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EIGENFUNCTIONS OF ANNIHILATION OPERATORS ASSOCIATED WITH WIGNER'S COMMUTATION RELATIONS

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Abstract. We consider linear continuous operators acting on the space of all entire functions with the uniform convergence topology and satisfying Wigner's commutation relations. These operators are closely connected with the Dunkl generalized convolution operators. We study the problem of description of eigenfunctions of these operators. It is shown that under some conditions the eigenfunctions of the operator under study can be described by Dunkl generalized translates of entire functions belonging to it's kernel. We also discuss the problem of completeness of the systems of eigenfunctions.

Keywords: commuation relations, Dunkl operator, eigenfunctions, entire functions.

1. Introduction

As usual, we will define creation and annihilation operators by a^+ and a correspondingly. By means of I we will define a single operator. For operators A, B we will consider the commutator $[A, B] = AB - BA$ and anticommutator $[A, B]_+ = AB + BA$.

In 1950 Wigner [1] showed, that not only classical Heisenberg commutation relations $[a, a^+] = I$, but also relations of more general character can result from movement equations of quantum mechanics, namely:

$$[a, a^+] = I + 2\alpha R, \tag{1}$$

where $\alpha \geq 0$ is some constant, R is some abstract operator, satisfying conditions: $[R, a]_+ = 0$, $[R, a^+]_+ = 0$, $R^2 = 1$, $R^{-1} = R$. Let us define the space of entire functions with the uniform convergence topology on the compacts by $H(\mathbb{C})$. It is known (see, for instance, [2]), commutation relations (1) can be realized in the space $H(\mathbb{C})$ the following way:

$$a^+ f(z) = zf(z),\ af(z) = \Lambda_\alpha f(z) = f'(z) + \alpha\Big(\frac{f(z) - f(-z)}{z}\Big),\ f \in H(\mathbb{C}). \tag{2}$$

Note, that the operator Λ_α is known as a Dunkl operator. Detailed information on Dunkl operators can be found, for instance, in the review [3].

Let $\varphi(z) = \sum\limits_{n=0}^{\infty} b_n z^n$ be an arbitrary exponential entire function, $\varphi \not\equiv \text{const}$. In paper [4] generalized convolution operators of the following form were studied:

$$M_{\alpha,\varphi}[f](z) = \sum_{n=0}^{\infty} b_n \Lambda_\alpha^n [f](z),\ f \in H(\mathbb{C}). \tag{3}$$

Convolution Dunkl operator (3) includes ordinary convolution operators on $H(\mathbb{C})$, corresponding the case $\alpha = 0$. Note, that $[M_{\alpha,\varphi}, \Lambda_\alpha] = 0$. Consequently, commutation relations (1) can

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be realized in the space $H(\mathbb{C})$ by the following operators:

$$a^{+}f(z) = \Lambda_\alpha f(z),\ af(z) = \widetilde{M}_{\alpha,\varphi},\ f \in H(\mathbb{C}),$$

where

$$\widetilde{M}_{\alpha,\varphi} f(z) = M_{\alpha,\varphi} f(z) - zf(z). \tag{4}$$

Operators (4) are linear continuous operators on $H(\mathbb{C})$.

For $\lambda \in \mathbb{C}$ we will define by S_λ shift operators on $H(\mathbb{C})$: $S_\lambda f(z) \equiv f(z+\lambda)$. Note, that operators of the form (4), corresponding the case $\alpha = 0$, possess the following interesting properties: **(A)** if $f \in \operatorname{Ker} \widetilde{M}_{0,\varphi}$, $f \not\equiv 0$, then the function $S_\lambda f$ is a eigenfunction of the operator $\widetilde{M}_{0,\varphi}$, satisfying its own value λ, i.e. the following holds true $\widetilde{M}_{0,\varphi} S_\lambda f = \lambda S_\lambda f$, $\forall \lambda \in \mathbb{C}$; **(B)**: if $f \in \operatorname{Ker} \widetilde{M}_{0,\varphi}$, $f \not\equiv 0$, then the shift system $\{S_\lambda f, \lambda \in \Lambda\}$ is complete in $H(\mathbb{C})$, where $\Lambda \subset \mathbb{C}$ is any set, containing an accumulation point.

