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SYMMETRY PROPERTIES FOR SYSTEMS OF TWO ORDINARY FRACTIONAL DIFFERENTIAL EQUATIONS

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Abstract. Lie point symmetries of two systems of ordinary fractional differential equations with the Riemann-Liouville derivatives are considered. Infinite algebra L of equivalence transformation operators is constructed. It is shown that all admitted operators generate some subalgebra in L and classification of systems with respect to point symmetries can be based on the optimal system of subalgebras. The optimal system of one-dimensional L subalgebras and the complete normalized optimal system for its finite-dimensional part L_6 are constructed.

Keywords: fractional derivatives, symmetries, group classification,
optimal system of subalgebras

1. Introduction

During the last years the apparatus of fractional integro-differentiation [1] has been more intensely used to construct mathematical models of different processes. Equations with fractional derivatives of different types are used in modeling processes with complex non-local dependencies, stochastic effects with the power distribution laws, in theory of automatic control, etc.

In papers [2, 3, 4] classical methods of group analysis of differential equations [5] are adapted to study equations with Riemann-Liouville and Caputo fractional derivatives.

In particular, it is shown in [2], that unlike ordinary differential equations of the first order, equations with the derivative of the order $0 < \alpha < 1$ have finite-dimensional groups of admissible transformations.

In paper [3], equations of the form $D^\alpha_x y(x) = f(x, y)$ are classified according to admitted groups of point transformations and classes of exact solutions are constructed. The present paper is devoted to investigation of systems of two equations of the same form

$$\begin{cases} D^\alpha u(t) = f(t, u, v), \\ D^\alpha v(t) = g(t, u, v) \end{cases}$$

with a fractional derivative of the Riemann-Liouville type. Equivalence transformations of the system are determined, and the problem of finding symmetries for given functions f, g is also solved here.

It is demonstrated that an algebra of admitted operators for the system (1) is a certain subalgebra in the algebra of operators L, generating equivalence transformations. Therefore, the problem of systems classification is reduced to construction of an optimal system of subalgebras $\Theta(L)$ [6, 7].

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2. Symmetries and equivalence transformations

The system of two differential equations with fractional derivatives

$$\begin{cases} D^{\alpha}u(t) = f(t,u,v), \\ D^{\alpha}v(t) = g(t,u,v) \end{cases} \tag{1}$$

is considered in the paper.

Here D^{α} is an operator of fractional Riemann-Liouville differentiation with respect to t:

$$D^{\alpha}u(t) \equiv D^{m}\left(I^{m-\alpha}u(t)\right) = \frac{1}{\Gamma(m-\alpha)}\frac{d^m}{dt^m}\int\limits_0^t \frac{u(\tau)}{(t-\tau)^{\alpha+1-m}}d\tau \tag{2}$$

with $0 < m-1 < \alpha \leqslant m, m \in \mathbb{N}$ ([1]).

Substitution of the variables

$$\bar{t} = \Phi(t,u,v), \ \ \bar{u} = \Psi^u(t,u,v), \ \ \bar{v} = \Psi^v(t,u,v) \tag{3}$$

is an equivalence transformation for the system (2) if the system has the same form in the new variables:

$$\begin{cases} D^{\alpha}\bar{u}(\bar{t}) = \bar{f}(\bar{t},\bar{u},\bar{v}), \\ D^{\alpha}\bar{v}(\bar{t}) = \bar{g}(\bar{t},\bar{u},\bar{v}). \end{cases}$$

The functions $\bar{f},\bar{g}$ are new functions of the arguments $\bar{t},\bar{u},\bar{v}$. If the functions remain unaltered, the transformation (3) is called an *admitted transformation* of the system (1).

One-parameter group of transformations can be described by its *infinitesimal operator*. For the equivalence transformations it has the following form:

$$X = \xi(t,u,v)\frac{\partial}{\partial t} + \eta^u(t,u,v)\frac{\partial}{\partial u} + \eta^v(t,u,v)\frac{\partial}{\partial v} + \nu^u(t,u,v,f,g)\frac{\partial}{\partial f} + \nu^v(t,u,v,f,g)\frac{\partial}{\partial g}. \tag{4}$$

According to the results [2], the action of infinitesimal transformations

$$\bar{t} = t + a\xi + o(a), \qquad \bar{u} = u + a\eta^u + o(a), \qquad \bar{v} = v + a\eta^v + o(a)$$

on fractional derivatives is defined by the *prolongation formula*:

$$D^{\alpha}_{\bar{t}}\bar{u}(\bar{t}) = D^{\alpha}_t u(t) + a\zeta^u_{\alpha} + o(a),$$

where ζ^u_{α} can be written in the form of a series

$$\zeta^u_{\alpha} = D^{\alpha}_t(\eta^u) - \alpha D_t(\xi)D^{\alpha}_t(u) + \sum_{n=1}^{\infty}\binom{\alpha}{n}\frac{n-\alpha}{n+1}D^{\alpha-n}_t(u)D^{n+1}_t(\xi). \tag{5}$$

Determining equations for finding coefficients of the infinitesimal operator (4) of equivalence transformations have the following form

$$(\zeta^u_{\alpha} - \nu^u)\big|_{D^{\alpha}u=f,D^{\alpha}v=g} = 0,$$
$$(\zeta^v_{\alpha} - \nu^v)\big|_{D^{\alpha}u=f,D^{\alpha}v=g} = 0,$$

where f and g are considered to be independent variables.

