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AN ALMOST EXPONENTIAL SEQUENCE OF EXPONENTIAL POLYNOMIALS

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Abstract. A special sequence of exponential polynomials, whose exponentials are divided into relatively small groups, is studied in the article. It is proved that this sequence is almost an exponential sequence for each convex domain of a complex plane. By means of this result necessary and sufficient conditions for the considered sequence to be a basis in a closed and invariant under differentiation subspace of the space of analytic functions in a convex domain are obtained. Two methods of description of the whole class of bases in an invariant subspace, whose elements are exponential polynomials, are given.

Keywords: exponential polynomial, invariant subspace, analytic function, convex domain, basis.

Let D be a convex domain in $\mathbb{C}$ and $\{K_p\}_{p=1}^{\infty}$ be the sequence of convex compacts, exhausting domain D, i.e. the following holds true: 1) $K_p \subset intK_{p+1}$ for all $p \geq 1$ (int defined a set interior), 2) $D = \bigcup_{p=1}^{\infty} K_p$. Let $H_M(z)$ define a support function of the set M (to be exact, complex conjugated with M set):

$$H_M(z) = \sup_{w \in M} Re(zw), \quad z \in \mathbb{C}.$$

Then it results from the condition 1) that for any $p \geq 1$ there is a number $\alpha_p > 0$ such that

$$H_{K_p}(z) + \alpha_p|z| \leqslant H_{K_{p+1}}(z), \quad z \in \mathbb{C}. \tag{1}$$

In paper [1] there was introduced the following notion. Sequence of functions $\{e_m\}_{m=1}^{\infty}$, analytical in the domain D, is called almost exponential, if there are numbers $\lambda_m \in \mathbb{C}$, $m \geq 1$, $|\lambda_m| \to \infty$ with $m \to \infty$, for which the following two conditions hold true: 1) for any $p \geq 1$ there is a constant $a > 0$ and a number s such that

$$\sup_{w \in K_p} |e_m(w)| \leqslant a \exp(H_{K_s}(\lambda_m)), \quad m = 1, 2, \dots;$$

2) for any $p \geq 1$ there is a constant $b > 0$ and a number s such that

$$b \exp(H_{K_p}(\lambda_m)) \leqslant \sup_{w \in K_s} |e_m(w)|, \quad m = 1, 2, \dots$$

Note, that the definition of an almost exponential sequence is bounded to a definite convex domain D. Therefore, it is more correct to call such a sequence almost exponential in the domain D. Numbers $\lambda_m \in \mathbb{C}$, $m \geq 1$ are called indexes of functions $\{e_m\}_{m=1}^{\infty}$. Examples of such almost exponential sequences can be sequences of exponents themselves, and also sequences of exponential monomials $\{z^n \exp(\lambda_m z)\}_{m=1,n=1}^{\infty,k_m}$ according to the condition $k_m/|\lambda_m| \to 0$ (see [1]). In paper [2] there was some more general sequence of functions $\{e_m\}_{m=1}^{\infty}$ considered, compiled of linear combinations of exponential monomials, which indexes are divided into so-called "relatively small groups". Such sequences appear naturally during studying classical problems of

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functions representation from the space, which is invariant to the action of some linear operator, by means of its own and associated functions of this operator. In paper [2] there were closed subspaces W studied, which are invariant with respect to the operator of differentiation, in the space $H(D)$ of functions, analytic in the convex domain D, with the topology of uniform convergence on compact subsets from D. If W is a nontrivial ($W \neq H(D)$ and $W \neq \{0\}$) subspace in $H(D)$, then the spectrum of the differentiation operator in W is not more that a countable set $\{\lambda_k\}$ (see [2]). Hence, if the spectrum is infinite, then its only accumulation point is ∞. Therefore, eigenfunctions of the differentiation operator in W are exponents with indexes λ_k. Associated functions will be, correspondingly, exponential monomials $z^n \exp(\lambda_k z)$, where $n = 1, \dots, n_k - 1$ (the natural number n_k can be defined as order of zero of some entire exponential function, connected with the subspace W [2]). In the case, when the set $\{\lambda_k\}$ is finite, the subspace W coincides with the space of linear differential equations solutions with constant coefficients (see, for instance, [3, chapter 4]). Then, according to the fundamental principle of L. Euler, every solution of such an equation is a linear combination of eigenfunctions and associated functions of the differentiation operator in W. Due to this, it is expedient to consider only invariant subspaces $W \subset H(D)$, with an infinite spectrum $\{\lambda_k\}_{k=1}^{\infty}$. For such subspaces we deal with an infinite system $\mathcal{E} = \{z^n \exp(\lambda_k z)\}_{k=1,n=0}^{\infty,n_k-1}$ of eigenfunctions and associated functions. If points of the spectrum are rather far from each other (see [4]), then in case of some supplementary conditions of the spectrum $\{\lambda_k\}_{k=1}^{\infty}$ and natural numbers n_k, $k = 1, 2, \dots$, (see [4],[5]) in the subspace W we also deal with the fundamental principle: every function from W is represented close due to the system $\mathcal{E}$, which absolutely and uniformly converges on compacts from the domain D. In case of spectrum points "sticking" such a representation is impossible [4]. However, even in this case in the subspace W there can exist a basis, compiled from linear combinations of eigenfunctions and associated functions of a differentiation operator, the indexes of which are divided into relatively small groups (see, for instance, [6]).

Let the sequence $\{\lambda_k\}$ be divided into groups U_m, $m = 1, 2, \dots$. Let us renumber the sequence members. The points λ_k, fallen in the group U_m, will be defined as $\lambda_{m,l}$, and their order (i.e. the number n_k) as $n_{m,l}$. The first index here m coincides with the group number, and the second index varies within the limits from 1 to M_m, where M_m is a number of the spectrum points, fallen into the group U_m. They say, that groups U_m, $m = 1, 2, \dots$, are relatively small, if the following holds true:

$$\lim_{m\to\infty} \max_{1\leqslant j,l\leqslant M_m} \frac{|\lambda_{m,j} - \lambda_{m,l}|}{|\lambda_{m,1}|} = 0.$$

Note, that the numbers $\lambda_{m,1}$ can be replaced here by any other representatives $\lambda_{m,j}$ of groups U_m. It immediately results from the correlation

$$\lim_{m\to\infty} \max_{1\leqslant j\leqslant M_m} \frac{|\lambda_{m,j}|}{|\lambda_{m,1}|} \leqslant \lim_{m\to\infty} \max_{1\leqslant j\leqslant M_m} \frac{|\lambda_{m,j} - \lambda_{m,1}|}{|\lambda_{m,1}|} + \lim_{m\to\infty} \frac{|\lambda_{m,1}|}{|\lambda_{m,1}|} = 1.$$

In new symbols the system of eigenfunctions and associated functions looks the following way $\mathcal{E} = \{z^n \exp(\lambda_{m,l} z)\}_{m=1,l=1,n=0}^{\infty,M_m,n_{m,l}-1}$. Let N_m be a number of spectrum points, fallen into the group U_m, $m = 1, 2, \dots$, according to their order, i.e. $N_m = \sum_{l=1}^{M_m} n_{m,l}$. Let us construct the function system $\tilde{\mathcal{E}} = \{e_{m,p}(z)\}_{m=1,p=1}^{\infty,N_m}$ according to the system $\mathcal{E}$. Assume, that

$$e_{m,p}(z) = \frac{(p-1)!}{2\pi i} \int\limits_{|\lambda-\lambda_{m,1}|=1} \frac{q_m(\lambda, z)d\lambda}{(\lambda - \lambda_{m,1})^p}, \quad p = 1, \dots, N_m, \quad m = 1, 2, \dots. \tag{2}$$

Therein

$$q_m(\lambda, z) = \frac{1}{2\pi i} \int\limits_{\Gamma_m} \frac{\exp(z\zeta)(\omega_m(\zeta) - \omega_m(\lambda))}{(\zeta - \lambda)\omega_m(\zeta)} d\zeta, \quad m = 1, 2, \ldots,$$

Γ_m is a contour, covering the points $\lambda_{m,l}$, $l = 1, 2, \ldots, M_m$, groups U_m, and $\omega_m(\lambda)$ is a monomial with these zeros with the account of their order and with the leading coefficient, is equal to 1, i.e.

$$\omega_m = \prod_{l=1}^{M_m} (\lambda - \lambda_{m,l})^{n_{m,l}}, \quad m = 1, 2, \ldots$$

From (2), applying the theorem of subtraction, we obtain the following equalities

$$e_{m,p}(z) = \sum_{l=1}^{M_m} \sum_{n=0}^{n_{m,l}-1} c_{m,p,l,n} z^n \exp(\lambda_{m,l} z), \quad m = 1, 2, \ldots, \quad p = 1, 2, \ldots, N_m. \tag{3}$$

In paper [2] according to the condition, that the sequence $\tilde{\mathcal{E}}$ is almost exponential, we obtain necessary and sufficient conditions of $\tilde{\mathcal{E}} = \{e_{m,p}(z)\}_{m=1,p=1}^{\infty,N_m}$ which is absolute and uniform basis in the subspace W. Under the same condition we can find description of all possible bases in W of the form (3), constructed on the basis of relatively small groups U_m.

In this connection there appears a problem of conditions clearing-up, when the system $\tilde{\mathcal{E}} = \{e_{m,p}(z)\}_{m=1,p=1}^{\infty,N_m}$ is an almost exponential sequence. The aim of this work is to show, that if the equality

$$\mathcal{N} = \overline{\lim_{m\to\infty}} \frac{N_m}{|\lambda_{m,1}|} = 0 \tag{4}$$

holds true, the sequence $\tilde{\mathcal{E}}$ will be exponential.

