\in\overline{\Delta}_a,$$

passing in the direction of the vector $(a,1)\in\mathbb{C}^2$. The domain Δ_a of the parameter change τ is a domain on the complex plane with a real-analytic boundary (in the surrounding of the boundary point 0).

Disintegrating the function $\varphi(x_1,x_2,x_3)$ in the expression (8) in the surrounding of the boundary point 0 into Taylor series, due to conditions for the function φ we will have

$$x_4=T(x_1,x_2,x_3)+o\left(|x'|^2\right),\quad |x'|\to 0,\quad x'=(x_1,x_2,x_3),\tag{9}$$

where $T(x_1,x_2,x_3)=c_{11}x_1^2+c_{22}x_2^2+c_{33}x_3^2+c_{12}x_1x_2+c_{13}x_1x_3+c_{23}x_2x_3$ is a positively defined (according to strict convexity of the function ρ) quadric form. The coefficient of the form $T(x_1,x_2,x_3)$ due to conditions (3) for the function $\varphi(x_1,x_2,x_3)$ holds true for the correlations

$$c_{12}=0,\quad c_{11}=c_{22}.$$

Let us point out a real and imaginary part of variables z_1,z_2 and write the expressions for x_1,x_2,x_3,x_4.

Assume $\tau=u+iv$, $a=a_1+ia_2$. Then

$$\frac{\tau}{1+|a|^2}a=\frac{(u+iv)(a_1+ia_2)}{1+|a|^2}=\frac{(ua_1-va_2)+i(ua_2+va_1)}{1+|a|^2},$$
$$\frac{\tau}{1+|a|^2}=\frac{u+iv}{1+|a|^2}.$$

Therefore,

$$x_1=\frac{ua_1-va_2}{1+|a|^2},\quad x_2=\frac{ua_2+va_1}{1+|a|^2},\quad x_3=\frac{u}{1+|a|^2},\quad x_4=\frac{v}{1+|a|^2}.$$

Let us write expressions for the quadric form $T(x_1, x_2, x_3)$.

$$T(x_1, x_2, x_3) = c_{11}x_1^2 + c_{11}x_2^2 + c_{33}x_3^2 + c_{13}x_1x_3 + c_{23}x_2x_3 =$$
$$= \frac{1}{\left(1+|a|^2\right)^2}\Big[c_{11}\left(u^2a_1^2 - 2uva_1a_2 + v^2a_2^2\right) + c_{11}\left(u^2a_2^2 + 2uva_1a_2 + v^2a_1^2\right) +$$
$$+ c_{33}u^2 + c_{13}\left(u^2a_1 - uva_2\right) + c_{23}\left(u^2a_2 + uva_1\right)\Big] = \frac{1}{\left(1+|a|^2\right)^2}\times$$
$$\times\Big[v^2\left(c_{11}a_2^2 + c_{11}a_1^2\right) + v\left(-2c_{11}ua_1a_2 + 2c_{11}ua_1a_2 - c_{13}ua_2 + c_{23}ua_1\right) +$$
$$+ \left(c_{11}u^2a_1^2 + c_{11}u^2a_2^2 + c_{33}u^2 + c_{13}u^2a_1 + c_{23}u^2a_2\right)\Big] = \frac{1}{\left(1+|a|^2\right)^2}\times$$
$$\times\Big[v^2\left(c_{11}a_2^2 + c_{11}a_1^2\right) + v\left(-c_{13}ua_2 + c_{23}ua_1\right) +$$
$$+ \left(c_{11}u^2a_1^2 + c_{11}u^2a_2^2 + c_{33}u^2 + c_{13}u^2a_1 + c_{23}u^2a_2\right)\Big].$$

Let us use the value obtained for x_4 and $T(x_1, x_2, x_3)$ in the equation (9) and give similar ones. We will obtain

$$v^2\left(c_{11}a_2^2 + c_{11}a_1^2\right) + v\left(-c_{13}ua_2 + c_{23}ua_1 - 1 - |a|^2\right) +$$
$$+ \left(c_{11}u^2a_1^2 + c_{11}u^2a_2^2 + c_{33}u^2 + c_{13}u^2a_1 + c_{23}u^2a_2\right) + o\left(|a|^2\right) = 0, \quad |a| \to +\infty.$$

Choosing $|a|$ rather large, i. e. replacing a by ta with $|a| = 1$, we will obtain

$$v^2\left(c_{11}a_2^2t^2 + c_{11}a_1^2t^2\right) + v\left(-c_{13}ua_2t + c_{23}ua_1t - 1 - |a|^2t^2\right) +$$
$$+ \left(c_{11}u^2a_1^2t^2 + c_{11}u^2a_2^2t^2 + c_{33}u^2 + c_{13}u^2a_1t + c_{23}u^2a_2t\right) + o\left(|t|^2\right) = 0, \quad t \to +\infty.$$