In this connection, the following question raises some interest: will these properties remain in the case $\alpha > 0$? It is proved in the paper, that with some supplementary limits, the analogue of the property **(A)** takes place when $\alpha > 0$, namely: eigenfunctions of the operator (4) can be described as generalized Dunkl shifts of entire functions from the operator kernel (4).

2. Systems of generalized shifts

It is known (see, for instance, [4]), that there is a single entire function $E_\alpha(z) = \sum_{n=0}^{\infty} c_{n,\alpha} z^n$, satisfying the conditions:

$$\Lambda_\alpha E_\alpha = E_\alpha,\ E_\alpha(0) = 1. \tag{5}$$

Let us define the set of all positive numbers by $\mathbb{Z}_+$, and the set of all entire nonnegative numbers by $\mathbb{Z}_{\geq 0}$. It is clear from (2), that $\Lambda_\alpha[z^n] = \psi(n) z^{n-1}$, where $\psi(n) = n + \alpha(1 - (-1)^n)$, $\forall z \in \mathbb{C}$, $n \in \mathbb{Z}_{\geq 0}$. Note, that the following relations hold true (see, for instance, [5]):

$$\psi(0) = 0;\ \psi(n) = \frac{c_{n-1,\alpha}}{c_{n,\alpha}},\ n \in \mathbb{Z}_+. \tag{6}$$

therefore, the operator (2) acts the entire function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ the following way:

$$\Lambda_\alpha[f](z) = \sum_{n=1}^{\infty} a_n \frac{c_{n-1,\alpha}}{c_{n,\alpha}} z^{n-1}. \tag{7}$$

It is seen from (7), that the operator (2) is a particular case of the Gelfand-Leontiev generalized differentiation operator [6].

It results from (5) and (6), that Taylor series coefficients of the function $E_\alpha(z)$ have the following form:

$$c_{0,\alpha} = 1,\ c_{n,\alpha} = \frac{1}{\psi(1)\psi(2)\cdots\psi(n)},\ n \in \mathbb{Z}_+.$$

With the help of operator (2) we will introduce on $H(\mathbb{C})$ a generalized shift operator

$$S_{\alpha,\lambda}[f](z) = \sum_{n=0}^{\infty} c_{n,\alpha} \Lambda_\alpha^n [f](z) \lambda^n,\ z, \lambda \in \mathbb{C}. \tag{8}$$

The operator (8) acts lineally and continuously from $H(\mathbb{C})$ to $H(\mathbb{C})$ (see, for instance, [4]). Note, that $S_{0,\lambda}[f](z) \equiv f(z+\lambda)$.

Let us prove the following statement.

Theorem 1. *Let arbitrary $\alpha \geq 0$ be given. Let the entire function f be so, that the following equality holds true for it:*

$$\Lambda_\alpha^n[zf(z)] = \psi(n)\Lambda_\alpha^{n-1}[f(z)] + z\Lambda_\alpha^n[f(z)],\ \forall\, n \in \mathbb{Z}_{\geq 0}. \tag{9}$$

Then the following holds true:

$$\widetilde{M}_{\alpha,\varphi}S_{\alpha,\lambda}f - S_{\alpha,\lambda}\widetilde{M}_{\alpha,\varphi}f = \lambda S_{\alpha,\lambda}f,\ \forall\lambda \in \mathbb{C}. \tag{10}$$

Proof. Note, that the operator (3) satisfies (see, for instance, [4]) the following commutation relations:

$$[M_{\alpha,\varphi}, S_{\alpha,\lambda}] = 0,\ \forall\lambda \in \mathbb{C}. \tag{11}$$

Let us take arbitrary $\lambda \in \mathbb{C}$. From (6), (8) and (9) we obtain:

$$S_{\alpha,\lambda}[zf(z)] = \sum_{n=0}^{\infty} c_{n,\alpha}\psi(n)\Lambda_\alpha^{n-1}[f](z)\lambda^n + z\sum_{n=0}^{\infty} c_{n,\alpha}\Lambda_\alpha^n[f](z)\lambda^n =$$

$$= \lambda\sum_{n=1}^{\infty} c_{n-1,\alpha}\Lambda_\alpha^{n-1}[f](z)\lambda^{n-1} + zS_{\alpha,\lambda}[f](z) = (\lambda + z)S_{\alpha,\lambda}[f](z).$$