It is proved (corollary from Lemma 5) in paper [2], that with $\mathcal{N} = 0$ for any $j \geq 1$ there is a constant C_j and a number $s > j$ such that

$$\sup_{w\in K_j} |e_{m,p}(w)| \leqslant C_j \exp H_{K_s}(\lambda_{m,1}), \quad m = 1, 2, \ldots, \quad p = 1, \ldots, N_m.$$

It means, that for the system $\tilde{\mathcal{E}} = \{e_{m,p}(z)\}_{m=1,p=1}^{\infty,N_m}$ item 1) holds true from the almost exponential sequence. Further we will show, that with $\mathcal{N} = 0$ for $\tilde{\mathcal{E}}$ item 2) also holds true. To prove this fact we will apply Theorem 1 from paper [2].

We will need some supplementary definitions and symbols. For a convex domain D and every $s = 1, 2, \ldots$ we will define a Banach space of entire exponential functions

$$P_s = \{f \in H(\mathbb{C}) : \|f\|_s = \sup_{\lambda\in\mathbb{C}} |f(\lambda)| \exp(-H_{K_s}(\lambda)) < \infty\},$$

and by $\mathcal{P}_D$ we will define an inductive limit of the spaces $\mathcal{P}_s$. Note (see, for instance, [3]), that the Laplace transform $L(\mu)(\lambda) = (\mu, \exp \lambda z)$ sets algebraic and topological isomorphism between the space $\mathcal{P}_D$ and the space $H^*(D)$ which are linear continuous functionals on $H(D)$.

For any $s = 1, 2, \ldots$ we will introduce a Banach space of complex sequences

$$R_s = \{b = \{b_{m,j}\} : \|b\|_s = \sup_{m,j}(|b_{m,j}| \exp(-H_{K_s}(\lambda_{m,1}))) < \infty\}.$$

Therein $m = 1, 2, \ldots$ and $j = 1, \ldots, N_m$. Let $R(D)$ be an inductive spaces limit R_s. For the entire function $f(\zeta)$ we will assume

$$q_m(\lambda, f) = \frac{1}{2\pi i} \int\limits_{\Gamma_m} \frac{f(\zeta)(\omega_m(\zeta) - \omega_m(\lambda))}{(\zeta - \lambda)\omega_m(\zeta)} d\zeta, \quad m = 1, 2, \ldots,$$

where the contour Γ_m and the monomial $\omega_m(\zeta)$ are similar to the described above. This formula defines a known interpolation polynomial of the degree not more than $N_m - 1$, which takes values, coinciding with the corresponding function values $f(\zeta)$ and its derivatives in points $\lambda_{m,l}$ together with its derivatives up to the order $n_{m,l} - 1$ inclusive, i.e.

$$q_m^{(n)}(\lambda_{m,l}, f) = f^{(n)}(\lambda_{m,l}), \quad l = 1, 2, \ldots, M_m, \quad n = 0, 1, \ldots, n_{m,l} - 1.$$

Let us decompose $q_m(\lambda, f)$ by monomials $(\lambda - \lambda_{m,1})^j$:

$$q_m(\lambda, f) = \sum_{j=0}^{N_m - 1} q_{m,j}(f) \frac{(\lambda - \lambda_{m,1})^j}{j!}, \quad m = 1, 2, \ldots$$

It is shown in paper [2] (Lemma 5), that for any function f from the space $\mathcal{P}_D$ the sequence of numbers $q(f) = \{q_{m,j-1}(f)\}_{m=1,j=1}^{\infty, N_m}$ belongs to the space $R(D)$.

Let $B(z, r)$ and $S(z, r)$ define an open circle and a circumference with the center in the point z and radius r correspondingly. To apply Theorem 1 from paper [2], as it was pointed out above, we need to prove the following auxiliary statements.

Lemma 1. *Let $h(z)$ be a positive uniform of the first order and continuous on the complex plane function. For any $\varepsilon > 0$ there is $\delta > 0$ such that the following inequality holds true*

$$\sup_{\lambda \in B(z, \delta|z|)} h(\lambda) \leqslant \inf_{\lambda \in B(z, \delta|z|)} h(\lambda) + \varepsilon \inf_{\lambda \in B(z, \delta|z|)} |\lambda|, \quad z \in \mathbb{C}.$$

Proof. Let us fix $\varepsilon > 0$. Due to the uniform function continuity $h(z)$ on the circle $S(0, 1)$ there is $\delta \in (0, 1/2)$ such that for any $z \in S(0, 1)$ and all $\lambda, w \in B(z, \delta)$ the following inequality holds true

$$|h(\lambda) - h(w)| \leqslant \varepsilon/2.$$

Hereof, considering the uniformity of the function $h(z)$ we obtain:

$$\sup_{\lambda \in B(z, \delta|z|)} h(\lambda) = |z| \sup_{\lambda \in B(z/|z|, \delta)} h(\lambda) \leqslant |z| \inf_{\lambda \in B(z/|z|, \delta)} (h(\lambda) + \varepsilon/2) = |z| \inf_{\lambda \in B(z/|z|, \delta)} h(\lambda) +$$

$$+ 2^{-1}\varepsilon|z| \leqslant \inf_{\lambda \in B(z, \delta|z|)} h(\lambda) + 2^{-1}(1 - \delta)^{-1}\varepsilon \inf_{\lambda \in B(z, \delta|z|)} |\lambda| \leqslant$$

$$\leqslant \inf_{\lambda \in B(z, \delta|z|)} h(\lambda) + \varepsilon \inf_{\lambda \in B(z, \delta|z|)} |\lambda|, \quad z \in \mathbb{C}.$$

The Lemma has been proved.

Lemma 2. *Let D be a convex domain, and sequence $\{\lambda_{m,l}\}$ be decomposed into relatively small groups U_m. Suppose, that $N_m/|\lambda_{m,1}| \leqslant 2^{-m}$ and $|\lambda_{m+1,1}| \geq 2|\lambda_{m,1}|$, $m = 1, 2, \ldots$ Then for every sequence $b = \{b_{m,j}\}$ from the space $R(D)$ there is the function $f \in \mathcal{P}_D$ such that $b_{m,j} = q_{m,j-1}(f)$, $m = 1, 2, \ldots$, $j = 1, \ldots, N_m$.*

Proof. Let the sequence $b = \{b_{m,j}\}$ belong to $R(D)$. Then, according to the definition of the space $R(D)$ there is a number s such that

$$\|b\|_s = \sup_{m,j}(|b_{m,j}| \exp(H_{K_s}(\lambda_{m,1}))) < \infty,$$

i.e. for some constant $C > 0$ the following inequalities hold true

$$|b_{m,j}| \leqslant C\exp(-H_{K_s}(\lambda_{m,1})), \quad m = 1, 2, \dots, \quad j = 1, \dots, N_m. \tag{5}$$

The construction of an entire function, which existence is stated in the Lemma, will be made in two stages. At the first stage we will construct a polynomial P_m for every group U_m, $m = 1, 2, \dots$, which will satisfy the necessary upper estimate and such that $q_{m,j-1}(P_m) = b_{m,j}$, $j = 1, \dots, N_m$. At the second stage, having improved the polynomials stated, we will "stick" them up to the needed entire function.

Let us proceed to the first stage. Assume

$$P_m(\lambda) = \sum_{j=0}^{N_m-1} b_{m,j+1}\frac{(\lambda - \lambda_{m,1})^j}{j!}, \quad m = 1, 2, \dots$$

For every $m = 1, 2, \dots$ we have:

$$q_m(\lambda, P_m) = \frac{1}{2\pi i}\int\limits_{\Gamma_m} \frac{P_m(\zeta)(\omega_m(\zeta) - \omega_m(\lambda))}{(\zeta - \lambda)\omega_m(\zeta)} d\zeta.$$

This equality defines the polynomial of the degree not more than $N_m - 1$, which in the points $\lambda_{m,l}$ together with its derivatives up to the order $n_{m,l} - 1$ inclusive, takes values, coinciding with the corresponding polynomial values $P_m(\lambda)$ and its derivatives. Whereas $P_m(\lambda)$ also possesses a degree not more than $N_m - 1$, and the number of points $\lambda_{m,l}$ in the group U_m subject to their order $n_{m,l}$ is equal to N_m, then polynomials $q_m(\lambda, P_m)$ and $P_m(\lambda)$ coincide. Then it is easy to obtain the following equalities from polynomial definitions $P_m(\lambda)$ and numbers $q_{m,j}(P_m)$

$$b_{m,j} = q_{m,j-1}(P_m), \quad m = 1, 2, \dots, \quad j = 1, \dots, N_m. \tag{6}$$

Now we will find upper estimates for all polynomial modules $P_m(\lambda)$. According to the condition $a_m = |\lambda_{m,1}|/N_m \geq 2^m$. Taking into account, that $j! \geq j^j/3^j$ with all $j \geq 1$, and the function $4x^{-1}\ln(3x)$ decreases with $x > 1$, for all $m = 1, 2, \dots$ and $4j = 0, 1, \dots, N_m - 1$ we have:

$$\frac{\ln(|\lambda_{m,1}|^j/j!)}{|\lambda_{m,1}|} \leqslant \frac{\ln(3^j|\lambda_{m,1}|^j/j^j)}{|\lambda_{m,1}|} = \frac{j\ln(3|\lambda_{m,1}|/j)}{|\lambda_{m,1}|} \leqslant \frac{N_m\ln(3|\lambda_{m,1}|/N_m)}{|\lambda_{m,1}|} = \frac{\ln(3a_m)}{a_m} = \varepsilon(m),$$

where $\varepsilon(m) \to 0$ with $m \to \infty$. Hereof we obtain

$$\frac{|\lambda_{m,1}|^j}{j!} \leqslant \exp(\varepsilon(m)|\lambda_{m,1}|), \quad m = 1, 2, \dots, \quad j = 0, \dots, N_m - 1.$$