Therefore, dividing to t^2 and proceeding in the given expression to the limit with $t \to +\infty$, we will obtain

$$v^2\left(c_{11}a_2^2 + c_{11}a_1^2\right) - v|a|^2 + c_{11}u^2a_1^2 + c_{11}u^2a_2^2 = 0,$$
$$c_{11}v^2|a|^2 - v|a|^2 + c_{11}u^2|a|^2 = 0,$$
$$c_{11}v^2 - v + c_{11}u^2 = 0.$$

Let us write the given equality in a complex form. We will obtain

$$c_{11}\left(v^2 - \frac{v}{c_{11}} + u^2\right) = 0,$$
$$\left(v^2 - 2v\frac{1}{2c_{11}} + \frac{1}{4c_{11}^2}\right) - \frac{1}{4c_{11}^2} + u^2 = 0,$$
$$u^2 + \left(v - \frac{1}{2c_{11}}\right)^2 = \left(\frac{1}{2c_{11}}\right)^2,$$
$$\left|\tau - \frac{i}{2c_{11}}\right|^2 = \left(\frac{1}{2c_{11}}\right)^2. \qquad (10)$$

Hence, we have shown, that the domain Δ of the change of parameter τ in the boundary case, when $|a| \to +\infty$, is a circle with the center in the point $\tau_0 = \dfrac{i}{2c_{11}}$ and radius $r_0 = \dfrac{1}{2c_{11}}$. Coefficient $c_{11} > 0$ according to positive definiteness of the quadric form $T(x_1, x_2, x_3)$. The correlation (10) sets the boundary $\partial\Delta$.

It should be noted, that the tangent to the boundary of the domain D, drawn in the boundary point 0, is the line Im $z_2 = 0$. It is easy to see, that, when $|a| \to +\infty$, sections $D_a(\tau)$ become closer to tangents of the boundary of the domain D in the boundary point 0, whereas

$$\operatorname{Im} z_2 = \frac{v}{1+|a|^2} \to 0, \quad \text{when } |a| \to +\infty.$$

Moreover, when $|a| \to +\infty$ of the section $D_a(\tau)$ are in the surrounding of the point $z_0 = 0$. Namely, if $z \in D_a(\tau)$

$$|z - z_0|^2 = \left|\frac{\tau}{1+|a|^2}a\right|^2 + \left|\frac{\tau}{1+|a|^2}\right|^2 = \frac{|\tau|^2|a|^2}{\left(1+|a|^2\right)^2} + \frac{|\tau|^2}{\left(1+|a|^2\right)^2} = \frac{|\tau|^2}{1+|a|^2} \to 0,$$

when $|a| \to +\infty$.

Applying real-analytic property of the function $\rho(z_1, z_2, \bar{z}_1, \bar{z}_2)$, we will solve the equation (4) with respect to the variable $\bar{z}_2$. Whereas $\rho(z, \bar{z})$ is a real analytic function, then it is disintegrated into a series in the surrounding of the point $(0,0) \in \mathbb{C}^4 = \mathbb{C}^2 \times \mathbb{C}^2$. Let us proceed from coordinates $\bar{z}$ to variables ζ, i. e. make a substitution

$$\bar{z}_1 = \zeta_1, \quad \bar{z}_2 = \zeta_2.$$

We will obtain the function $\widehat{\rho}(z, \zeta)$, analytic from z and ζ with conditions

$$\begin{cases} \widehat{\rho}(z,\zeta) &= 0, \\ \zeta &= \bar{z}. \end{cases}$$

Whereas the gradient of the function $\widehat{\rho}(z_1, z_2, \zeta_1, \zeta_2)$ differs from zero, then the derivative of one of the variables differs from zero, for instance, the derivative $\dfrac{\partial \widehat{\rho}}{\partial \zeta_2} \neq 0$. Then, applying the theorem about an implicit function for holomorphic functions (Theorem 3 from chapter 1, §4 from [10]), we will define the variable ζ_2 by the rest of variables:

$$\begin{cases} \zeta_2 &= \psi(z_1, z_2, \zeta_1), \\ \bar{z}_1 &= \zeta_1, \\ \bar{z}_2 &= \zeta_2. \end{cases}$$

Then $f(z_1, z_2, \bar{z}_1, \bar{z}_2) = f(z_1, z_2, \bar{z}_1, \psi(z_1, z_2, \zeta_1))$ is a real analytic function, which disintegrates into the series by variables $z_1, z_2, \zeta_1 = \bar{z}_1$, which converges in the neighborhood of the boundary point $(0,0)$. Namely,

$$f(z_1, \bar{z}_1, z_2) = \sum_{l=0}^{+\infty} \sum_{h+k+2m=l} b_{h,k,m} z_1^h \bar{z}_1^k z_2^m,$$

where we redefined the element in power degree (giving power 2 by z_2).