By analogy to the algorithm for constructing coordinates of admitted operators, suggested in [2, 3], let us find symmetries and equivalence transformations from the following class:

$$\begin{aligned} &\xi = \xi(t), \qquad \xi(0) = 0, \\ &\eta^u = p^{uu}(t)u + p^{uv}(t)v + q^u(t), \qquad \eta^v = p^{vu}(t)u + p^{vv}(t)v + q^v(t). \end{aligned} \tag{6}$$

In this case, $D^{\alpha}(\eta^u), D^{\alpha}(\eta^v)$ can be represented via fractional derivatives and integrals $D^{\alpha-n}u, D^{\alpha-n}v$ in the prolongation formula (5) and its analogue for ζ^v_{α} by means of the generalized Leibniz rule (there are no compact chain rule formulae for fractional differentiation).

As a result, the determining equations split with respect to the variables $D^{\alpha-n}u$, $D^{\alpha-n}v$. Then, solving the resulting infinite system of equations, one obtains expressions for coordinates of the operator (4):

$$\begin{cases} \xi = (C_1 + C_2 t)t, \\ \eta^u = (\alpha-1)C_2 tu + C_3 u + C_4 v + q^u(t), \\ \eta^v = (\alpha-1)C_2 tv + C_5 u + C_6 v + q^v(t), \\ \nu^u = -\alpha f C_1 - (\alpha+1)C_2 tf + C_3 f + C_4 g + D^\alpha q^u(t), \\ \nu^v = -\alpha g C_1 - (\alpha+1)C_2 tg + C_5 f + C_6 g + D^\alpha q^v(t), \end{cases} \tag{7}$$

where $C_1, \ldots, C_6$ are arbitrary constants, and q^u, q^v are arbitrary functions of t.

In search of admitted operators

$$X = \xi(t,u,v)\frac{\partial}{\partial t} + \eta^u(t,u,v)\frac{\partial}{\partial u} + \eta^v(t,u,v)\frac{\partial}{\partial v},$$

the determining equations take the following form

$$\begin{aligned} (\zeta^u_\alpha - \xi f_t - \eta^u f_u - \eta^v f_v)|_{D^\alpha u = f(t,u,v), D^\alpha v = g(t,u,v)} &= 0, \\ (\zeta^v_\alpha - \xi g_t - \eta^u g_u - \eta^v g_v)|_{D^\alpha u = f(t,u,v), D^\alpha v = g(t,u,v)} &= 0. \end{aligned}$$

Solving them with the same restrictions on the class of symmetries (6), one obtains the coordinates ξ, η^u, η^v of the same form (7), but with additional conditions

$$\begin{cases} (C_1 + C_2 t)tf_t + [(\alpha-1)C_2 tu + C_3 u + C_4 v + q^u(t)]f_u + \\ \qquad + [(\alpha-1)C_2 tv + C_5 u + C_6 v + q^v(t)]f_v = \\ \qquad\qquad = D_t^\alpha q^u(t) + (C_3 - \alpha C_1 - (\alpha+1)C_2 t)f + C_4 g, \\ (C_1 + C_2 t)tg_t + [(\alpha-1)C_2 tu + C_3 u + C_4 v + q^u(t)]g_u + \\ \qquad + [(\alpha-1)C_2 tv + C_5 u + C_6 v + q^v(t)]g_v = \\ \qquad\qquad = D_t^\alpha q^v(t) + (C_6 - \alpha C_1 - (\alpha+1)C_2 t)g + C_5 f. \end{cases} \tag{8}$$

Thus, when the functions $f(t,u,v), g(t,u,v)$ are given, symmetries of the system (1) can be found by solving the system (8). The admitted operators form a subalgebra in the Lie algebra $L = L_6 + L_\infty$, where the algebra L_6, and the infinite-dimensional algebra L_∞ are generated by the basis operators

$$\begin{gathered} X_1 = t\frac{\partial}{\partial t}, \qquad X_2 = t^2\frac{\partial}{\partial t} + (\alpha-1)tu\frac{\partial}{\partial u} + (\alpha-1)tv\frac{\partial}{\partial v}, \\ X_3 = u\frac{\partial}{\partial u}, \qquad X_4 = v\frac{\partial}{\partial u}, \qquad X_5 = u\frac{\partial}{\partial v}, \qquad X_6 = v\frac{\partial}{\partial v}, \end{gathered} \tag{9}$$

and by operators of the form

$$X_{q^u} = q^u(t)\frac{\partial}{\partial u}, \quad X_{q^v} = q^v(t)\frac{\partial}{\partial v}, \tag{10}$$

respectively.

Note, that in the our case all possible symmetries of the system (1) can be obtained from the algebra L, generating equivalence transformations. Meanwhile, if two systems of the type (1) are connected by an equivalence transformation, then their operators can be obtained from each other by the same transformation (by substitution of variables in a differential operator). A set of such transformations in the Lie algebra L corresponds to a group of *inner automorphisms* of this algebra [5].

Therefore, to solve the equations' classification problems with respect to admitted transformation groups (one-, two-parameter, etc.) it is sufficient to construct classes of dissimilar subalgebras of the algebra L with respect to equivalence transformations. In our case this is equivalent to the problem of construction of an optimal system of subalgebras of the algebra L (finding dissimilar subalgebras with respect to inner automorphisms).