Therefore, for all $m = 1, 2, \dots$ and $\lambda \in B(\lambda_{m,1}, |\lambda_{m,1}|)$ the following estimate holds true

$$|P_m(\lambda)| \leqslant \sum_{j=0}^{N_m-1} |b_{m,j+1}| \frac{|(\lambda - \lambda_{m,1})^j|}{j!} \leqslant \sum_{j=0}^{N_m-1} |b_{m,j+1}| \frac{|\lambda_{m,1}|^j}{j!} \leqslant \exp(\varepsilon(m)|\lambda_{m,1}|) \sum_{j=0}^{N_m-1} |b_{m,j+1}|.$$

Hereof, subject to (5) we obtain:

$$|P_m(\lambda)| \leqslant CN_m \exp(H_{K_s}(\lambda_{m,1}) + \varepsilon(m)|\lambda_{m,1}|), \quad m = 1, 2, \dots, \quad \lambda \in B(\lambda_{m,1}, |\lambda_{m,1}|).$$

Whereas $N_m/|\lambda_{m,1}| \to 0$, then we can consider, that for some number m_0 the following inequalities hold true

$$N_m \leqslant \exp(2^{-1}\alpha_s|\lambda_{m,1}|), \quad m \geq m_0,$$

where α_s is a constant from formula (1). Moreover, $\varepsilon(m) \to 0$ with $m \to \infty$. Therefore, we can also consider, that

$$2\varepsilon(m) \leqslant \alpha_s, \quad m \geq m_0.$$

Consequently, from the written above and (1) we obtain:

$$|P_m(\lambda)| \leqslant C \exp(H_{K_s}(\lambda_{m,1}) + \alpha_s|\lambda_{m,1}|) \leqslant C \exp(H_{K_{s+1}}(\lambda_{m,1})),$$

$$m \geq m_0, \quad \lambda \in B(\lambda_{m,1}, |\lambda_{m,1}|).$$

Increasing the constant $C > 0$ if it is necessary, we can consider, that

$$|P_m(\lambda)| \leqslant C \exp(H_{K_{s+1}}(\lambda_{m,1})), \quad m \geq 1, \quad \lambda \in B(\lambda_{m,1}, |\lambda_{m,1}|). \tag{7}$$

Let us proceed to the second stage, as a result of which there will be the needed entire function f in the Lemma. First of all, let us note, that the series, made from inverse values of points modules $\lambda_{m,l}$ subject to their orders $n_{m,l}$, converge. Indeed, we have:

$$\sum_{m=1}^{\infty} \sum_{l=1}^{M_m} \frac{n_{m,l}}{|\lambda_{m,l}|} = \sum_{m=1}^{\infty} \frac{1}{|\lambda_{m,1}|} \sum_{l=1}^{M_m} \frac{n_{m,l}|\lambda_{m,1}|}{|\lambda_{m,l}|} \leqslant \sum_{m=1}^{\infty} \frac{1}{|\lambda_{m,1}|} \sum_{l=1}^{M_m} \frac{n_{m,l}}{b_m} =$$

$$= \sum_{m=1}^{\infty} \frac{1}{|\lambda_{m,1}| b_m} \sum_{l=1}^{M_m} n_{m,l} = \sum_{m=1}^{\infty} \frac{N_m}{|\lambda_{m,1}| b_m},$$

where $b_m = \min_{1 \leqslant l \leqslant M_m} |\lambda_{m,l}|/|\lambda_{m,1}|$. Whereas groups U_m are relatively small, then

$$b_m = \min_{1 \leqslant l \leqslant M_m} \frac{|\lambda_{m,l} - \lambda_{m,1} + \lambda_{m,1}|}{|\lambda_{m,1}|} \geq \min_{1 \leqslant l \leqslant M_m} \frac{|\lambda_{m,1}| - |\lambda_{m,l} - \lambda_{m,1}|}{|\lambda_{m,1}|} =$$

$$= \min_{1 \leqslant l \leqslant M_m} \left(1 - \frac{|\lambda_{m,l} - \lambda_{m,1}|}{|\lambda_{m,1}|}\right) = 1 - \max_{1 \leqslant l \leqslant M_m} \frac{|\lambda_{m,l} - \lambda_{m,1}|}{|\lambda_{m,1}|} \geq 1 - \max_{1 \leqslant j,l \leqslant M_m} \frac{|\lambda_{m,j} - \lambda_{m,l}|}{|\lambda_{m,1}|} =$$

$$= 1 - \delta(m) \to 1, \quad m \to \infty.$$

Hereof, subject to the Lemma conditions we obtain:

$$\sum_{m=1}^{\infty} \sum_{l=1}^{M_m} \frac{n_{m,l}}{|\lambda_{m,l}|} \leqslant \sum_{m=1}^{\infty} \frac{N_m}{|\lambda_{m,1}| b_m} \leqslant \sum_{m=1}^{\infty} \frac{1}{2^m b_m} < \infty.$$

Convergence of this series means, that the canonical entire function φ of the set $\{\lambda_{m,l}, n_{m,l}\}$ possesses an exponential minimal type (see [7, Theorem 3.9]). This function transforms into zero only in the points $\lambda_{m,l}$ with the order $n_{m,l}$ and is defined by the formula

$$\varphi(\lambda) = \prod_{m=1}^{\infty} \prod_{l=1}^{M_m} \left(1 - \frac{\lambda}{\lambda_{m,l}}\right)$$

(if some point $\lambda_{k,j}$ coincides with the origin of coordinates, then the factor $(1 - \lambda/\lambda_{k,j})$ in this product should be replaced by the factor $\lambda^{n_{k,j}}$). Then according to Theorem 2.3 from paper [7] density of the zero function set $\varphi(\lambda)$ is equal to zero. It results from here, that this set is a correctly distributed one (see [7, ch. I, § 6, p. 3]). Note, that due to the function type minimality $\varphi(\lambda)$, its indicatrix of growth (see [7, ch. I, § 5, p. 4]) is uniformly is equal to zero. Hence, according to Theorem 6.2 from paper [7] the following correlation holds true

$$\lim_{|\lambda|\to\infty, \lambda\notin E} \frac{\ln|\varphi(\lambda)|}{|\lambda|} = 0, \tag{8}$$

where E is a set of circles $B(\xi_p, r_p)$ of a zero linear density, i.e.

$$\lim_{r\to\infty} \frac{1}{r} \sum_{|\xi_p|<r} r_p = 0. \tag{9}$$

Note, that the set E covers the zero function set $\varphi(\lambda)$, and the correlation (8) holds true on its bounds. To construct a needed entire function we will require a similar cover, possessing some supplementary property: every bounded component of the cover contains only one group U_m of zeros φ. We are going to start construction of such a set cover $\{\lambda_{m,l}\}$.

First of all we should note, that the circles $B(\lambda_{m,1}, 4^{-1}|\lambda_{m,1}|)$, $m = 1, 2, \ldots$, do not meet pairwise. Indeed, due to the Lemma condition $|\lambda_{m+1,1}| \geq 2|\lambda_{m,1}|$, $m = 1, 2, \ldots$. Therefore, the distance between centers of neighboring circles has the following low estimate:

$$|\lambda_{m+1,1} - \lambda_{m,1}| \geq |\lambda_{m+1,1}| - |\lambda_{m,1}| \geq \frac{|\lambda_{m+1,1}|}{4} + \frac{6|\lambda_{m,1}|}{4} - |\lambda_{m,1}| > \frac{|\lambda_{m+1,1}|}{4} - \frac{|\lambda_{m,1}|}{4}.$$

It results from here, that these circles do not meet. Whereas the sequence of centers modules is growing, then any two circles do not meet.

Let us now choose a growing sequence of natural numbers $m(6) < m(7) < \ldots < m(k) < \ldots$ such that the following two conditions hold true: 1) for any $k \geq 6$ the group U_m with $m \geq m(k)$ is completely in the circle $B(\lambda_{m,1}, k^{-1}|\lambda_{m,1}|)$, 2) for every $k \geq 6$ and all $m = m(k), m(k)+1, \ldots, m(k+1)-1$ there is a number τ_m from the segment $[k^{-1}, (k-1)^{-1}]$ such that the circle $S(\lambda_{m,1}, \tau_m|\lambda_{m,1}|)$ does not meet the set E.

The first condition will be satisfied, as groups U_m are relatively small, i.e.

$$\max_{1\leqslant l\leqslant M_m} \frac{|\lambda_{m,1} - \lambda_{m,l}|}{|\lambda_{m,1}|} \to 0, \quad m \to \infty.$$

The satisfying of the second condition is achieved by correlation (9). Indeed, the relative length of the segment $[k^{-1}|\lambda_{m,1}|, (k-1)^{-1}|\lambda_{m,1}|]$, i.e. the value

$$\frac{|(k-1)^{-1}|\lambda_{m,1}| - k^{-1}|\lambda_{m,1}||}{|\lambda_{m,1}|} = \frac{1}{k-1} - \frac{1}{k}$$

for every fixed $k \geq 6$ is constant when $m \to \infty$. At the same time, according to (9), the relative sum of all circles radiuses $B(\xi_p, r_p)$, having a nonempty meeting with the circle $B(\lambda_{m,1}, (k-1)^{-1}|\lambda_{m,1}|)$, approaches to zero when $m \to \infty$.