Choosing $|a|$ rather large, we will consider moments N on the sections $D_a(\tau)$:

$$G(a, N) = \int_{\partial\Delta_a} \tau^N f\Big(D_a(\tau)\Big)\, d\tau =$$

$$= \int_{\partial\Delta_a} \tau^N \sum_{l=0}^{+\infty} \sum_{h+k+2m=l} b_{h,k,m} \left(\frac{\tau}{1+|a|^2}a\right)^h \left(\overline{\frac{\tau}{1+|a|^2}a}\right)^k \left(\frac{\tau}{1+|a|^2}\right)^m d\tau.$$

Let us prove, that the coefficients $b_{h,k,m} = 0$ for $k > 0$. Let l_0 be the lowest power degree with the property, that $b_{h,k,m} \neq 0$ for $k > 0$ and k_0 is the lowest degree by $\bar{z}_1$, for which it holds true.

We obtain, that $G(a,N)=0$ for all N and a, in particular, for ta with $|a|=1$ and $t\to+\infty$. Let us consider the limit

$$\lim_{t\to+\infty} G(ta,N)\,t^{l_0} =$$
$$= \lim_{t\to+\infty}\int\limits_{\partial\Delta_a}\tau^N\sum_{l=l_0}^{+\infty}\sum_{h+k+2m=l} b_{h,k,m}\left(\frac{\tau}{1+|ta|^2}ta\right)^h\left(\overline{\frac{\tau}{1+|ta|^2}ta}\right)^k\times$$
$$\times\left(\frac{\tau}{1+|ta|^2}\right)^m t^{l_0}d\tau = \lim_{t\to+\infty}\int\limits_{\partial\Delta_a}\tau^N\sum_{l=l_0}^{+\infty}\sum_{h+k+2m=l} b_{h,k,m}\tau^h\bar\tau^k\tau^m t^h t^k t^{l_0}a^h\bar a^k\times$$
$$\times\left(\frac{1}{\frac{1}{t^2}+|a|^2}\right)^h\frac{1}{t^{2h}}\left(\frac{1}{\frac{1}{t^2}+|a|^2}\right)^k\frac{1}{t^{2k}}\left(\frac{1}{\frac{1}{t^2}+|a|^2}\right)^m\frac{1}{t^{2m}}\,d\tau =$$
$$= \lim_{t\to+\infty}\sum_{l=l_0}^{+\infty}\sum_{h+k+2m=l} b_{h,k,m}\int\limits_{\partial\Delta_a}\tau^N\tau^h\bar\tau^k\tau^m\,d\tau\cdot t^{h+k+l_0-(2h+2k+2m)}a^h\bar a^k\times$$

$$\times\left(\frac{1}{\frac{1}{t^2}+|a|^2}\right)^{h+k+m+m-m} = \lim_{t\to+\infty}\sum_{l=l_0}^{+\infty}\sum_{h+k+2m=l} b_{h,k,m}\int\limits_{\partial\Delta_a}\tau^N\tau^h\bar\tau^k\tau^m\,d\tau\cdot t^{l_0-l}\times$$
$$\times a^h\bar a^k\left(\frac{1}{t^2}+|a|^2\right)^m\frac{1}{\left(\frac{1}{t^2}+|a|^2\right)^l} = \sum_{h+k+2m=l_0} b_{h,k,m}\int\limits_{\partial\Delta}\tau^N\tau^h\bar\tau^k\tau^m\,d\tau\frac{a^h\bar a^k\,|a|^{2m}}{|a|^{2l_0}} = 0,$$

where $\partial\Delta$ is defined by the correlation (10).

We will calculate the value of the integral $\int\limits_{\partial\Delta}\tau^N\tau^h\bar\tau^k\tau^m\,d\tau$, expressing $\bar\tau$ as a fractional-linear function from the correlation (10). We will obtain

$$\left|\tau-\frac{i}{2c_{11}}\right|^2 = \left(\tau-\frac{i}{2c_{11}}\right)\left(\bar\tau+\frac{i}{2c_{11}}\right) = \left(\tau-\frac{i}{2c_{11}}\right)\bar\tau+\frac{i}{2c_{11}}\tau+\frac{1}{4c_{11}^2}.$$

Then

$$\left(\tau-\frac{i}{2c_{11}}\right)\bar\tau+\frac{i}{2c_{11}}\tau+\frac{1}{4c_{11}^2} = \frac{1}{4c_{11}^2},$$
$$\bar\tau = \frac{-\frac{i}{2c_{11}}\tau}{\tau-\frac{i}{2c_{11}}}.$$

We will substitute the value obtained for $\bar\tau$ into the subintegral expression. We will obtain

$$\int\limits_{\partial\Delta} \tau^N \tau^h \bar\tau^k \tau^m \, d\tau = \int\limits_{\partial\Delta} \tau^{N+h+m} \bar\tau^k \, d\tau = \left(-\frac{i}{2c_{11}}\right)^k \int\limits_{\partial\Delta} \frac{\tau^{N+h+m}\tau^k}{\left(\tau - \dfrac{i}{2c_{11}}\right)^k} \, d\tau =$$

$$= \left(-\frac{i}{2c_{11}}\right)^k \int\limits_{\partial\Delta} \frac{\tau^{N+h+m+k}}{\left(\tau - \dfrac{i}{2c_{11}}\right)^k} \, d\tau.$$

$$\int\limits_{\partial\Delta} \frac{\tau^{N+h+m+k}}{\left(\tau - \dfrac{i}{2c_{11}}\right)^k} \, d\tau = 2\pi i \frac{1}{(k-1)!} \lim_{\tau\to\tau_0} \frac{d^{k-1}}{d\tau^{k-1}} \tau^{N+h+m+k} =$$

$$= 2\pi i \frac{1}{(k-1)!} \lim_{\tau\to\tau_0} \frac{(N+h+m+k)!}{(N+h+m+1)!} \tau^{N+h+m+1} =$$

$$= 2\pi i \frac{1}{(k-1)!} \frac{(N+h+m+k)!}{(N+h+m+1)!} \left(\frac{i}{2c_{11}}\right)^{N+h+m+1}.$$