3. Optimal system of subalgebras

To construct an optimal system of subalgebras $\Theta(L)$ it is convenient to introduce the basis

$$Y_1 = X_1, \quad Y_2 = X_2, \quad Y_3 = X_3 - X_6, \quad Y_4 = X_4 \quad Y_5 = X_5 \quad Y_6 = X_3 + X_6.$$

The table of commutators has the following form

	Y_1	Y_2	Y_3	Y_4	Y_5	Y_6	Y_{q^u}	Y_{q^v}
Y_1	0	Y_2	0	0	0	0	$\langle t\dot{q^u}\rangle_u$	$\langle t\dot{q^v}\rangle_v$
Y_2		0	0	0	0	0	$\langle t^2\dot{q^u} - (\alpha-1)tq^u\rangle_u$	$\langle t^2\dot{q^v} - (\alpha-1)tq^v\rangle_v$
Y_3			0	$-2Y_4$	$2Y_5$	0	$\langle -q^u\rangle_u$	$\langle q^v\rangle_v$
Y_4				0	$-Y_3$	0	0	$\langle -q^v\rangle_u$
Y_5					0	0	$\langle -q^u\rangle_v$	0
Y_6						0	$\langle -q^u\rangle_u$	$\langle -q^v\rangle_v$
Y_{ν^u}							0	0
Y_{ν^v}								0

The part of the table below the main diagonal is constructed due to skew-symmetry of the commutator. An abbreviated notation of operators is used here

$$\langle q\rangle_u = q\frac{\partial}{\partial u}, \quad \langle q\rangle_v = q\frac{\partial}{\partial v}.$$

One can see that the set of operators $\{Y_{q^u}, Y_{q^v}\}$ with arbitrary functions $q^u(t), q^v(t)$ is an infinite Abelian ideal L_∞ in the algebra L, and the algebra has the following structure:

$$L = L_\infty \oplus \{Y_1, Y_2\} \oplus \{Y_3, Y_4, Y_5\} \oplus \{Y_6\}.$$

Subalgebras $\{Y_6\}$ and $\{Y_1, Y_2\}$ are a center and an ideal in the algebra $L_6 = \{Y_1, \ldots, Y_6\}$, respectively.

Every operator $Z \in L$ generates an inner automorphism of the algebra L under consideration. It can be constructed as a solution of the Cauchy problem

$$\frac{d\bar{Y}}{ds} = [Z, \bar{Y}], \quad \bar{Y}\Big|_{s=0} = Y, \tag{11}$$

where operators are defined by their coordinates in the given basis:

$$\bar{Y} = \bar{k}^1 Y_1 + \ldots + \bar{k}^6 Y_6 + Y_{\bar{q}^u} + Y_{\bar{q}^v}, \quad \bar{k}^i = \bar{k}^i(s, k^1, \ldots, k^6, q^u, q^v).$$

Note, that the inner automorphism, constructed for the operators Z from the center, is always an identity transformation in L_6.

Solving the system of equations (11) for $Y_1, \ldots, Y_5$, one obtains inner automorphisms in the form of operator coordinates transformations:

	$\bar{k}^1$	$\bar{k}^2$	$\bar{k}^3$	$\bar{k}^4$	$\bar{k}^5$	$\bar{k}^6$
A_1	k^1	a_1k^2	k^3	k^4	k^5	k^6
A_2	k^1	$k^2 - a_2k^1$	k^3	k^4	k^5	k^6
A_3	k^1	k^2	k^3	a_3k^4	k^5/a_3	k^6
A_4	k^1	k^2	$k^3 - a_4k^5$	$k^4 + 2a_4k^3 - {a_4}^2k^5$	k^5	k^6
A_5	k^1	k^2	$k^3 + a_5k^4$	k^4	$k^5 - 2a_5k^3 - {a_5}^2k^4$	k^6

Here a_i are arbitrary parameters. Taking discrete automorphisms (equivalence transformations $\bar{u} = -u$) into account allows one to avoid imposing the limitation $a^3 > 0$. The time reversal transformation $\bar{t} = -t$ changes the Riemann-Liouville operator and is not considered as a discrete equivalence transformation. Hence, in what follows it is assumed that $a^1 > 0$.

The action of automorphisms $A_1 \dots A_5, A_6$ on the coordinates q^u, q^v looks as follows:

$$
\begin{aligned}
&A_1: \ \bar{q}^u = q^u(a_1 t), \quad \bar{q}^v = q^v(a_1 t),\\
&A_2: \ \bar{q}^u = (1-a_2 t)^{\alpha-1} q^u\left(\frac{t}{1-at}\right), \quad \bar{q}^v = (1-a_2 t)^{\alpha-1} q^v\left(\frac{t}{1-at}\right),\\
&A_3: \ \bar{q}^u = \tilde{a}_3 q^u, \quad \bar{q}^v = q^v/\tilde{a}_3, \quad (\tilde{a}_3 = \pm\sqrt{|a_3|}, \tilde{a}_3 a_3 > 0),\\
&A_4: \ \bar{q}^u = q^u - a_4 q^v, \quad \bar{q}^v = q^v,\\
&A_5: \ \bar{q}^u = q^u, \quad \bar{q}^v = q^v - a_5 q^u,\\
&A_6: \ \bar{q}^u = a_6 q^u, \quad \bar{q}^v = a_6 q^v, \quad a_6 > 0,
\end{aligned}
$$

and the combination of automorphisms A_{ν^u}, A_{ν^v} has the form

$$
\begin{aligned}
\bar{q}^u &= q^u - (k^1 t + k^2 t^2)\dot{\nu}^u + (k^3 + k^6 + (\alpha-1)k^2 t)\nu^u + k^4 \nu^v,\\
\bar{q}^v &= q^v + k^5 \nu^u - (k^1 t + k^2 t^2)\dot{\nu}^v + (-k^3 + k^6 + (\alpha-1)k^2 t)\nu^v.
\end{aligned}
\tag{12}
$$

Peculiarities of constructing automorphisms and an optimal system of subalgebras for operators with arbitrary functions are illustrated, e.g., in [7].