We can consider, that the first number $m(6)$ is chosen rather large, that the following inequality holds true

$$\frac{|\lambda_{m(6),1}|}{5} + 1 \leqslant \frac{|\lambda_{m(6),1}|}{4}.$$

Then the circles $B(\lambda_{m,1}, 1+\tau_m|\lambda_{m,1}|)$, $m \geq m(6)$, do not meet pairwise.

Suppose $\Omega_m = B(\lambda_{m,1}, \tau_m|\lambda_{m,1}|)$, $m \geq m(6)$. By its construction the set Ω_m completely contains the group U_m, and its swelling $\Omega_m + B(0,1)$ does not meet the set $\Omega_j + B(0,1)$ for all $j \neq m$, $j \geq m(6)$. In particular, $\Omega_m + B(0,1)$ does not possess points of any group U_j, $j \neq m$, $j \geq m(6)$. While increasing the number, if it is needed, $m(6)$, we can consider, that for all $m \geq m(6)$ the set $\Omega_m + B(0,1)$ also does not possess a single point of the group U_j when $j < m(6)$. Then for any $m < m(6)$ we can choose an open set Ω_m such that Ω_m completely contains the group U_m, and sets $\overline{\Omega}_j$, $j < m(6)$, which do not meet pairwise and have an empty meeting with sets $\Omega_k + B(0,1)$, $k \geq m(6)$.

Let us fix an arbitrary number $\varepsilon \in (0,1)$. Whereas the bound $\partial\Omega_m$ of the set Ω_m for all $m \geq m(6)$ does not possess points E, then, according to (8), there is a constant $a > 0$ such that

$$|\varphi(\lambda)| \geq a\exp(-\varepsilon|\lambda|), \quad \lambda \in \partial\Omega_m, \quad m = 1, 2, \ldots \tag{10}$$

The function $\varphi(\lambda)$ is a minimal exponential one. Then, due to Theorem 1.2 from paper [7], its derivative $\varphi'(\lambda)$ possesses the same property. Therefore, there is a constant $b > 0$, for which the following inequalities hold true

$$|\varphi(\lambda)| \leqslant b\exp(\varepsilon|\lambda|), \quad \lambda \in \mathbb{C}. \tag{11}$$

$$|\varphi'(\lambda)| \leqslant b\exp(\varepsilon|\lambda|), \quad \lambda \in \mathbb{C}. \tag{12}$$

Let $w \in \partial\Omega_m$ and $\lambda \in B(w, \exp(-3\varepsilon|\lambda_{m,1}|))$. According to the formula for the primitive, taking into account (12), we obtain:

$$|\varphi(\lambda) - \varphi(w)| = |\int_w^\lambda \varphi'(\xi)d\xi| \leqslant \max_{\xi\in[w,\lambda]} |\varphi'(\xi)||\lambda - w| \leqslant b \max_{\xi\in[w,\lambda]} \exp(\varepsilon|\xi|)|\lambda - w| \leqslant$$

$$\leqslant b\exp(\varepsilon(|w|+1))\exp(-3\varepsilon|\lambda_{m,1}|) = b\exp(\varepsilon(|w|+1-3|\lambda_{m,1}|)).$$

Hereof, due to (10) we have:

$$|\varphi(\lambda)| \geq |\varphi(w)| - b\exp(\varepsilon(|w|+1-3|\lambda_{m,1}|)) \geq a\exp(-\varepsilon|w|) - b\exp(\varepsilon(|w|+1-3|\lambda_{m,1}|)) =$$

$$= \exp(-\varepsilon|w|)(a - b\exp(\varepsilon(2|w|+1-3|\lambda_{m,1}|)).$$

As to the construction, for all $m \geq m(6)$ the following embedding holds true $\Omega_m \subset B(\lambda_{m,1}, 5^{-1}|\lambda_{m,1}|)$. Consequently, the inequality holds true $|w - \lambda_{m,1}| \leqslant 5^{-1}|\lambda_{m,1}|$. Then, from the described above we obtain:

$$|\varphi(\lambda)| \geq \exp(-\varepsilon|w|)(a - b\exp(\varepsilon(12|\lambda_{m,1}|/5 + 1 - 3|\lambda_{m,1})))=$$

$$= \exp(-\varepsilon|w|)(a - b\exp(\varepsilon(-3|\lambda_{m,1}|/5+1))), \quad \lambda \in B(w, \exp(-3\varepsilon|\lambda_{m,1}|)),$$

where $w \in \partial\Omega_m$ and $m \geq m(6)$. Choose the number $m_0 \geq m(6)$ such that for all $m \geq m_0$ the inequality holds true: $b\exp(\varepsilon(-3|\lambda_{m,1}|/5+1)) \leqslant a/2$. Taking into account, that $|\lambda - w| \leqslant \exp(-3\varepsilon|\lambda_{m,1}|) < 1$ and $\varepsilon \in (0,1)$, we obtain:

$$|\varphi(\lambda)| \geq (2e)^{-1}a\exp(-\varepsilon|\lambda|), \quad \lambda \in B(w, \exp(-3\varepsilon|\lambda_{m,1}|)), \quad w \in \partial\Omega_m, \quad m \geq m_0.$$

According to construction, the sets $\overline{\Omega_m + B(0, \exp(-3\varepsilon|\lambda_{m,1}|))}$, $m \geq m(6)$, do not meet pairwise and do not meet with the sets $\overline{\Omega_j}$, $j < m(6)$. Hence, there is a constant $\gamma > 0$ such that the sets $\overline{\Omega_m + B(0, \exp(-3\varepsilon|\lambda_{m,1}|))}$ do not meet pairwise for all $m \geq 1$. Whereas all zeros of the function φ are in the unification $\bigcup_{m\geq 1}\Omega_m$, then, decreasing the needed number $a>0$, we can consider, that the following estimates hold true:

$$|\varphi(\lambda)| \geq (2e)^{-1}a\exp(-\varepsilon|\lambda|), \qquad \lambda\in B(w,\exp(-3\varepsilon|\lambda_{m,1}|)), \quad w\in\partial\Omega_m, \quad m\geq 1. \tag{13}$$

For any $m=1,2,\ldots$ we will define the function with the following properties by β_m: 1) $\beta_m\in C^\infty(\mathbb{C})$, 2) $0\leqslant\beta_m(z)\leqslant 1$, $z\in\mathbb{C}$, 3) $\beta_m(z)=1$, $z\in\Omega_m$, 4) $\beta_m(z)=0$, $z\notin\Omega_m+B(0,\gamma\exp(-3\varepsilon|\lambda_{m,1}|))$, 5) $|(d\beta_m(z)/d\bar{z}|\leqslant\exp(3\varepsilon|\lambda_{m,1}|)$, $z\in\mathbb{C}$, where the constant $\alpha>0$ does not depend on the number $m\geq 1$ (information about such functions is presented, for instance, in [8], Theorem 1.4.1 and Formula (1.4.2)).

Let us consider the function

$$\beta(\lambda)=\sum_{m=1}^{\infty}\beta_m(\lambda)P_m(\lambda).$$

It is defined in all the complex plane, differs from zero only in sets $\Omega_m+B(0,\gamma\exp(-3\varepsilon|\lambda_{m,1}|))$, $m\geq 1$, and it coincides with the function $\beta_m(\lambda)P_m(\lambda)$ in each of these sets. Likewise, in the set Ω_m it coincides with the function $P_m(\lambda)$. Therefore, due to analyticity $P_m(\lambda)$, $m\geq 1$, the function $d\beta(\lambda)/d\bar{\lambda}$ differs from zero only in sets $(\Omega_m+B(0,\gamma\exp(-3\varepsilon|\lambda_{m,1}|)))\setminus\Omega_m$, $m\geq 1$, and in each of these sets it coincides with the function $P_m(\lambda)d\beta_m(\lambda)/d\bar{\lambda}$. According to inequality (7) and property 5 of functions β_m we obtain the estimate:

$$\left|\frac{d\beta(\lambda)}{d\bar{\lambda}}\right|\leqslant C\exp(H_{K_{s+1}}(\lambda_{m,1})+3\varepsilon|\lambda_{m,1}|), \qquad \lambda\in B(\lambda_{m,1},|\lambda_{m,1}|), \quad m\geq 1. \tag{14}$$

Due to construction the set diameter Ω_m approaches to zero, when $m\to\infty$. Consequently, there is a number m_1 such that for all $m\geq m_1$ the following embedding holds true

$$\Omega_m+B(0,\gamma\exp(-3\varepsilon|\lambda_{m,1}|))\subset B(\lambda_{m,1},\delta|\lambda_{m,1}|), \tag{15}$$

where $\delta>0$ is defined by the number $\varepsilon>0$ in Lemma 1. According to this Lemma, we have:

$$H_{K_{s+1}}(\lambda_{m,1})+3\varepsilon|\lambda_{m,1}|\leqslant H_{K_{s+1}}(\lambda)+4\varepsilon|\lambda|, \qquad \lambda\in B(\lambda_{m,1},\delta|\lambda_{m,1}|), \quad m\geq 1.$$

Therefore, due to (14) and (15), there is a constant $C_1>0$, for which the following inequalities hold true

$$\left|\frac{d\beta(\lambda)}{d\bar{\lambda}}\right|\leqslant C_1\exp(H_{K_{s+1}}(\lambda)+4\varepsilon|\lambda|), \qquad \lambda\in\Omega_m+B(0,\gamma\exp(-3\varepsilon|\lambda_{m,1}|)), \quad m\geq 1.$$