Therefore,

$$\int\limits_{\partial\Delta} \tau^N \tau^h \bar\tau^k \tau^m \, d\tau = \left(-\frac{i}{2c_{11}}\right)^k 2\pi i \frac{1}{(k-1)!} \frac{(N+h+m+k)!}{(N+h+m+1)!} \left(\frac{i}{2c_{11}}\right)^{N+h+m+1} =$$

$$= (-1)^k \left(\frac{i}{2c_{11}}\right)^{N+h+m+k+1} 2\pi i \binom{N+h+m+k}{k-1}.$$

Let us complete the proof of the theorem. Whereas

$$\sum_{h+k+2m=l_0} b_{h,k,m} \int\limits_{\partial\Delta} \tau^N \tau^h \bar\tau^k \tau^m \, d\tau \cdot \frac{a^h \bar a^k |a|^{2m}}{|a|^{2l_0}} = 0,$$

then, substituting into the given expression the value of the integral obtained, we will have

$$\sum_{h+k+2m=l_0} b_{h,k,m} (-1)^k \left(\frac{i}{2c_{11}}\right)^{N+h+m+k+1} 2\pi i \binom{N+h+m+k}{k-1} \frac{a^h \bar a^k |a|^{2m}}{|a|^{2l_0}} = 0.$$

Choosing $N = k_0 - 1$, we will obtain the following correlation for coefficients $b_{h,k,m}$

$$\sum_{h+k_0+2m=l_0} (-1)^{k_0} \left(\frac{i}{2c_{11}}\right)^{2k_0+h+m} 2\pi i \binom{2k_0+h+m-1}{k_0-1} b_{h,k_0,m} a^{h+m} \bar a^{k_0+m} = 0.$$

Substituting $a = e^{i\theta}$, we will obtain

$$\sum_{h+k_0+2m=l_0} (-1)^{k_0} \left(\frac{i}{2c_{11}}\right)^{2k_0+h+m} 2\pi i \binom{2k_0+h+m-1}{k_0-1} b_{h,k_0,m} e^{i\theta(h-k_0)} = 0,$$

that means, that $b_{h,k_0,m} = 0$ for $h + k_0 + 2m = l_0$. Thus, for $k \geq 1$ we have $b_{h,k,m} = 0$ for any power degree l.

Therefore, we have shown, that the function f is holomorphic in the surrounding of the point 0. Due to the theorem condition, the function f holomorphically extends to the meeting of the domain D with every complex line, passing through the boundary point 0. Consequently, according to the Hartogs theorem about continuation (Theorem 1 from chapter 3, §11, p. 32 from [10]) and application of a fractional-linear transform (when the boundary point passes to an infinite one, and lines, passing through the boundary point, pass to the parallel lines) the function f will holomorphically extend along all the domain $D \subset \mathbb{C}^2$. These consideration complete the proof of the theorem in the two-dimensional case. $\square$

3. Multidimensional case

Let us consider n-dimensional complex space $\mathbb{C}^n$, points of which we will define by $w = (w_1, \dots, w_n)$, $z = (z_1, \dots, z_n)$ and etc. We should remind, that the domain D is called strictly convex, if the function $\rho(w_1, \dots, w_n)$, setting the boundary ∂D of the domain D, i. e. $D = \{w \mid \rho(w) < 0\}$, satisfies the condition

$$\sum_{p,j=1}^{n} \frac{\partial^2 \rho}{\partial w_p \partial w_j}(w^0)\, \xi_p \xi_j + \sum_{p,j=1}^{n} \frac{\partial^2 \rho}{\partial \bar{w}_p \partial \bar{w}_j}(w^0)\, \bar{\xi}_p \bar{\xi}_j + \sum_{p,j=1}^{n} \frac{\partial^2 \rho}{\partial w_p \partial \bar{w}_j}(w^0)\, \xi_p \bar{\xi}_j > 0$$

$\forall \xi \neq 0,\ w^0 \in \partial D$.

Let D be a bounded strictly convex domain in $\mathbb{C}^n$ $(n > 1)$ with a real-analytic boundary ∂D, i.e. $D = \{w \mid \rho(w) < 0\}$, where the function $\rho(w_1, \dots, w_n)$ is real-analytic in some neighborhood of the domain closure $\overline{D}$. With this, $grad\,\rho = \left(\frac{\partial \rho}{\partial w_1}, \dots, \frac{\partial \rho}{\partial w_n}\right) \neq 0$ on ∂D. Further all indexes $p, j, r, s \in \{1, \dots, n\}$.