In accordance with the procedure [6], bases of the required r-dimensional subalgebras of the algebra L are written in the form of matrices, where the lines represent coordinates of the basis of the subalgebra in the basis Y. Matrix elements should satisfy the *subalgebra conditions,* i.e. the space should be closed under the commutation operation. The action of the group of inner automorphisms A (certain linear transformations of columns) and the group B of transformations of the subalgebra basis (all linear nondegenerate transformations of lines) is considered on the set of matrices. Those matrices that are dissimilar with respect to these transformations define elements of the optimal system $\Theta(L)$. Classifying matrices by means of transformations A, B, one achieves the maximal possible number of zero coordinates and the minimal number of arbitrary constants.

It is always possible to construct an optimal system, satisfying the additional requirement of *normalization.* The largest subalgebra of the algebra L, for which K is an ideal, i.e. $[X, Y] \in K$ holds for all $X \in K$ and $Y \in \mathrm{Nor}_L K$, is termed as the normalizer $\mathrm{Nor}_L K$ of the subalgebra K in L. Normalized optimal system should contain the normalizer $\mathrm{Nor}_L K \in \Theta_A L$ together with every subalgebra $K \in \Theta_A L$.

The construction starts with the algebra $L_4 = \{Y_3, Y_4, Y_5, Y_6\}$. Only automorphisms A_3, A_4, A_5 act there. Expressions $k^6, k^3k^3 + k^4k^5$ are invariant under the groups of inner automorphisms. Calculations carried out according to the above algorithm provide a normalized optimal system of subalgebras $\Theta(L_4)$, given in Table 1. The abbreviations $\{4-5+6\} = \{Y_4 - Y_5 + \gamma Y_6\}$ are used in the tables, the sign ,,=" in the column Nor indicates that the given subalgebra is self-normalized.

Table 1. Optimal system $\Theta(L_4)$

No	Subalgebra	Nor
4.1	$3,4,5,6$	$=$
3.1	$3,4,6$	$=$
3.2	$3,4,5$	4.1
2.1	$3,6$	$=$
2.2	$4-5,6$	$=$
2.3	$4,6$	3.1
2.4	$3+\beta 6,4$	3.1
1.1	$3+\gamma 6$	2.1
1.2	6	4.1
1.3	$4+6$	2.3
1.4	4	3.1
1.5	$4-5+\gamma 6$	2.2
	$\gamma \geq 0, \beta \in \mathbf{R}$	

The optimal system $\Theta(L_6)$ is constructed using the decomposition $L_6 = J \oplus N$, where $N = L_4$ is a subalgebra, $J = \{Y_1, Y_2\}$ is an ideal. For any subalgebra N_p from the optimal system $\Theta_{A_N}(N)$ (in our case, from Table 1) there is a stabilizer $A_p \subset A$ in L_6, i.e. automorphisms L_6, which do not change this subalgebra (but can change the form of the corresponding matrix). The stabilizer A_p in this case includes A_1, A_2 and some combinations A_3, A_4, A_5.

By means of transformations from A_p, the arbitrary subalgebra from $J \oplus N_p$ (N_p with operators from the ideal added arbitrarily) is simplified and the optimal system $\Theta_{A_p}(J \oplus N_p) = \{K_{p,q}\}$ is constructed. The set of all subalgebras obtained for different N_p makes up the optimal system $\Theta_A(L_6)$.

A normalized optimal system constructed is shown in Tables 2-5 together with corresponding indices of N_p and normalizers.

Decomposing the algebra L into the ideal L_∞ and the subalgebra L_6, one can construct $\Theta(L)$ starting with the optimal system $\Theta(L_6)$ according to the same procedure. Automorphisms A_{ν^u}, A_{ν^v} of the form (12) change only the components $\langle q^u \rangle_u$ and $\langle q^v \rangle_v$ of the operator Y. If at least one condition

$$k^1 \neq 0, \quad k^2 \neq 0,$$
$$(k^6)^2 - (k^3)^2 - k^4 k^5 \neq 0$$

holds true for the operator coefficients, one can turn the arbitrary functions q^u, q^v to zero by the choice of the functions $\nu^u(t)$, $\nu^v(t)$. Thus, only elements 1.1 with $\gamma = 1$ and 1.4 of the optimal system $\Theta(L_6)$ (and the zero subalgebra as well) generate new elements $\Theta(L)$. The corresponding subalgebras $1.18 - 1.20$ are also given in Table 2.

Likewise, one can also construct subalgebras of a higher dimension, containing $\langle q^u \rangle_u$ and $\langle q^v \rangle_v$. Subalgebra conditions are written in the form of differential relations.

For every subalgebra K from optimal system one can obtain all functions $f(t,u,v), g(t,u,v)$ such that the system (1) admits the given operators. This is done by solving equations (8) with the known coefficients $C_1, \ldots C_6$ and functions $q^u(t), q^v(t)$ simultaneously. In this case invariants of subalgebra K will be arbitrary elements in these functions (as one can see from the structure of equations (8)). All systems that have admitted algebras of operators similar to K can be reduced to this form by equivalence transformations.

The results of calculations are given in the corresponding columns of Tables 2-5, where F and G are arbitrary functions of invariants J_i. For the sake of convenience, polar coordinates r, ϕ: $u = r\cos\phi, u = v\sin\phi$ are sometimes used in tables.

4. Conclusion

Equivalence transformations of the system (1) are constructed, including a general nondegenerate linear transformation of unknown functions u and v, dilation of the independent variable t, addition of fixed functions $q(t)$ to u and v, and the projective transformation of a special form.

It is shown that admitted operators of the system form a subalgebra of the algebra $L = L_\infty \oplus L_6$, generating equivalence transformations, and the problem of classification of the systems (1) with respect to admitted groups of point transformations is reduced to construction of an optimal system of subalgebras $\Theta(L)$.

Classical algorithms [6, 7] are applied to construct $\Theta(L)$. As a result, a complete normalized optimal system of subalgebras L_6 and an optimal system of one-dimensional subalgebras L are calculated.