Hereof, taking into account (13) and everything said above about the function $d\beta(\lambda)/d\bar{\lambda}$ we obtain:

$$|v(\lambda)|=\left|\frac{1}{\varphi(\lambda)}\frac{d\beta(\lambda)}{d\bar{\lambda}}\right|\leqslant C_2\exp(H_{K_{s+1}}(\lambda)+5\varepsilon|\lambda|), \qquad \lambda\in\mathbb{C},$$

where $C_2=2ea^{-1}C_1$. This implies, that

$$\int\limits_{\mathbb{C}}|v(\lambda)|^2\exp(-2H_{K_{s+1}}(\lambda)-11\varepsilon|\lambda|)d\sigma(\lambda)=C_3<\infty,$$

where $d\sigma$ is a planar Lebesgue measure. The function $H_{K_{s+1}}(\lambda)$ is convex, and, consequently, subharmonic. Then, as it is known (see, for instance, [9, ch. 3, § 6, p.2, Theorem 3.6.2]), in the space of locally integrated with the square root of the function module in $\mathbb{C}$ there is an element g, which (in general) satisfies the equality

$$dg/d\overline{\lambda} = v \tag{16}$$

and, Moreover, the estimate

$$\int\limits_{\mathbb{C}} |g(\lambda)|^2 \exp(-2H_{K_{s+1}}(\lambda) - 12\varepsilon|\lambda|)d\sigma(\lambda) = C_4 < \infty. \tag{17}$$

Let us show, that it is an element of the space P_D. According to (16), the generalized derivative f on $\overline{\lambda}$ is equal to zero everywhere on the plane. It is well known, that it means analyticity f in all the complex plane. Let us find the upper estimate for the entire function module $f(\lambda)$. Due to the function subharmonicity $|f(\lambda)|$ we have:

$$|f(\lambda)| \leqslant \frac{1}{\pi} \int\limits_{B(\lambda,1)} |f(w)|d\sigma(w) \leqslant \frac{1}{\pi} \int\limits_{B(\lambda,1)} |\beta(w)|d\sigma(w) + \frac{1}{\pi} \int\limits_{B(\lambda,1)} |\varphi(w)g(w)|d\sigma(w). \tag{18}$$

Applying (7), Lemma 1, and also properties of the functions $\beta_m(\lambda)$ and sets Ω_m, as for the function above $d\beta(\lambda)/d\overline{\lambda}$, we obtain the inequality

$$|\beta(w)| \leqslant C_5 \exp(H_{K_{s+1}}(w) + \varepsilon|w|), \quad w \in \mathbb{C},$$

where C_5 is some positive constant. Then for all $\lambda \in \mathbb{C}$

$$\frac{1}{\pi} \int\limits_{B(\lambda,1)} |\beta(w)|d\sigma(w) \leqslant C_5 \sup_{w\in B(\lambda,1)} \exp(H_{K_{s+1}}(w) + \varepsilon|w|) \leqslant C_6 \exp(H_{K_{s+1}}(\lambda) + 2\varepsilon|\lambda|).$$

In the latter inequality we again applied Lemma 1. Likewise, applying (11), we obtain:

$$\sup_{w\in B(\lambda,1)} |\varphi(w)| \leqslant C_7 \exp(2\varepsilon|\lambda|), \quad \lambda \in \mathbb{C}.$$

Therefore, subject to the previous inequality due to (18) we have:

$$|f(\lambda)| \leqslant C_6 \exp(H_{K_{s+1}}(\lambda) + 2\varepsilon|\lambda|) + C_7 \exp(2\varepsilon|\lambda|) \frac{1}{\pi} \int\limits_{B(\lambda,1)} |g(w)|d\sigma(w). \tag{19}$$

To estimate the last integral we will apply the Cauchy-Bunyakovsky inequality. Due to (17), we obtain:

$$\int\limits_{B(\lambda,1)} |g(w)|d\sigma(w) \leqslant$$

$$\left(\int\limits_{B(\lambda,1)} |g(w)|^2 \exp(-2H_{K_{s+1}}(w) - 128\varepsilon|w|)d\sigma \int\limits_{B(\lambda,1)} \exp(2H_{K_{s+1}}(w) + 12\varepsilon|w|)d\sigma \right)^{1/2} \leqslant$$

$$\leqslant \left(C_4 \int\limits_{B(\lambda,1)} \exp(2H_{K_{s+1}}(w) + 12\varepsilon|w|)d\sigma \right)^{1/2} \leqslant \pi\sqrt{C_4} \exp\left(\sup_{w\in B(\lambda,1)} (H_{K_{s+1}}(w) + 6\varepsilon|w|) \right) \leqslant$$

$$\leqslant \pi C_8 \exp(H_{K_{s+1}}(\lambda) + 6\varepsilon|\lambda|), \quad \lambda \in \mathbb{C},$$

where C_8 is some positive constant (while obtaining the last inequality we applied Lemma 1). Thus, subject to (19) we obtain:

$$|f(\lambda)| \leqslant C_6 \exp(H_{K_{s+1}}(\lambda) + 2\varepsilon|\lambda|) + C_7 C_8 \exp(H_{K_{s+1}}(\lambda) + 8\varepsilon|\lambda|) \leqslant$$

$$\leqslant C_9 \exp(H_{K_{s+1}}(\lambda) + 8\varepsilon|\lambda|), \quad \lambda \in \mathbb{C}.$$

Whereas the number $\varepsilon > 0$ can be arbitrarily small, then, due to (1), the following estimate will hold true

$$|f(\lambda)| \leqslant C_9 \exp H_{K_{s+2}}(\lambda), \quad \lambda \in \mathbb{C},$$

which means, that the function $f(\lambda)$ is an element of the space P_D.

It is remained to verify the equalities

$$b_{m,j} = q_{m,j-1}(f), \quad m = 1, 2, \dots, \quad j = 1, \dots, N_m.$$

By definition, the numbers $q_{m,j-1}(f)$ are coefficients of the polynomial

$$q_m(\lambda, f) = \frac{1}{2\pi i} \int\limits_{\Gamma_m} \frac{f(\zeta)(\omega_m(\zeta) - \omega_m(\lambda))}{(\zeta - \lambda)\omega_m(\zeta)} d\zeta,$$

decomposed on degrees $\lambda - \lambda_{m,1}$. In the latter formula Γ_m is an arbitrary contour, totally covering the group U_m. Due to construction the set Ω_m contains the group U_m. Hence, as a contour Γ_m we can take the frontier $\partial\Omega_m$ of the set Ω_m. In the set Ω_m, and in its frontier, $\partial\Omega_m$, the function $f(\lambda)$ coincides with the function $P_m(\lambda) - \varphi(\lambda)g(\lambda)$. Therefore, $g(\lambda) = (P_m(\lambda) - f(\lambda))/\varphi(\lambda)$ is a function, analytical on Ω_m and possibly having some poles in points of the group U_m. However, existence of at least one such pole contradicts inequality (17). Consequently, $g(\lambda)$ does not possess special points in Ω_m. Then we obtain:

$$q_m(\lambda, f) = \frac{1}{2\pi i} \int\limits_{\partial\Omega_m} \frac{f(\zeta)(\omega_m(\zeta) - \omega_m(\lambda))}{(\zeta - \lambda)\omega_m(\zeta)} d\zeta = \frac{1}{2\pi i} \int\limits_{\partial\Omega_m} \frac{P_m(\zeta)(\omega_m(\zeta) - \omega_m(\lambda))}{(\zeta - \lambda)\omega_m(\zeta)} d\zeta -$$

$$- \frac{1}{2\pi i} \int\limits_{\partial\Omega_m} \frac{\varphi(\zeta)g(\zeta)(\omega_m(\zeta) - \omega_m(\lambda))}{(\zeta - \lambda)\omega_m(\zeta)} d\zeta.$$

The polynomial $\omega_m(\zeta)$ transforms to zero only in the points of the group U_m. The function $\varphi(\zeta)$ also transforms to zero in these points. Therefore, $\varphi(\zeta)/\omega_m(\zeta)$ is an entire function. The fraction $\omega_m(\zeta) - \omega_m(\lambda)/(\zeta - \lambda)$ is also an entire function. Hence, according to the Cauchy theorem, the latter integral is equal to zero and we have:

$$q_m(\lambda, f) = \frac{1}{2\pi i} \int\limits_{\partial\Omega_m} \frac{f(\zeta)(\omega_m(\zeta) - \omega_m(\lambda))}{(\zeta - \lambda)\omega_m(\zeta)} d\zeta =$$

$$= \frac{1}{2\pi i} \int\limits_{\partial\Omega_m} \frac{P_m(\zeta)(\omega_m(\zeta) - \omega_m(\lambda))}{(\zeta - \lambda)\omega_m(\zeta)} d\zeta = q_m(\lambda, P_m).$$

Hereof, subject to (6) we obtain

$$q_{m,j-1}(f) = q_{m,j-1}(P_m) = b_{m,j}, \quad m = 1, 2, \dots, \quad j = 1, \dots, N_m.$$

The Lemma has been proved.

Theorem 3. *Let D be a convex domain in $\mathbb{C}$, the sequence $\{\lambda_{m,l}\}$ is divided into relatively small groups U_m so, that $\mathcal{N} = 0$, and sequence of functions $\tilde{\mathcal{E}} = \{e_{m,p}(z)\}_{m=1,p=1}^{\infty,N_m}$ is defined by* *formula (2). Then $\tilde{\mathcal{E}}$ is an almost exponential sequence in the domain D with the index $\lambda_{m,1}$ (to be more precise, with the indexes $\lambda'_{m,j}$, where $\lambda'_{m,j} = \lambda_{m,1}$, $j = 1, \dots, N_m$).*

Proof. As it was already said above, in paper [2] it was stated (corollary of Lemma 5), that with $\mathcal{N} = 0$ for the system $\tilde{\mathcal{E}}$ the following item holds true 1) from the definition of an almost exponential sequence. Let us show, that when $\mathcal{N} = 0$ for $\tilde{\mathcal{E}}$ item 2) also holds true.