Let us assume, that for all points of the boundary the following conditions hold true:

1. $p < j,\ r < s$

$$\begin{aligned} &4\frac{\partial \rho}{\partial w_p}\frac{\partial \rho}{\partial w_j}\frac{\partial^2 \rho}{\partial w_r \partial w_s} - 2\frac{\partial \rho}{\partial w_r}\frac{\partial \rho}{\partial w_p}\frac{\partial^2 \rho}{\partial w_s \partial w_j} - 2\frac{\partial \rho}{\partial w_s}\frac{\partial \rho}{\partial w_p}\frac{\partial^2 \rho}{\partial w_r \partial w_j} - \\ &-2\frac{\partial \rho}{\partial w_r}\frac{\partial \rho}{\partial w_j}\frac{\partial^2 \rho}{\partial w_s \partial w_p} - 2\frac{\partial \rho}{\partial w_s}\frac{\partial \rho}{\partial w_j}\frac{\partial^2 \rho}{\partial w_r \partial w_p} + 4\frac{\partial \rho}{\partial w_r}\frac{\partial \rho}{\partial w_s}\frac{\partial^2 \rho}{\partial w_p \partial w_j} = 0. \end{aligned} \tag{11}$$

2. $p < j$

$$\begin{aligned} &2\frac{\partial \rho}{\partial w_p}\frac{\partial \rho}{\partial w_j}\frac{\partial^2 \rho}{\partial w_r^2} - 2\frac{\partial \rho}{\partial w_r}\frac{\partial \rho}{\partial w_p}\frac{\partial^2 \rho}{\partial w_r \partial w_j} - 2\frac{\partial \rho}{\partial w_r}\frac{\partial \rho}{\partial w_j}\frac{\partial^2 \rho}{\partial w_r \partial w_p} + \\ &+2\left(\frac{\partial \rho}{\partial w_r}\right)^2 \frac{\partial^2 \rho}{\partial w_p \partial w_j} = 0. \end{aligned} \tag{12}$$

3. $p < r$

$$\left(\frac{\partial \rho}{\partial w_p}\right)^2 \frac{\partial^2 \rho}{\partial w_r^2} - 2\frac{\partial \rho}{\partial w_r}\frac{\partial \rho}{\partial w_p}\frac{\partial^2 \rho}{\partial w_r \partial w_p} + \left(\frac{\partial \rho}{\partial w_r}\right)^2 \frac{\partial^2 \rho}{\partial w_p^2} = 0. \tag{13}$$

Let us also define the family of complex lines, passing through the point w_0, $w_0 \in \partial D$ by $\mathfrak{L}_{w_0}$.

Lemma 2. *If boundaries of all two-dimensional sections of the domain D satisfy the condition* (2), *then in n dimensional case the following group of correlations* (11) – (13) *holds true.*

Proof. In the equation

$$\rho(w_1, \dots, w_n) = 0$$

we will put the following parametrization

$$\begin{cases} w_1 &= \alpha_1 \xi + \beta_1 \eta \\ w_2 &= \alpha_2 \xi + \beta_2 \eta \\ \ \vdots & \\ w_n &= \alpha_n \xi + \beta_n \eta. \end{cases}$$

We will obtain a two-dimensional section, defined by vectors $\alpha = (\alpha_1, \dots, \alpha_n)$ and $\beta = (\beta_1, \dots, \beta_n)$. The group of correlations (11) – (13) will be obtained from the fact, that the boundary $\rho(\xi, \eta)$ of the two-dimensional section should satisfy the condition (2) in variables ξ and η for any vectors α and β, i.e. the following condition should hold true

$$\left(\frac{\partial\rho}{\partial\eta}(\xi,\eta)\right)^2 \frac{\partial^2\rho}{\partial\xi^2}(\xi,\eta) - 2\frac{\partial\rho}{\partial\xi}(\xi,\eta)\frac{\partial\rho}{\partial\eta}(\xi,\eta)\frac{\partial^2\rho}{\partial\xi\partial\eta}(\xi,\eta) + \left(\frac{\partial\rho}{\partial\xi}(\xi,\eta)\right)^2 \frac{\partial^2\rho}{\partial\eta^2}(\xi,\eta) = 0. \quad (14)$$