Symmetries of systems can also be used to obtain their solutions. System of the form (1) also occur when constructing solutions of fractional order partial differential equations, e.g., by the method of invariant subspaces [8].

Table 2. Optimal system $\Theta_1(L_6 \oplus L_\infty)$

No	Subalgebra	Nor	N_p	f, g	Invariants
1.1	$3+\gamma 6$	4.2		$f = uF(t, v^{1+\gamma}u^{1-\gamma})$	$J_1 = t$
				$g = vG(t, v^{1+\gamma}u^{1-\gamma})$	$J_2 = v^{1+\gamma}u^{1-\gamma}$
1.2	6	6.1		$f = uF(t, v/u)$	$J_1 = t$
				$g = uG(t, v/u)$	$J_2 = v/u$
1.3	$4+6$	4.4		$f = v(F + G\ln v)$	$J_1 = t$
				$g = vG$	$J_2 = ve^{-u/v}$
1.4	4	5.1		$f = F(t, v) + uG(t, v)$	$J_1 = t$
				$g = vG(t, v)$	$J_2 = v$
1.5	$4-5+\gamma 6$	4.3		$f = uF - vG$	$J_1 = t$
				$g = vF + uG$	$J_2 = re^{\gamma\phi}$
1.6	1	5.3	0	$f = t^{-\alpha}F(u, v)$	$J_1 = u$
				$g = t^{-\alpha}G(u, v)$	$J_2 = v$
1.7	2	6.1	0	$f = t^{-2\alpha}uF$	$J_1 = u/v$
				$g = t^{-2\alpha}vG$	$J_2 = vt^{1-\alpha}$
1.8	$k1+3+\gamma 6$	3.8	1.1	$f = t^{-\alpha}uF$	$J_1 = u^k t^{-1-\gamma}$
				$g = t^{-\alpha}vG$	$J_2 = v^k t^{1-\gamma}$
1.9	$k1+6$	5.3	1.2	$f = v^{1-\alpha k}F$	$J_1 = vt^{-1/k}$
				$g = v^{1-\alpha k}G$	$J_2 = u/v$
1.10	$k1+4+6$	3.10	1.3	$f = t^{-\alpha}v(F + G\ln t)$	$J_1 = vt^{-1/k}$
				$g = kt^{-\alpha}vG$	$J_2 = ve^{-u/v}$
1.11	$1+4$	3.10	1.4	$f = t^{-\alpha}(F + G\ln t)$	$J_1 = v$
				$g = t^{-\alpha}G$	$J_2 = u - v\ln t$
1.12	$k1+\gamma 6+$	3.9	1.5	$f = t^{-\alpha}(uF - vG)$	$J_1 = t^{1/k}e^{\phi}$
	$+4-5$			$g = t^{-\alpha}(vF + uG)$	$J_2 = re^{\gamma\phi}$
1.13	$\pm 2+3+\gamma 6$	$3.12_{0,0}$	1.1	$f = t^{-1-\alpha}e^{\mp 1/t}F$	$J_1 = ut^{1-\alpha}e^{\pm(\gamma+1)/t}$
				$g = t^{-1-\alpha}e^{\mp 1/t}G$	$J_2 = vt^{1-\alpha}e^{\pm(\gamma-1)/t}$

No	Subalgebra	Nor	N_p	f, g	Invariants
1.14	$\pm 2 + 6$	5.4_0	1.2	$f = t^{-2\alpha} uF$	$J_1 = ut^{1-\alpha} e^{\pm 1/t}$
				$g = t^{-2\alpha} vG$	$J_2 = u/v$
1.15	$\pm 2 + 4 + 6$	3.15_0	1.3	$f = t^{-2\alpha} v(F \mp G/t)$	$J_1 = u/v \pm 1/t$
				$g = t^{-2\alpha} vG$	$J_2 = vt^{1-\alpha} e^{\pm 1/t}$
1.16	$2 + 4$	$4.8_{-2,0}$	1.4	$f = t^{-2\alpha} v(F - G/t)$	$J_1 = u/v + 1/t$
				$g = t^{-2\alpha} vG$	$J_2 = vt^{1-\alpha}$
1.17	$\pm 2 + \gamma 6 +$	$3.13_{0,0}$	1.5	$f = t^{-2\alpha}(uF - vG)$	$J_1 = \phi \mp 1/t$
	$+4 - 5$			$g = t^{-2\alpha}(vF + uG)$	$J_2 = rt^{1-\alpha} e^{\pm \gamma/t}$
1.18	$\langle q^u \rangle_u + \langle q^v \rangle_v$		0	$f = u\dfrac{D^\alpha q^u(t)}{q^u(t)} + F$	$J_1 = t$
				$g = u\dfrac{D^\alpha q^v(t)}{q^u(t)} + G$	$J_2 = q^v(t) - vq^u(t)$
1.19	$3 + 6 + \langle q^v \rangle_v$		1.1_1	$f = uF$	$J_1 = t$
				$g = \frac{1}{2} D^\alpha(q^v) \ln \lvert u \rvert + G$	$J_2 = 2v - q^v \ln \lvert u \rvert$
1.20	$4 + \langle q^v \rangle_v$		1.4	$f = u\dfrac{D^\alpha q^v(t)}{q^v(t)} + F + v\dfrac{G}{q^v}$	$J_1 = t$
				$g = \dfrac{D^\alpha q^v(t)}{q^v(t)} v + G$	$J_2 = 2uq^v(t) - v^2$

$k \neq 0, \gamma \geq 0,$ $4.8_{-2,0}$ is a subalgebra 4.8 when $\delta = -2, \beta = 0$.