Let us assume, that item 2) is not valid. Then there is a number p such that for any $s = 1, 2, \dots$ there are numbers $m(s) \to \infty$, when $s \to \infty$, and $l(s)$, when the following inequality holds true

$$s^{-1} \exp(H_{K_p}(\lambda_{m,1})) > \sup_{w \in K_s} |e_{(m(s),l(s)}(w)|. \tag{20}$$

Thus, we obtain the sequence of functions $\{e_{m(s),l(s)}\}_{s=1}^{\infty}$, possessing properties (20). Whereas $|\lambda_{m,1}|$ unlimitedly grows with $m \to$, then, proceeding to the sequence, we can consider, that $|\lambda_{m(s)+1,1}| \geq 2|\lambda_{m(s),1}|$, $s = 1, 2, \dots$. According to the condition $\mathcal{N} = 0$, i.e. $N_m/|\lambda_{m,1}| \to 0$ when $m \to \infty$. Hence, we can also assume, that $N_{m(s)}/|\lambda_{m(s),1}| \leqslant 2^{-s}$, $s = 1, 2, \dots$. Then, due to Lemma 2 for any sequence $b = \{b_{s,j}\}$ from the space $R(D)$ there is the function $f \in \mathcal{P}_D$ such that $b_{s,j} = q_{m(s),j-1}(f)$, $s = 1, 2, \dots$, $j = 1, \dots, N_{m(s)}$.

Let W' be a closure in the space $H(D)$ of the linear capsule of the system of functions $\{z^n \exp(\lambda_{m(s),l} z)\}_{s=1,l=1,n=0}^{\infty, M_{m(s)}, n_{m(s),l}-1}$. Then, as it is easy to notice, W' is closed and invariant with respect to operator of the subspace differentiation in $H(D)$, and functions of the system $\{z^n \exp(\lambda_{m(s),l} z)\}_{s=1,l=1,n=0}^{\infty, M_{m(s)}, n_{m(s),l}-1}$ are eigenfunctions and associated functions of this operator in W'. Due to construction the subspace W is not empty and differs from $H(D)$. Indeed, let z be some point in the domain D and number t is such that the compact K_t contains z. Let us consider the function $\tilde{\varphi}(\lambda) = \varphi(\lambda) \exp(\lambda z)$, $\lambda \in \mathbb{C}$, where, like in Lemma 2, $\varphi(\lambda)$ is a function, which transforms to zero only in the points $\lambda_{m(s),j}$ with the order $n_{m(s),j}$, $s = 1, 2, \dots$, $j = 1, \dots, N_{m(s)}$. Due to (11) and the compact choice K_t, the following inequality holds true

$$|\tilde{\varphi}(\lambda)| \leqslant b \exp(\varepsilon|\lambda| + Re(z\lambda)) \leqslant b \exp(\varepsilon|\lambda| + H_{K_t}(\lambda)), \quad \lambda \in \mathbb{C}.$$

Whereas $\varepsilon > 0$ can be arbitrarily small, due to (1) we obtain:

$$|\tilde{\varphi}(\lambda)| \leqslant b \exp H_{K_{t+1}}(\lambda), \quad \lambda \in \mathbb{C}.$$

This estimate means, that the function $\tilde{\varphi}(\lambda)$ belongs to the space $\mathcal{P}_D$. Then, as it was stated above, there is a functional $\mu \in H^*(D)$, for which $\tilde{\varphi}(\lambda)$ is the Laplace transform: $\tilde{\varphi}(\lambda) = (\mu, \exp \lambda w)$. Differentiating the latter equality, we obtain:

$$0 = \tilde{\varphi}^{(n)}(\lambda_{m(s),l}) = (\mu, z^n \exp(\lambda_{m(s),l} w), \quad s = 1, 2, \dots, \quad l = 1, \dots, M_{m(s)}, \quad n = 0, \dots, n_{m(s),l} - 1.$$

Consequently, the zero functional μ transforms to zero in all system functions $\{z^n \exp(\lambda_{m(s),l} z)\}_{s=1,l=1,n=0}^{\infty, ?M_{m(s)}, n_{m(s)}}$ and it means, by linearity and continuity on all the subspace W'. Hence, W' cannot coincide with the space $H(D)$.

Therefore, all conditions of Theorem 1 from paper [2] are satisfied. According to this, existence of the pointed out entire function $f \in \mathcal{P}_D$ for every sequence $b = \{b_{s,l}\}$ from the space $R(D)$ is equivalent to the system of functions $\{e_{m(s),j}\}_{s=1,j=1}^{\infty, N_{m(s)}}$ is an almost exponential basis in the subspace W with indexes $\lambda_{m(s),1}$, $s = 1, 2, \dots$ (to be more precise, with indexes $\lambda'_{m(s),j}$, where $\lambda'_{m(s),j} = \lambda_{m(s),1}$, $j = 1, \dots, N_{m(s)}$). In particular, $\{e_{m(s),j}\}_{s=1,j=1}^{\infty, N_{m(s)}}$ is an almost exponential sequence in the domain D with indexes $\lambda_{m(s),1}$, $s = 1, 2, \dots$. Due to property 2) for such a sequence for the number p there is a constant $c > 0$ and a number $s(p)$ such that

$$c\exp(H_{K_p}(\lambda_{m(s),1})) \leqslant \sup_{w\in K_{s(p)}} |e_{m(s),j}(w)|, \quad s=1,2,\ldots, \quad j=1,\ldots,N_{m(s)}.$$

Whereas K_s is a growing sequence of compacts, this implies, that

$$c\exp(H_{K_p}(\lambda_{m(s),1})) \leqslant \sup_{w\in K_s} |e_{m(s),l(s)}(w)|, \quad s\geq s(p).$$

This inequality for all $s\geq s(p)$ such that $s^{-1}<c$, contradicts (20). Thus, our assumption, that item 2) from the definition of an almost exponential sequence for the system $\tilde{\mathcal{E}}$ does not hold true, is wrong. The theorem has been proved.

Remark. The condition $\mathcal{N}=0$ in Theorem 3 is significant. To prove it we will consider the following example. Let $\varepsilon\in(0,1)$ and $\{\lambda_m\}_{m=1}^{\infty}$ be an unlimitedly growing sequence of positive numbers with orders n_m, is equal to integer parts $[\varepsilon\lambda_m]$ of numbers $\varepsilon\lambda_m$, $m=1,2,\ldots$. Let us divide the sequence $\{\lambda_m\}_{m=1}^{\infty}$ into relatively small groups U_m so, that every group U_m will contain only point λ_m. Then $N_m=n_m$, $m=1,2,\ldots$, and, therefore,

$$\mathcal{N}=\lim_{m\to\infty}\frac{n_m}{\lambda_m}=\lim_{m\to\infty}\frac{[\varepsilon\lambda_m]}{\lambda_m}=\varepsilon>0.$$

In this case the system of functions $\tilde{\mathcal{E}}=\{e_{m,p}(z)\}_{m=1,j=1}^{\infty,N_m}$ is easily defined. Indeed, for all $m=1,2,\ldots$ we obtain: $\omega_m(\zeta)=(\zeta-\lambda_m)^{n_m}$. Consequently, the function

$$q_m(\lambda,z)=\frac{1}{2\pi i}\int\limits_{\Gamma_m}\frac{\exp(z\zeta)(\omega_m(\zeta)-\omega_m(\lambda))}{(\zeta-\lambda)\omega_m(\zeta)}d\zeta$$

with every fixed z for the variable λ is a polynomial of the degree not more than n_m-1, j derivative of which for all $j=0,\ldots,n_m-1$ in the point λ_m coincides with the corresponding derivative function $\exp(z\lambda)$, calculated in the point λ_m. The latter is equal to $z^j\exp(\lambda_m z)$. By definition the function $e_{m,j}(z)$ is $(j-1)$ derivative $q_m(\lambda,z)$, calculated in the point λ_m. Therefore,

$$e_{m,j}(z)=z^{j-1}\exp(\lambda_m z), \quad j=1,\ldots,n_m, \quad m=1,2,\ldots.$$

As a domain D we will take a triangle with the vertex in the points $(0,0)$, $(-1,1)$ and $(1,-1)$. For all $s=1,2,\ldots$, $m=1,2,\ldots$ and $j=2,\ldots,n_m$ we have:

$$\sup_{w\in K_s}|e_{m,j}(w)|\leqslant\sup_{w\in D}|e_{m,j}(w)|=\sup_{w\in D}|z^{j-1}\exp(\lambda_m w)|=$$
$$=\sup_{x\in(1,0)}\exp((j-1)\ln(-\sqrt{2}x)+x\lambda_m).$$

By means of simple calculations we obtain, that the latter supremum is achieved in the point $x=(1-j)/\lambda_m$, and it is equal to $\exp((j-1)\ln(\sqrt{2}((j-1)/\lambda_m))+1-j)$. Hereof, with $j=n_m$, rather large m, and all $s=1,2,\ldots$, we obtain:

$$\sup_{w\in K_s}|e_{m,n_m}(w)|\leqslant\exp(([\varepsilon\lambda_m]-1)\ln(\sqrt{2}(([\varepsilon\lambda_m]-1)/\lambda_m))+1-[\varepsilon\lambda_m])\leqslant$$
$$\leqslant\exp(([\varepsilon\lambda_m]-1)\ln(\sqrt{2}(([\varepsilon\lambda_m]-1)/\lambda_m))+1-[\varepsilon\lambda_m])\leqslant\exp(1-[\varepsilon\lambda_m])\leqslant$$
$$\leqslant\exp(2-\varepsilon\lambda_m)\leqslant 9\exp(-\varepsilon\lambda_m). \tag{21}$$

Let us choose the number p so, that the following inequality holds true

$$H_{K_p}(1)\geq H_D(1)-\varepsilon/2=-\varepsilon/2.$$

Then for all $m = 1, 2, \ldots$ we have:

$$H_{K_p}(\lambda_m) = \lambda_m H_{K_p}(\lambda_m/\lambda_m) = \lambda_m H_{K_p}(1) \geq -\varepsilon\lambda_m/2.$$

Subject to (21) it means, that item 2) from the definition of an almost exponential sequence does not hold true for the system $\tilde{\mathcal{E}}$.