Let us find derivatives, included in the equation (14). We will obtain

$$\frac{\partial\rho}{\partial\xi} = \frac{\partial\rho}{\partial w_1}\frac{\partial w_1}{\partial\xi} + \frac{\partial\rho}{\partial w_2}\frac{\partial w_2}{\partial\xi} + \ldots + \frac{\partial\rho}{\partial w_n}\frac{\partial w_n}{\partial\xi} =$$

$$= \alpha_1\frac{\partial\rho}{\partial w_1} + \alpha_2\frac{\partial\rho}{\partial w_2} + \ldots + \alpha_n\frac{\partial\rho}{\partial w_n} = \sum_{r=1}^{n}\frac{\partial\rho}{\partial w_r}\alpha_r.$$

$$\frac{\partial\rho}{\partial\eta} = \frac{\partial\rho}{\partial w_1}\frac{\partial w_1}{\partial\eta} + \frac{\partial\rho}{\partial w_2}\frac{\partial w_2}{\partial\eta} + \ldots + \frac{\partial\rho}{\partial w_n}\frac{\partial w_n}{\partial\eta} =$$

$$= \beta_1\frac{\partial\rho}{\partial w_1} + \beta_2\frac{\partial\rho}{\partial w_2} + \ldots + \beta_n\frac{\partial\rho}{\partial w_n} = \sum_{p=1}^{n}\frac{\partial\rho}{\partial w_p}\beta_p.$$

$$\frac{\partial^2\rho}{\partial\xi^2} = \alpha_1\left(\alpha_1\frac{\partial^2\rho}{\partial w_1\partial w_1} + \alpha_2\frac{\partial^2\rho}{\partial w_1\partial w_2} + \ldots + \alpha_n\frac{\partial^2\rho}{\partial w_1\partial w_n}\right) +$$

$$+ \alpha_2\left(\alpha_1\frac{\partial^2\rho}{\partial w_2\partial w_1} + \alpha_2\frac{\partial^2\rho}{\partial w_2\partial w_2} + \ldots + \alpha_n\frac{\partial^2\rho}{\partial w_2\partial w_n}\right) + \ldots +$$

$$+ \alpha_n\left(\alpha_1\frac{\partial^2\rho}{\partial w_n\partial w_1} + \alpha_2\frac{\partial^2\rho}{\partial w_n\partial w_2} + \ldots + \alpha_n\frac{\partial^2\rho}{\partial w_n\partial w_n}\right) = \sum_{r,s=1}^{n}\frac{\partial^2\rho}{\partial w_r\partial w_s}\alpha_r\alpha_s.$$

$$\frac{\partial^2\rho}{\partial\eta^2} = \beta_1\left(\beta_1\frac{\partial^2\rho}{\partial w_1\partial w_1} + \beta_2\frac{\partial^2\rho}{\partial w_1\partial w_2} + \ldots + \beta_n\frac{\partial^2\rho}{\partial w_1\partial w_n}\right) +$$

$$+ \beta_2\left(\beta_1\frac{\partial^2\rho}{\partial w_2\partial w_1} + \beta_2\frac{\partial^2\rho}{\partial w_2\partial w_2} + \ldots + \beta_n\frac{\partial^2\rho}{\partial w_2\partial w_n}\right) + \ldots +$$

$$+ \beta_n\left(\beta_1\frac{\partial^2\rho}{\partial w_n\partial w_1} + \beta_2\frac{\partial^2\rho}{\partial w_n\partial w_2} + \ldots + \beta_n\frac{\partial^2\rho}{\partial w_n\partial w_n}\right) = \sum_{p,j=1}^{n}\frac{\partial^2\rho}{\partial w_p\partial w_j}\beta_p\beta_j.$$

$$\frac{\partial^2\rho}{\partial\xi\partial\eta} = \alpha_1\left(\beta_1\frac{\partial^2\rho}{\partial w_1\partial w_1} + \beta_2\frac{\partial^2\rho}{\partial w_1\partial w_2} + \ldots + \beta_n\frac{\partial^2\rho}{\partial w_1\partial w_n}\right) +$$

$$+ \alpha_2\left(\beta_1\frac{\partial^2\rho}{\partial w_2\partial w_1} + \beta_2\frac{\partial^2\rho}{\partial w_2\partial w_2} + \ldots + \beta_n\frac{\partial^2\rho}{\partial w_2\partial w_n}\right) + \ldots +$$

$$+ \alpha_n\left(\beta_1\frac{\partial^2\rho}{\partial w_n\partial w_1} + \beta_2\frac{\partial^2\rho}{\partial w_n\partial w_2} + \ldots + \beta_n\frac{\partial^2\rho}{\partial w_n\partial w_n}\right) = \sum_{s,j=1}^{n}\frac{\partial^2\rho}{\partial w_s\partial w_j}\alpha_s\beta_j.$$

Then, taking into account the above results,

$$\left(\frac{\partial\rho}{\partial\eta}\right)^2 = \sum_{p,j=1}^{n}\frac{\partial\rho}{\partial w_p}\frac{\partial\rho}{\partial w_j}\beta_p\beta_j.$$

$$\left(\frac{\partial\rho}{\partial\xi}\right)^2 = \sum_{r,s=1}^{n}\frac{\partial\rho}{\partial w_r}\frac{\partial\rho}{\partial w_s}\alpha_r\alpha_s.$$