TABLE 3. Optimal system $\Theta_2(L_6)$

No	Subalgebra	Nor	N_p	f, g	Invariants
2.1	$3, 6$	4.2		$f = uF(t)$	$J_1 = t$
				$g = vG(t)$	
2.2	$4 - 5, 6$	4.3		$f = uF(t) - vG(t)$	$J_1 = t$
				$g = vF(t) + uG(t)$	
2.3	$4, 6$	5.1		$f = uF(t) + vG(t)$	$J_1 = t$
				$g = vF(t)$	
2.4	$3 + \beta 6, 4$	5.1		$f = uF(t) + v^{\frac{\beta+1}{\beta-1}} G(t)$	$J_1 = t$
				$g = vF(t)$	
	$(\beta = 1)$	5.1		$f = uF(t, v)$	$J_1 = t$
				$g = vF(t, v)$	$J_2 = v$
2.5	$1, 2$	6.1	0	$f = t^{-\alpha} u^{1/(1-\alpha)} F(u/v)$	$J_1 = u/v$
				$g = t^{-\alpha} u^{1/(1-\alpha)} G(u/v)$	
2.6	$1, 3 + \gamma 6$	3.8	1.1	$f = t^{-\alpha} uF(v^{1+\gamma} u^{1-\gamma})$	$J_1 = v^{1+\gamma} u^{1-\gamma}$
				$g = t^{-\alpha} vG(v^{1+\gamma} u^{1-\gamma})$	
2.7	$1, 6$	5.3	1.2	$f = t^{-\alpha} uF(u/v)$	$J_1 = u/v$
				$g = t^{-\alpha} uG(u/v)$	
2.8	$1, 4 + 6$	3.10	1.3	$f = t^{-\alpha} v(F + G \ln v)$	$J_1 = ve^{-u/v}$
				$g = t^{-\alpha} vG$	
2.9	$1, 4$	4.6	1.4	$f = t^{-\alpha}(uF(v) + G(v))$	$J_1 = v$
				$g = t^{-\alpha} vF(v)$	
2.10	$1, 4 - 5 + \gamma 6$	3.9	1.5	$f = t^{-\alpha}(uF - vG)$	$J = re^{\gamma \phi}$
				$g = t^{-\alpha}(vF + uG)$	
2.11	$2, \beta 1 + 3 + \gamma 6$	4.2	1.1	$f = t^{-2\alpha}(u/v)^{\alpha\beta/2} uF$	$J_1 = u^{1-\gamma-\beta+\alpha\beta} \cdot$
				$g = t^{-2\alpha}(u/v)^{\alpha\beta/2} vG$	$\cdot v^{1+\gamma+\beta-\alpha\beta}$