Let W be a closure in the space $H(D)$ of the linear capsule of the functions system $\{z^n \exp(\lambda_{m,l}z)\}_{s=1,l=1,n=0}^{\infty,M_m,n_{m,l}-1}$. Then, as it is easy to notice, W is closed and invariant with respect to the operator of subspace differentiation in $H(D)$. Further we will consider, that the subspace W is nontrivial.

The following result follows directly from Theorem 3 and Theorem 1 in paper [2].

Theorem 4. *Let D be a convex domain in $\mathbb{C}$, the sequence $\{\lambda_{m,l}\}$ is divided into relatively small groups U_m so, that $\mathcal{N} = 0$, and sequence of functions $\tilde{\mathcal{E}} = \{e_{m,j}(z)\}_{m=1,j=1}^{\infty,N_m}$ is defined by Formula (2). Then the following statements are equivalent:*

1) the system of functions $\tilde{\mathcal{E}}$ is a basis in the space W;

2) for every sequence $b = \{b_{m,j}\}$ from the space $R(D)$ there is a function $f \in \mathcal{P}_D$ such that $b_{m,p} = q_{m,j-1}(f)$, $m = 1, 2, \ldots$, $j = 1, \ldots, N_m$.

We will say (see[2]), that the system of functions $\tilde{\mathcal{E}} = \{e_{m,j}(z)\}_{m=1,j=1}^{\infty,N_m}$ possesses a Kathe group property, if for any number p there is a number s and a constant C, satisfying the following condition: for every $m = 1, 2, \ldots$ and every function h_m of the form

$$h_m(z) = \sum_{j=1}^{N_m} a_{m,j} e_{m,j}(z)$$

the following inequality holds true

$$\sum_{j=1}^{N_m} |a_{m,j}| \sup_{z \in K_p} |e_{m,j}(z)| \leqslant C \sup_{z \in K_s} |h_m(z)|.$$

Theorem 5. *Let D be a convex domain in $\mathbb{C}$, sequence $\{\lambda_{m,l}\}$ is divided into relatively small groups U_m so, that $\mathcal{N} = 0$, and sequence of functions $\tilde{\mathcal{E}} = \{e_{m,j}(z)\}_{m=1,j=1}^{\infty,N_m}$ is defined by Formula (2). Then $\tilde{\mathcal{E}}$ possesses a Kathe group property.*

Proof. Assume, that the system $\tilde{\mathcal{E}}$ does not possess a Kathe group property. Then there is a number p such that for every $s = 1, 2, \ldots$ there is a number $m(s) \to \infty$, when $s \to \infty$, and the function h_s of the form

$$h_s(z) = \sum_{j=1}^{N_{m(s)}} a_{m(s),j} e_{m(s),j}(z),$$

for which the following inequality holds true

$$\sum_{j=1}^{N_{m(s)}} |a_{m(s),j}| \sup_{z \in K_p} |e_{m(s),j}(z)| > s \sup_{z \in K_s} |h_s(z)|. \tag{22}$$

Whereas $|\lambda_{m,1}|$ unlimitedly grows with $m \to \infty$, then, proceeding to the subsequence, we can consider, that $|\lambda_{m(s)+1,1}| \geq 2|\lambda_{m(s),1}|$, $s = 1, 2, \ldots$. According to the condition $\mathcal{N} = 0$, i.e. $N_m/|\lambda_{m,1}| \to 0$ when $m \to \infty$. Therefore, we can consider, that $N_{m(s)}/|\lambda_{m(s),1}| \leqslant 2^{-s}$, $s = 1, 2, \ldots$. Then, due to Lemma 2 for every sequence $b = \{b_{s,j}\}$ from the space $R(D)$ there is a function $f \in \mathcal{P}_D$ such that $b_{s,j} = q_{m(s),j-1}(f)$, $s = 1, 2, \ldots$, $j = 1, \ldots, N_{m(s)}$.

Let W' be a closure in the space $H(D)$ of a linear capsule of the system of functions $\{z^n \exp(\lambda_{m(s),l} z)\}_{s=1,l=1,n=0}^{\infty,M_{m(s)},n_{m(s),l}-1}$. Like in Theorem 3, all conditions of Theorem 1 from paper [2] are satisfied. Then, according to this Theorem the system of functions $\{e_{m(s),j}\}_{s=1,j=1}^{\infty,N_{m(s)}}$ is an almost exponential basis in the subspace W' with indexes $\lambda_{m(s),1}$, $s = 1, 2, \ldots$.
Whereas $N_{m(s)}/|\lambda_{m(s),1}| \leqslant 2^{-s}$, $s = 1, 2, \ldots$, then the series

$$\sum_{s=1}^{\infty} \frac{N_{m(s)}}{|\lambda_{m(s),1}|}$$

converge. It results from here, that, the value $\mathcal{J}(\Lambda)$, defined in paper [1], for the sequence $\Lambda = \{\lambda_k\}_{k=1}^{\infty}$, consisting of points $\lambda_{m(s),1}$, $s = 1, 2, \ldots$, where every point $\lambda_{m(s),1}$ is applied in it $N_{m(s)}$ times, is equal to zero. Then, according to the corollary of Theorem 3 in paper [1] the system of functions $\{e_{m(s),j}\}_{s=1,j=1}^{\infty,N_{m(s)}}$ is a Kathe basis in W'. In particular, for the number p there is a number $s(p)$ and a constant $B > 0$ such that for any function $g \in W'$ the following inequality holds true

$$\sum_{s=1,j=1}^{\infty,N_{m(s)}} |d_{m(s),j}| \sup_{z \in K_p} |e_{m(s),j}(z)| \leqslant B \sup_{z \in K_{s(p)}} |g(z)|,$$

where

$$g(z) = \sum_{s=1,j=1}^{\infty,N_{m(s)}} d_{m(s),j} e_{m(s),j}(z), \qquad z \in D.$$

Whereas K_s is a growing sequence of compacts, it implies, that

$$\sum_{s=1,j=1}^{\infty,N_{m(s)}} |d_{m(s),j}| \sup_{z \in K_p} |e_{m(s),j}(z)| \leqslant B \sup_{z \in K_s} |g(z)|, \qquad s \geq s(p).$$

This inequality for all $s \geq s(p)$ such that $s > B$, contradicts (22). Thus, our assumption, that the system $\tilde{\mathcal{E}}$ does not possess a Kathe group property, is wrong. The Theorem has been proved.

Alongside with the system $\tilde{\mathcal{E}}$ we will consider other systems of functions $\mathcal{E}' = \{e'_{m,j}(z)\}_{m=1,j=1}^{\infty,N_m}$. Assume

$$e'_{m,j}(z) = \sum_{k=1}^{N_m} a_{m,j,k} e_{m,k}(z), \qquad m = 1, 2, \ldots, \qquad j = 1, \ldots, N_m. \tag{23}$$

We will state, that the system $\tilde{\mathcal{E}}'$ is normalized, if for all $m = 1, 2, \ldots$

$$\max_{1 \leqslant k \leqslant N_m} |a_{m,j,k}| = 1, \qquad j = 1, \ldots, N_m.$$

The following results were obtained directly from Theorems 3 and 5, and also Lemma 8 and Theorem 2 in paper [2].

Theorem 6. *Let D be a convex domain in $\mathbb{C}$, sequence $\{\lambda_{m,l}\}$ is divided into relatively small groups U_m so, that $\mathcal{N} = 0$. Then any normalized system $\tilde{\mathcal{E}}' = \{e'_{m,j}(z)\}_{m=1,j=1}^{\infty,N_m}$, defined by Formulae (23) and (2), is an almost exponential sequence in the domain D with indexes $\lambda_{m,1}$.*

Theorem 7. *Let D be a convex domain in $\mathbb{C}$, sequence $\{\lambda_{m,l}\}$ is divided into relatively small groups U_m so, that $\mathcal{N} = 0$, and the system $\tilde{\mathcal{E}} = \{e_{m,j}(z)\}_{m=1,j=1}^{\infty,N_m}$ is defined by Formula (2). If in the subspace W there is a basis of the form (23), then the system $\tilde{\mathcal{E}}$ is also a basis in W.*

Theorem 7 reduces the problem of the basis existence by relatively small groups in a subspace W leads to verifying of the system basis $\tilde{\mathcal{E}}$ in this subspace. In conclusion, we will present description of all possible bases in W.