Therefore,

$$\begin{gathered} S_{\alpha,\lambda}\widetilde{M}_{\alpha,\varphi}f = S_{\alpha,\lambda}M_{\alpha,\varphi}f - (z+\lambda)S_{\alpha,\lambda}f, \\ \widetilde{M}_{\alpha,\varphi}S_{\alpha,\lambda}f = M_{\alpha,\varphi}S_{\alpha,\lambda}f - zS_{\alpha,\lambda}f. \end{gathered} \tag{12}$$

From (11) and (12) we obtain (10). □

Note, that when $\alpha = 0$ equality (9) holds true for any entire function f. Therefore, it results from Theorem 1, in particular, the property **(A)** for the operator $\widetilde{M}_{0,\varphi}$. When $\alpha > 0$ the equality (12) will not hold true for the arbitrary entire function. in the following theorem we set the class of entire functions, for which (9) holds true with any $\alpha \geq 0$.

Theorem 2. *Let $f \in H(\mathbb{C})$ be an even function. Then the relation (9) holds true for the function f with any $\alpha \geq 0$.*

Theorem 2 validity results from the following Lemma.

Lemma 1. *Let $f \in H(\mathbb{C})$. Then with any $\alpha \geq 0$ and for all $n \in \mathbb{Z}_{\geq 0}$ the following relation holds true*

$$\Lambda_\alpha^n[zf(z)] = \psi(n)\Lambda_\alpha^{n-1}[f(z)] + z\Lambda_\alpha^n[f(z)] - \alpha(1-(-1)^n)\Lambda_\alpha^{n-1}[f(z) - f(-z)].$$

Before we present the proof of Lemma 1, we will prove the following auxiliary Lemma.

Lemma 2. *With any $n, k \in \mathbb{Z}_{\geq 0}$ and $\alpha > 0$ the following holds true:*

$$\psi(n+k) = \psi(n) + \psi(k) - \alpha(1-(-1)^n)(1-(-1)^{k+n-1}). \tag{13}$$

Proof. Let us take arbitrary $\alpha > 0$. There are 4 cases possible: 1) n is an even number, k is an uneven number; then $n+k$ is an uneven number, $\psi(n) = n$, $\psi(k) = k + 2\alpha$, $\psi(n+k) = n + k + 2\alpha = \psi(n) + \psi(k)$; 2) n is uneven, k is even; then $n+k$ is uneven, $\psi(n) = n + 2\alpha$, $\psi(k) = k$, $\psi(n+k) = n+k+2\alpha = \psi(n)+\psi(k)$; 3) n is even, k is even; then $n+k$ is even, $\psi(n) = n$, $\psi(k) = k$, $\psi(n+k) = n+k = \psi(n)+\psi(k)$; 4) n is uneven, k is uneven; then $n+k$ is even, $\psi(n) = n+2\alpha$, $\psi(k) = k+2\alpha$, $\psi(n+k) = n+k = \psi(n)+\psi(k)-4\alpha$. Therefore, if at least one of the numbers n and k is even, then $\psi(n+k) = \psi(n)+\psi(k)$, otherwise, $\psi(n+k) = \psi(n) + \psi(k) - 4\alpha$. On the basis of this, we obtain the following formula:

$$\psi(n+k) = \psi(n) + \psi(k) - \alpha(1-(-1)^n)(1-(-1)^k). \tag{14}$$

Let us demonstrate, that (14) is equivalent to (13). Indeed,

$$(1-(-1)^n)(1-(-1)^k) = (1-(-1)^n)(1-(-1)^k\cdot(-1)^{2n}) =$$
$$= (1-(-1)^n)(1-(-1)^{k+n}\cdot(-1)^n) =$$
$$= 1-(-1)^{k+n}\cdot(-1)^n-(-1)^n+(-1)^{k+n}\cdot(-1)^{2n} =$$
$$= 1-(-1)^n+(-1)^{k+n}(1-(-1)^n) =$$
$$= (1-(-1)^n)(1+(-1)^{k+n}) = (1-(-1)^n)(1-(-1)^{k+n-1}).$$

□

Let us present the proof of Lemma 1.