No	Subalgebra	Nor	N_p	f, g	Invariants
2.12	$2, \beta 1+6$	6.1	1.2	$f = t^{-\alpha-\frac{\alpha}{\lambda}} u^{\frac{\beta+1}{\lambda}} F$	$J_1 = u/v$
				$g = t^{-\alpha-\frac{\alpha}{\lambda}} v^{\frac{\beta+1}{\lambda}} G$	$\lambda = \beta+1-\alpha\beta$
	$\beta = 1/(\alpha-1)$			$f = 0, \quad g = 0$	
2.13	$2, \beta 1+4+6$	4.4	1.3	$f = t^{-\alpha-\frac{\alpha}{\lambda}} v^{\frac{\beta+1}{\lambda}} (F + uG/v)$	$J_1 = u/v$
				$g = t^{-\alpha-\frac{\alpha}{\lambda}} v^{\frac{\beta+1}{\lambda}} G$	$\lambda = \beta+1-\alpha\beta$
	$\beta = 1/(\alpha-1)$			$f = t^{-2\alpha} e^{\frac{\alpha u}{(\alpha-1)v}} (vF+uG)$	$J_1 = vt^{1-\alpha}$
				$g = t^{-2\alpha} e^{\frac{\alpha u}{(\alpha-1)v}} vG$	
2.14	$2, 4$	5.1	1.4	$f = t^{-2\alpha}(vF+uG)$	$J_1 = vt^{1-\alpha}$
				$g = t^{-2\alpha} vG$	
2.15	$2, 1+4$	4.4	1.4	$f = t^{-2\alpha} e^{\alpha u/v} (vF+uG)$	$J_1 = vt^{1-\alpha} e^{(\alpha-1)u/v}$
				$g = t^{-2\alpha} e^{\alpha u/v} vG$	
2.16	$2, \beta 1+4-5+\gamma 6$	4.3	1.5	$f = t^{-2\alpha} e^{-\alpha\beta\phi}(uF - vG)$	$J = re^{\gamma\phi}$
				$g = t^{-2\alpha} e^{-\alpha\beta\phi}(vF + uG)$	
2.17	$\gamma 1+3, \beta 1+6$	3.8	2.1	$f = t^{-\alpha} uF(t^2 u^{-\gamma-\beta} v^{\gamma-\beta})$	$J_1 = t^2 u^{-\gamma-\beta} v^{\gamma-\beta}$
				$g = t^{-\alpha} vG(t^2 u^{-\gamma-\beta} v^{\gamma-\beta})$	
2.18	$3, \pm 2+6$	$3.12_{0,0}$	2.1	$f = t^{-2\alpha} uF$	$J_1 = uvt^{2-2\alpha} e^{\pm 2/t}$
				$g = t^{-2\alpha} vG$	
2.19	$2+3, \beta 2+6$	$3.12_{0,0}$	2.1	$f = t^{-2\alpha} uF$	$J_1 = u^{\beta+1} v^{\beta-1} t^{2\beta(1-\alpha)} e^{2/t}$
				$g = t^{-2\alpha} vG$	
2.20	$\gamma 1+4-5, \beta 1+6$	3.9	2.2	$f = t^{-\alpha}(uF - vG)$	$J_1 = tr^{-\beta} e^{\gamma\phi}$
				$g = t^{-\alpha}(vF + uG)$	
2.21	$4-5, \pm 2+6$	$3.13_{0,0}$	2.2	$f = t^{-2\alpha}(uF - vG)$	$J_1 = rt^{1-\alpha} e^{1/(\beta t)}$
				$g = t^{-2\alpha}(vF + uG)$	
2.22	$2+4-5, \beta 2+6$	$3.13_{0,0}$	2.2	$f = t^{-2\alpha}(uF - vG)$	$J_1 = r^\beta t^{\beta(1-\alpha)} e^{-\phi+1/t}$
				$g = t^{-2\alpha}(vF + uG)$	
2.23	$4, k1+6$	4.6	2.3	$f = t^{-\alpha}(vF + uG)$	$J_1 = vt^{-1/k}$
				$g = t^{-\alpha} vG$	
2.24	$1+4, \beta 1+6$	3.10	2.3	$f = t^{-\alpha}(vF + uG)$	$J_1 = tv^{-\beta} e^{-u/v}$
				$g = t^{-\alpha} vG$	
2.25	$4, \pm 2+6$	$4.8_{0,0}$	2.3	$f = t^{-2\alpha}(vF + uG)$	$J_1 = vt^{1-\alpha} e^{\pm 1/t}$
				$g = t^{-2\alpha} vG$	
2.26	$2+4, \pm 2+6$	3.15_0	2.3	$f = t^{-2\alpha}(vF + uG)$	$J_1 = vt^{1-\alpha} e^{\pm(1/t+u/v)}$
				$g = t^{-2\alpha} vG$	
2.27	$2+4, 6$	$4.8_{-2,0}$	2.3	$f = t^{-2\alpha}(vF + uG)$	$J_1 = u/v + 1/t$
				$g = t^{-2\alpha} vG$	
2.28	$k1+3+\beta 6, 4$	4.6	2.4	$f = t^{-\alpha}(t^{2/k} vF + uG)$	$J_1 = vt^{(1-\beta)/k}$
				$g = t^{-\alpha} vG$	
2.29	$\pm 2+3+\beta 6, 4$	$4.8_{0,0}$	2.4	$f = t^{-2\alpha}(e^{\mp 2/t} vF + uG)$	
				$g = t^{-2\alpha} vG$	$J_1 = vt^{1-\alpha} e^{\pm(\beta-1)/t}$
2.30	$(-2)1+3+\beta 6,$	3.17	2.4	$f = t^{-\alpha-1} \left(\frac{u}{v} + \frac{1}{t}\right)^{\frac{\beta-3}{2}} \left(\left(\frac{u}{v} + \frac{1}{t}\right) F - G/t\right)$	
	$2+4$			$g = t^{-\alpha-1} \left(\frac{u}{v} + \frac{1}{t}\right)^{\frac{\beta-3}{2}} G$	$J_1 = \frac{t^{2\alpha-2}}{v^2} \left(\frac{u}{v} + \frac{1}{t}\right)^{\beta-3+2\alpha}$
	$k \neq 0, \gamma \geq 0$				