For every $m = 1, 2, \dots$ we will define the matrix, compiled from coefficients of function disintegration $e'_{m,j}(z)$ due to the system $\tilde{\mathcal{E}}_m = \{e_{m,j}(z)\}_{j=1}^{N_m}$ by $\mathcal{A}_m = (a_{m,j,k})$. Let $\mathcal{A}_m$ be a nongenerated and $\mathcal{A}_m^{-1} = (b_{m,j,k})$ be a matrix, reciprocal to $\mathcal{A}_m$. Assume

$$\mathfrak{a}(\mathbb{A}) = \overline{\lim_{m\to\infty}} \max_{1\leqslant j,k\leqslant N_m} \frac{\ln |b_{m,j,k}|}{|\lambda_{m,1}|}.$$

Note, that in the case, when D is a limited convex domain, the value $\mathfrak{a}(\mathbb{A})$ coincides with the value $\mathfrak{a}_D(\mathbb{A})$, introduced in paper [2].

Theorem 8. *Let D be a convex domain in $\mathbb{C}$, sequence $\{\lambda_{m,l}\}$ is divided into relatively small groups U_m so, that $\mathcal{N} = 0$. Suppose, that the system $\tilde{\mathcal{E}} = \{e_{m,j}(z)\}_{m=1,j=1}^{\infty,N_m}$, defined by Formula (2), is a basis in the subspace W, and the system $\tilde{\mathcal{E}}' = \{e'_{m,j}(z)\}_{m=1,j=1}^{\infty,N_m}$, defined by Formula (32), is a normalized sequence. then the following statements are equivalent.*

1) system $\tilde{\mathcal{E}}'$ is the basis in W.

2) system $\tilde{\mathcal{E}}'$ possesses a Kathe group property.

If D is a limited domain, statements 1) and 2) are equivalent.

3) $\mathfrak{a}(\mathbb{A}) = 0$.

Proof. Equivalency of statements 1) and 3) is proved in Theorem 3 in paper [2]. Let us prove equivalency of 1) and 2).

Assume, that the system $\tilde{\mathcal{E}}'$ is a basis in the subspace W. Whereas $\tilde{\mathcal{E}}'$ is a normalized sequence, then due to Theorem 6 it is an almost exponential basis in W with indexes $\lambda_{m,1}$ (to be more precise, with indexes $\lambda'_{m,j}$, where $\lambda'_{m,j} = \lambda_{m,1}$, $j = 1, \dots, N_m$). According to nontriviality of the subspace W there is a non-zero functional $\mu \in H^*(D)$, which transforms to zero in all functions from W. In particular, it concerns functions of the system $\{z^n \exp(\lambda_{m,l} z)\}_{s=1,l=1,n=0}^{\infty,M_m,n_{m,l}-1}$. Let $\psi(\lambda)$ be a Laplace transform of the functional μ. Then the following equalities hold true

$$0 = \psi^{(n)}(\lambda_{m,l}) = (\mu, z^n \exp(\lambda_{m,l} w), \quad m = 1, 2, \dots, \quad l = 1, \dots, M_m, \quad n = 0, \dots, n_{m,l} - 1,$$

i.e. the function $\psi(\lambda)$ vanishes in points $\lambda_{m,l}$ with the order not less than $n_{m,l}$, $m = 1, 2, \dots$, $l = 1, \dots, M_m$. Whereas $\psi(\lambda)$ is an entire exponential function, then due to Theorem 2.3 from paper [7, ch. I] the density of its zero set is finite. According to the fact, that groups U_m are relatively small, the sequence $\Lambda = \{\lambda_k\}_{k=1}^{\infty}$, compiled from points $\lambda_{m,1}$, $m = 1, 2, \dots$ will also have a finite density, and every point $\lambda_{m,1}$ is applied in it N_m times. It results from here, that, the value $\mathfrak{J}(\lambda)$, defined in paper [1], is equal to zero. Then, according to the corollary of Theorem 3 in paper [1] the system of functions $\tilde{\mathcal{E}}'$ is a Kathe basis in W, i.e. for every number p there is a number s and a constant $B > 0$ such that for any function $g \in W$ the following inequality holds true

$$\sum_{s=1,j=1}^{\infty,N_m} |d_{m,j}| \sup_{z\in K_p} |e'_{m,j}(z)| \leqslant B \sup_{z\in K_s} |g(z)|,$$

where

$$g(z) = \sum_{s=1,j=1}^{\infty,N_m} d_{m,j} e'_{m,j}(z), \quad z \in D.$$

In particular, for any number $m = 1, 2, \dots$ and any function h_m of the form

$$h_m(z) = \sum_{j=1}^{N_m} a_{m,j} e'_{m,j}(z)$$

the following inequality holds true

$$\sum_{j=1}^{N_m} |a_{m,j}| \sup_{z \in K_p} |e'_{m,j}(z)| \leqslant B \sup_{z \in K_s} |h_m(z)|. \tag{24}$$

It means, that the system $\tilde{\mathcal{E}}'$ possesses a Kathe group property.

And, let the system $\tilde{\mathcal{E}}'$ possess a Kathe group property. Then for every $m = 1, 2, \ldots$ matrix $\mathcal{A}_m = (a_{m,j,k})$ is a nongenerated. Indeed, on the contrary, for some number $m = 1, 2, \ldots$ there is a set of coefficients $a_{m,1}, \ldots, a_{m,N_m}$, which are not equal to zero simultaneously and such that

$$h_m(z) = \sum_{j=1}^{N_m} a_{m,j} e'_{m,j}(z) \equiv 0.$$

Then for every $p, s = 1, 2, \ldots$ we obtain:

$$\sum_{j=1}^{N_m} |a_{m,j}| \sup_{z \in K_p} |e'_{m,j}(z)| > 0 = \sup_{z \in K_s} |h_m(z)|.$$

This contradicts inequality (24).

According to the condition, the system $\tilde{\mathcal{E}} = \{e_{m,j}(z)\}_{m=1,j=1}^{\infty,N_m}$ is a basis in the subspace W, and due to Theorem 3 the system $\tilde{\mathcal{E}}$ will be an almost exponential basis in W with indexes $\lambda_{m,1}$. Then, like in the case with the system $\tilde{\mathcal{E}}'$, for any number p there is a number s and a constant $C > 0$ such that for any function $g \in W$ the following inequality holds true

$$\sum_{s=1,j=1}^{\infty,N_m} |d_{m,j}| \sup_{z \in K_p} |e_{m,j}(z)| \leqslant C \sup_{z \in K_s} |g(z)|, \tag{25}$$

where

$$g(z) = \sum_{s=1,j=1}^{\infty,N_m} d_{m,j} e_{m,j}(z), \qquad z \in D.$$

Let $\mathcal{A}_m^{-1} = (b_{m,j,k})$ be matrix, reciprocal to $\mathcal{A}_m$, $m = 1, 2, \ldots$. For any function $g \in W$ we have:

$$g(z) = \sum_{s=1,j=1}^{\infty,N_m} d_{m,j} e_{m,j}(z) = \sum_{m=1,j=1}^{\infty,N_m} d_{m,j} \sum_{k=1}^{N_m} b_{m,j,k} e'_{m,k}(z) =$$

$$= \sum_{m=1,k=1}^{\infty,N_m} e'_{m,k}(z) \sum_{j=1}^{N_m} d_{m,j} b_{m,j,k} = \sum_{m=1,k=1}^{\infty,N_m} d'_{m,k} e'_{m,k}(z), \quad z \in D. \tag{26}$$

Therefore, we deal with the function disintegration $g(z)$ according to the system $\tilde{\mathcal{E}}'$. Whereas $g(z)$ is an arbitrary function from the subspace W, then to set the system basis $\tilde{\mathcal{E}}'$ in W it is enough to prove, that the latter series uniformly converge on compacts in the domain D, and the function disintegration $g(z)$ according to the system $\tilde{\mathcal{E}}'$ is unique.

Let us fix $p \geq 1$. We obtain:

$$\sum_{m=1,k=1}^{\infty,N_m} |d_{m,k}| \sup_{z\in K_p} |e'_{m,k}(z)| = \sum_{m=1,k=1}^{\infty,N_m} \sup_{z\in K_p} |e'_{m,k}(z)| \left|\sum_{j=1}^{N_m} d_{m,j} b_{m,j,k}\right| \leqslant$$

$$\leqslant \sum_{m=1,k=1}^{\infty,N_m} \sup_{z\in K_p} |e'_{m,k}(z)| \sum_{j=1}^{N_m} |d_{m,j} b_{m,j,k}| = \sum_{m=1,k=1}^{\infty,N_m} |d_{m,j}| \sum_{j=1}^{N_m} |b_{m,j,k}| \sup_{z\in K_p} |e'_{m,k}(z)|.$$

According to the condition the system $\tilde{\mathcal{E}}'$ possesses a Kathe group property. Consequently, due to (24), there is a number s and a constant B such that

$$\sum_{k=1}^{N_m} |b_{m,j,k}| \sup_{z\in K_p} |e'_{m,k}(z)| \leqslant B \sup_{z\in K_s} |e_{m,j}(z)|, \quad m = 1,2,\ldots, \quad j = 1,\ldots,N_m.$$

Hereof and from the above results, subject to (25) we obtain

$$\sum_{m=1,k=1}^{\infty,N_m} |d'_{m,k}| \sup_{z\in K_p} |e'_{m,k}(z)| \leqslant B \sum_{m=1,k=1}^{\infty,N_m} |d_{m,j}| \sup_{z\in K_s} |e_{m,j}(z)| \leqslant C \sup_{z\in K_r} |g(z)|.$$

It means, that the series under consideration uniformly converges on compacts K_p, $p = 1,2,\ldots$. Whereas these compacts exhaust the domain D, then we obtain the needed statement. Moreover, it results from the latter estimate, that the function $g(z)$, which is equivalent to zero, possesses only trivial disintegration. Therefore, coefficients $d'_{m,k}$ in (26) are defined uniquely. The Theorem has been proved.

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