Therefore,

$$\left(\frac{\partial\rho}{\partial\eta}\right)^2\frac{\partial^2\rho}{\partial\xi^2}=\sum_{p,j,r,s=1}^{n}\frac{\partial\rho}{\partial w_p}\frac{\partial\rho}{\partial w_j}\frac{\partial^2\rho}{\partial w_r\partial w_s}\beta_p\beta_j\alpha_r\alpha_s,$$

$$\frac{\partial\rho}{\partial\xi}\frac{\partial\rho}{\partial\eta}\frac{\partial^2\rho}{\partial\xi\partial\eta}=\sum_{p,j,r,s=1}^{n}\frac{\partial\rho}{\partial w_r}\frac{\partial\rho}{\partial w_p}\frac{\partial^2\rho}{\partial w_s\partial w_j}\beta_p\beta_j\alpha_r\alpha_s,$$

$$\left(\frac{\partial\rho}{\partial\xi}\right)^2\frac{\partial^2\rho}{\partial\eta^2}=\sum_{p,j,r,s=1}^{n}\frac{\partial\rho}{\partial w_r}\frac{\partial\rho}{\partial w_s}\frac{\partial^2\rho}{\partial w_p\partial w_j}\beta_p\beta_j\alpha_r\alpha_s.$$

Let us substitute values of derivatives in the correlation (14). We will obtain, that for any vectors α and β the following condition should be satisfied

$$\sum_{p,j,r,s=1}^{n}\left(\frac{\partial\rho}{\partial w_p}\frac{\partial\rho}{\partial w_j}\frac{\partial^2\rho}{\partial w_r\partial w_s}-2\frac{\partial\rho}{\partial w_r}\frac{\partial\rho}{\partial w_p}\frac{\partial^2\rho}{\partial w_s\partial w_j}+\frac{\partial\rho}{\partial w_r}\frac{\partial\rho}{\partial w_s}\frac{\partial^2\rho}{\partial w_p\partial w_j}\right)\beta_p\beta_j\alpha_r\alpha_s=0.$$

Further we will consider the similar ones in the given sum, with the indexes $p,j,r,s\in\{1,\dots,n\}$. We will obtain

$$\sum_{\substack{p<j\\ r,s}}\left(2\frac{\partial\rho}{\partial w_p}\frac{\partial\rho}{\partial w_j}\frac{\partial^2\rho}{\partial w_r\partial w_s}-2\frac{\partial\rho}{\partial w_r}\frac{\partial\rho}{\partial w_p}\frac{\partial^2\rho}{\partial w_s\partial w_j}-2\frac{\partial\rho}{\partial w_r}\frac{\partial\rho}{\partial w_j}\frac{\partial^2\rho}{\partial w_s\partial w_p}+\right.$$
$$\left.+2\frac{\partial\rho}{\partial w_r}\frac{\partial\rho}{\partial w_s}\frac{\partial^2\rho}{\partial w_p\partial w_j}\right)\beta_p\beta_j\alpha_r\alpha_s+$$
$$+\sum_{\substack{p=j\\ r,s}}\left(\left(\frac{\partial\rho}{\partial w_p}\right)^2\frac{\partial^2\rho}{\partial w_r\partial w_s}-2\frac{\partial\rho}{\partial w_r}\frac{\partial\rho}{\partial w_p}\frac{\partial^2\rho}{\partial w_s\partial w_p}+\frac{\partial\rho}{\partial w_r}\frac{\partial\rho}{\partial w_s}\frac{\partial^2\rho}{\partial w_p^2}\right)\beta_p^2\alpha_r\alpha_s=0.$$

$$\sum_{\substack{p<j\\ r<s}}\left(4\frac{\partial\rho}{\partial w_p}\frac{\partial\rho}{\partial w_j}\frac{\partial^2\rho}{\partial w_r\partial w_s}-2\frac{\partial\rho}{\partial w_r}\frac{\partial\rho}{\partial w_p}\frac{\partial^2\rho}{\partial w_s\partial w_j}-2\frac{\partial\rho}{\partial w_s}\frac{\partial\rho}{\partial w_p}\frac{\partial^2\rho}{\partial w_r\partial w_j}-\right.$$
$$\left.-2\frac{\partial\rho}{\partial w_r}\frac{\partial\rho}{\partial w_j}\frac{\partial^2\rho}{\partial w_s\partial w_p}-2\frac{\partial\rho}{\partial w_s}\frac{\partial\rho}{\partial w_j}\frac{\partial^2\rho}{\partial w_r\partial w_p}+4\frac{\partial\rho}{\partial w_r}\frac{\partial\rho}{\partial w_s}\frac{\partial^2\rho}{\partial w_p\partial w_j}\right)\beta_p\beta_j\alpha_r\alpha_s+$$
$$+\sum_{\substack{p<j\\ r=s}}\left(2\frac{\partial\rho}{\partial w_p}\frac{\partial\rho}{\partial w_j}\frac{\partial^2\rho}{\partial w_r^2}-2\frac{\partial\rho}{\partial w_r}\frac{\partial\rho}{\partial w_p}\frac{\partial^2\rho}{\partial w_r\partial w_j}-2\frac{\partial\rho}{\partial w_r}\frac{\partial\rho}{\partial w_j}\frac{\partial^2\rho}{\partial w_r\partial w_p}+\right.$$
$$\left.+2\left(\frac{\partial\rho}{\partial w_r}\right)^2\frac{\partial^2\rho}{\partial w_p\partial w_j}\right)\beta_p\beta_j\alpha_r^2+\sum_{\substack{p=j\\ r<s}}\left(2\left(\frac{\partial\rho}{\partial w_p}\right)^2\frac{\partial^2\rho}{\partial w_r\partial w_s}-2\frac{\partial\rho}{\partial w_r}\frac{\partial\rho}{\partial w_p}\frac{\partial^2\rho}{\partial w_s\partial w_p}-\right.$$
$$\left.-2\frac{\partial\rho}{\partial w_s}\frac{\partial\rho}{\partial w_p}\frac{\partial^2\rho}{\partial w_r\partial w_p}+2\frac{\partial\rho}{\partial w_r}\frac{\partial\rho}{\partial w_s}\frac{\partial^2\rho}{\partial w_p^2}\right)\beta_p^2\alpha_r\alpha_s+$$
$$+\sum_{\substack{p=j\\ r=s}}\left(\left(\frac{\partial\rho}{\partial w_p}\right)^2\frac{\partial^2\rho}{\partial w_r^2}-2\frac{\partial\rho}{\partial w_r}\frac{\partial\rho}{\partial w_p}\frac{\partial^2\rho}{\partial w_r\partial w_p}+\left(\frac{\partial\rho}{\partial w_r}\right)^2\frac{\partial^2\rho}{\partial w_p^2}\right)\beta_p^2\alpha_r^2=0.$$