Proof. Let $f(z)=\sum\limits_{k=0}^{\infty}a_kz^k$. Then $zf(z)=\sum\limits_{k=1}^{\infty}a_{k-1}z^k$. From (6) and (7) we obtain:

$$\Lambda_\alpha^n[zf(z)] = \sum_{k=n}^{\infty}a_{k-1}\frac{c_{k-n,\alpha}}{c_{k,\alpha}}z^{k-n} =$$
$$= \sum_{k=n}^{\infty}a_{k-1}z^{k-n}\psi(k-n+1)\psi(k-n+2)\cdots\psi(k) =$$
$$= \sum_{k=0}^{\infty}a_{k+n-1}z^k\psi(k+1)\psi(k+2)\cdots\psi(k+n).$$

From the latter equality and (13) we obtain:

$$\Lambda_\alpha^n[zf(z)] = \Sigma_1+\psi(n)\Sigma_2-\alpha(1-(-1)^n)\Sigma_3, \tag{15}$$

where $\Sigma_1=\sum\limits_{k=1}^{\infty}a_{k+n-1}z^k\psi(k)\psi(k+1)\cdots\psi(k+n-1)$,

$$\Sigma_2 = \sum_{k=0}^{\infty}a_{k+n-1}z^k\psi(k+1)\psi(k+2)\cdots\psi(k+n-1),$$

$$\Sigma_3 = \sum_{k=0}^{\infty}a_{k+n-1}z^k(1-(-1)^{k+n-1})\psi(k+1)\psi(k+2)\cdots\psi(k+n-1).$$

Note, that summarizing in Σ_1 starts with $k=1$ according to $\psi(0)=0$. We have:

$$\Sigma_1 = \sum_{k=n}^{\infty}a_kz^{k-n+1}\psi(k-n+1)\psi(k-n+2)\cdots\psi(k) = z\Lambda_\alpha^n[f(z)];$$
$$\Sigma_2 = \sum_{k=n-1}^{\infty}a_kz^{k-n+1}\psi(k-n+2)\psi(k-n+3)\cdots\psi(k) = \Lambda_\alpha^{n-1}[f(z)]; \tag{16}$$
$$\Sigma_3 = \sum_{k=n-1}^{\infty}a_k(1-(-1)^k)z^{k-n+1}\psi(k-n+2)\cdots\psi(k) = \Lambda_\alpha^{n-1}[f(z)-f(-z)].$$

The Lemma statement results from (15) and (16). □

Let us formulate the basic result of the article.

Theorem 3. *Let the given arbitrary $\alpha\geq 0,\lambda\in\mathbb{C}\setminus\{0\}$. Let the function $f\in H(\mathbb{C})$ satisfy the following conditions: 1) $f\in\ker\widetilde{M}_{\alpha,\varphi}$ for some φ, 2) f is an even function, 3) $S_{\alpha,\lambda}f\not\equiv 0$. Then the function $S_{\alpha,\lambda}f$ is an eigenfunctions of the operator $\widetilde{M}_{\alpha,\varphi}$, which is equivalent to its own value λ.*

Proof. Whereas f is an even function, then it results from Theorems 1 and 2, that for f the following relation holds true (10). From (10), taking into account, that $f \in \ker \widetilde{M}_{\alpha,\varphi}$, we obtain: $\widetilde{M}_{\alpha,\varphi} S_{\alpha,\lambda} f = \lambda S_{\alpha,\lambda} f$. □

Corollary 1. *Let the given arbitrary $\alpha \geq 0$, $\lambda \in \mathbb{C} \setminus \{0\}$. Let the function $f \in H(\mathbb{C})$ satisfy the conditions 1) and 2) of Theorem 3 and the condition $f(\lambda) \neq 0$. Then the function $S_{\alpha,\lambda} f$ is an eigenfunctions of the operator $\widetilde{M}_{\alpha,\varphi}$, which is equivalent to its own value λ.*