Whereas the given equality is satisfied for any vectors α and β and there are given the similar ones (all sums consist of different components), then the equality to zero means, that all expressions with sum signs are equal to zero. Note, that the second and the third sums in the latter correlation are obtained from each other by means of indexes transforms. Therefore, we have shown the validity of the correlations (11) – (13). $\square$

Theorem 5. *Let the function $f \in C^w(\partial D)$ possess one-dimensional holomorphic extension property along all the complex lines from $\mathfrak{L}_{w_0}$, meeting D, then the function f holomorphically extends in D.*

Remark 2. *If the point w_0 is fixed in advance, then the condition* (11) – (13) *can be satisfied only in the point w_0.*

Proof. We will make a two-dimensional section of the domain D, passing through the boundary point w_0 and the point 0, being in the domain D. The function, setting the boundary of such a two-dimensional section, will satisfy the condition (2), hence the function f according to the previous item, will holomorphically extend in the interior of such two-dimensional sections and define the function F in them. Whereas holomorphic extension of function f in two-dimensional sections is given by a two-dimensional the Bochner-Martinelli integral F, then in all two-dimensional sections holomorphic extensions coincide. These two-dimensional sections cover all the domain D. Hence, the function F is defined in all the domain D.

Whereas holomorphic extension of function f in two-dimensional sections is set by the Bochner-Martinelli integral, depending real-analytically on vectors α and β, then holomorphic extension of the function f is a real analytic function. Therefore, the function F belongs to the class C^∞ in the neighborhood of the point 0.

Whereas the two-dimensional section is defined by two complex lines, then the function F, being holomorphic in all the two-dimensional section, will also be holomorphic on complex lines, lying in the same section. Thus, the function F will be holomorphic in the meeting of the domain D with every complex line, passing through the point 0.

Thus, we have shown, that the function F belongs to the class C^∞ in the neighborhood of the point 0 and is holomorphic in the meeting of the domain D with every complex line, passing through the point 0. We are in the condition of the Forelli theorem (Theorem 4.4.5 from [11]) and, applying this theorem, we will obtain, that the function F will be holomorphic in some surrounding of the point 0.

Whereas the function F is holomorphic in some surrounding of the point 0 and in the meeting of the domain D with every complex line, passing through the given point, then, according to the Hartogs theorem on extension (Theorem 1 from chapter 3, §11, p. 32 from [10]) by analogy to the previous item, it is holomorphic in al the domain D. □

4. Examples

In the given item we consider examples of domains, for which the following theorem holds true 5.

Example 1. $D = \mathbb{B}^n$ is a ball with the center in the zero of the radius R, i. e. $D = \{\zeta : |\zeta| < R\}$.

Example 2. Let $\zeta_j = \dfrac{L_j(w)}{L(w)}$, $j = 1, \ldots, n$, where $L_j(w)$, $L(w)$ are linear functions, then the image of the ball $\mathbb{B}^n$ with the given mapping (if it is not singular) is a domain, for which the following theorem holds true 5.

Example 3. Let the function ρ, setting the domain boundary D, have the form

$$\rho(w_1, \ldots, w_n) = |w_1|^2 + \ldots + |w_n|^2 - R^2 + \sum_j |L_j(w)|^2,$$

where $L_j(w)$ are linear functions. Then for the domain $D = \{w \,|\, \rho(w) < 0\}$ the following conditions of the theorem hold true 5.

Example 4. Let the function $\rho(w, \bar{w})$ lineally depending on w and on $\bar{w}$ is arbitrary . Then the domain $D = \{w \mid \rho(w) < 0\}$ satisfies the theorem conditions 5.

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