Proof. Let us prove, that condition 3) of Theorem 3 results from the condition $f(\lambda) \neq 0$. Let $f(\lambda) \neq 0$. Assume, that condition 3) of Theorem 3 does not hold true. Consequently, the function f satisfies the equation $S_{\alpha,\lambda} f \equiv 0$. Then, according to [7, ch. III, §3] the function f can be presented in the form:

$$f(z) = \lim_{n \to \infty} \sum_{|\mu_k| < q_n} \sum_{j=0}^{m_k - 1} p_{kj} z^j E_\alpha^{(j)}(\mu_k z), \tag{17}$$

where $\{\mu_k\}$ are zeros of the function $E_\alpha(\lambda z)$, m_k is the root order μ_k, $\{q_n\}$ is a growing sequence of positive numbers, p_{kj} are some constants. It follows from the representation (17), that $f(\lambda) = 0$, that contradicts the initial assumption. □

Let us give some examples, satisfying conditions of Theorem 3.

Example 1. Let $\varphi(z) = z$. In this case $\widetilde{M}_{\alpha,\varphi} = \Lambda_\alpha - zI$. Then the function $f(z) = e^{z^2/2}$ satisfies conditions 1) and 2) of Theorem 3. Moreover, according to corollary 1, the function f satisfies conditions 3) of Theorem 3 with any $\lambda \in \mathbb{C}$.

Example 2. Let $\varphi(z) = z^3$. In this case $\widetilde{M}_{\alpha,\varphi} = \Lambda_\alpha^3 - zI$. Let us find an even entire solution f of the equation $\Lambda_\alpha^3[f](z) - zf(z) = 0$. Whereas f is an even function, then the latter equation can be substituted by the following differential equation:

$$f'''(z) + 2\alpha \frac{zf''(z) - f'(z)}{z^2} - zf(z) = 0.$$

Then, as a desired solution we can take, for instance, a generalized hypergeometric function $f(z) = {}_0F_2(\{\}, \{\frac{1}{2}, \frac{3}{4} + \frac{\alpha}{2}\}, \frac{z^4}{64})$. The function satisfies conditions 1) and 2) of Theorem 3. Also, according to corollary 1, the function f satisfies conditions 3) of Theorem 3 at least for those $\lambda \in \mathbb{C}$, when $f(\lambda) \neq 0$.

3. Notice on completeness of eigenfunctions

As it has already been said the the introduction, in the case $\alpha = 0$ the operator $\widetilde{M}_{\alpha,\varphi}$ possesses the property of eigenfunctions completeness (property **(B)**). This property was proved by the author in paper [8]. According to Godefroy-Shapiro criterion [9, p. 6], hypercyclicity of the operator results from this property $\widetilde{M}_{0,\varphi}$. Let us remind, that the linear continuous operator Φ on the topological vector space X is called hypercyclic, if there is such an element $x \in X$, that its orbit $\{\Phi^n x,\ n = 0, 1, 2, ...\}$ is complete in X. A more detailed description of hypercyclic operators theory can be found, for instance, in monograph [9].

Note, that the analogue of the property **(B)** takes place for the case $\varphi(z) = z$ and with $\alpha > 0$. Indeed, for this case the property **(B)** implies completeness in $H(\mathbb{C})$ of the generalized shifts system $\{S_{\lambda,\alpha} e^{z^2/2}, \lambda \in \Lambda\}$, where $\Lambda \subset \mathbb{C}$ is any set, containing an accumulation point. The latter, as it is easy to see, is equivalent to completeness in $H(\mathbb{C})$ of the system $\{\Lambda_\alpha^n(e^{z^2/2}),\ n = 0, 1, \cdots\}$. Note, that $\Lambda_\alpha^n(e^{z^2/2}) = e^{z^2/2} P_{n,\alpha}(z)$, where $P_{n,\alpha}$ is a polynomial of the degree n. The system of polynomials $P_{n,\alpha}$, is obviously, complete in $H(\mathbb{C})$. Consequently, the system $\{\Lambda_\alpha^n(e^{z^2/2}),\ n = 0, 1, \cdots\}$ is also complete. Hence, according to Godefroy-Shapiro criterion, the operator $\widetilde{M}_{\alpha,\varphi}$ is a hypercyclic operator in the space $H(\mathbb{C})$ for the case $\varphi(z) = z$.

In connection with the written above, let us formulate the following unsolved problem: to study the problem of the operator eigenfunctions completeness $\widetilde{M}_{\alpha,\varphi}$ for other functions φ.

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