TABLE 4. Optimal system $\Theta_3(L_6)$

No	Subalgebra	Nor	N_p	f, g
3.1	$3,4,6$	5.1		$f = uF(t), \quad g = vF(t)$
3.2	$3,4,5$	6.1		$f = uF(t), \quad g = vF(t)$
3.3	$1,2,3+\gamma 6$	4.2	1.1	$f = t^{-\alpha} u^{\frac{1}{1-\alpha}} C_1 \left(u/v\right)^{\frac{\alpha(\gamma+1)}{2(\alpha-1)}}$
				$g = t^{-\alpha} v^{\frac{1}{1-\alpha}} C_2 \left(u/v\right)^{\frac{\alpha(\gamma-1)}{2(\alpha-1)}}$
3.4	$1,2,6$	6.1	1.2	$f = 0, \quad g = 0$
3.5	$1,2,4+6$	4.4	1.3	$f = t^{-\alpha} v^{\frac{\alpha}{1-\alpha}} e^{\frac{\alpha}{\alpha-1}\frac{u}{v}} (C_2 u + C_1 v)$
				$g = t^{-\alpha} v^{\frac{\alpha}{1-\alpha}} e^{\frac{\alpha}{\alpha-1}\frac{u}{v}} C_2 v$
3.6	$1,2,4$	5.1	1.4	$f = t^{-\alpha} v^{\frac{\alpha}{1-\alpha}} (C_2 u + C_1 v)$
				$g = t^{-\alpha} v^{\frac{\alpha}{1-\alpha}} C_2 v$
3.7	$1,2,4-5+\gamma 6$	4.3	1.5	$f = t^{-\alpha} (re^{\gamma\phi})^{\frac{\alpha}{1-\alpha}} (C_1 u - C_2 v)$
				$g = t^{-\alpha} (re^{\gamma\phi})^{\frac{\alpha}{1-\alpha}} (C_1 v + C_2 u)$
3.8	$1,3,6$	$=$	2.1	$f = C_1 t^{-\alpha} u, \qquad g = C_2 t^{-\alpha} v$
3.9	$1,4-5,6$	$=$	2.2	$f = t^{-\alpha}(C_1 u - C_2 v)$
				$g = t^{-\alpha}(C_2 u + C_1 v)$
3.10	$1,4,6$	4.6	2.3	$f = t^{-\alpha}(C_2 u + C_1 v), \quad g = t^{-\alpha} C_2 v$
3.11	$1,3+\beta 6,4$	4.6	2.4	$f = t^{-\alpha}(C_2 u + C_1 v^{\frac{\beta+1}{\beta-1}}), \quad g = t^{-\alpha} C_2 v$
	$\beta = 1$			$f = uF(v), \quad g = vF(v)$
3.12	$2,\gamma 1+3,\beta 1+6$	4.2	2.1	$f = t^{-\alpha} \left(t^2 u^{-\gamma-\beta} v^{\gamma-\beta}\right)^{-\alpha/(\beta+1-\alpha\beta)/2} C_1 u$
				$g = t^{-\alpha} \left(t^2 u^{-\gamma-\beta} v^{\gamma-\beta}\right)^{-\alpha/(\beta+1-\alpha\beta)/2} C_2 v$
	$\beta = 1/(\alpha-1)$			$f = 0, \quad g = 0$
3.13	$2,\gamma 1+4-5,\beta 1+6$	4.3	2.2	$f = t^{-\alpha} \left(tr^{-\beta} e^{\gamma\phi}\right)^{-\alpha/(\beta+1-\alpha\beta)} (C_1 u - C_2 v)$
				$g = t^{-\alpha} \left(tr^{-\beta} e^{\gamma\phi}\right)^{-\alpha/(\beta+1-\alpha\beta)} (C_1 v + C_2 u)$
	$\beta = 1/(\alpha-1)$			$f = 0, \quad g = 0$
3.14	$2,1+4,\beta 1+6$	4.4	2.3	$f = t^{-2\alpha}(vt^{1-\alpha})^{\alpha\beta/\lambda} e^{\alpha/\lambda \cdot u/v}(C_2 u + C_1 v)$
				$g = t^{-2\alpha}(vt^{1-\alpha})^{\alpha\beta/\lambda} e^{\alpha/\lambda \cdot u/v} C_2 v, \ \lambda = \beta + 1 - \alpha\beta$
	$\beta = 1/(\alpha-1)$			$f = 0, \quad g = 0$
3.15	$2,4,\beta 1+6$	5.1	2.3	$f = t^{-\alpha-\alpha/\lambda} v^{\alpha\beta/\lambda}(C_2 u + C_1 v)$
				$g = t^{-\alpha-\alpha/\lambda} v^{\alpha\beta/\lambda} C_2 v, \quad \lambda = \beta + 1 - \alpha\beta$
	$\beta = 1/(\alpha-1)$			$f = 0, \quad g = 0$
3.16	$2,4,\delta 1+3+\beta 6$	5.1	2.4	$f = t^{-2\alpha}(C_1 u + C_2 v (vt^{1-\alpha})^{2/\lambda})(vt^{1-\alpha})^{\alpha\delta/\lambda}$
				$g = t^{-2\alpha} C_2 v (vt^{1-\alpha})^{\alpha\delta/\lambda}$
	$\beta = 1, \delta = 0$			$f = t^{-2\alpha} uF(vt^{1-\alpha}), \quad g = t^{-2\alpha} vF(vt^{1-\alpha})$
	$\beta = -1 + 2/\alpha, \delta = -2/\alpha$			$f = t^{-2\alpha} vF(vt^{1-\alpha}), \quad g = 0$
	$\beta = 1 + \alpha\delta - \delta$			$f = 0, \quad g = 0$
3.17	$(-2)1+3,2+4,6$	$=$	3.1	$f = t^{-\alpha-1}(v+tu)^{-\alpha}(C_1 v^{\alpha}(v+tu) - C_2 v^{\alpha+1})$
				$g = C_2 t^{-\alpha} v^{\alpha+1}(v+tu)^{-\alpha}$
3.18	$\delta 1+3,4,\beta 1+6$	4.6	3.1	$f = t^{-\alpha} \left(C_1 t^{\frac{2}{\delta+\beta}} v^{\frac{\delta-\beta}{\delta+\beta}} + C_2 u\right), \quad g = t^{-\alpha} C_2 v$
	$\delta + \beta = 0$			$f = t^{-\alpha} uF(tv^{-\beta}), \quad g = t^{-\alpha} vF(tv^{-\beta})$
3.19	$3,4,\pm 2+6$	$4.8_{0,0}$	3.1	$f = t^{-2\alpha} C_1 u + C_2 e^{\pm 2/t}/(vt^2)$
				$g = t^{-2\alpha} C_1 v$
3.20	$\pm 2+3,4,\beta 2+6$	$4.8_{0,0}$	3.1	$f = t^{-2\alpha} \left(C_1 u + C_2 v \left(v^{-\beta} t^{(\alpha-1)\beta} e^{-1/t}\right)^{\frac{2}{\beta\pm 1}}\right)$
				$g = t^{-2\alpha} C_1 v$
	$\beta = \pm 1$			$f = t^{-\alpha} uF(vt^{1-\alpha} e^{\mp 1/t})$
				$g = t^{-\alpha} vF(vt^{1-\alpha} e^{\mp 1/t})$

TABLE 5. Optimal system $\Theta_{4,5,6}(L_6)$

No	Subalgebra	Nor	N_p	f	g
4.1	$3,4,5,6$	6.1		$f=uF(t)$	$g=vF(t)$
4.2	$1,2,3,6$	$=$	2.1	$f=0$	$g=0$
4.3	$1,2,4-5,6$	$=$	2.2	$f=0$	$g=0$
4.4	$1,2,4,6$	5.1	2.3	$f=0$	$g=0$
4.5	$1,2,3+\beta 6,4$	5.1	2.4	$f=0$	$g=0$
	$\beta=1$			$f=Ct^{-\alpha}uv^{\alpha/(1-\alpha)}$	$g=Ct^{-\alpha}v^{1/(1-\alpha)}$
	$\beta=2/\alpha-1$			$f=Ct^{-\alpha}v^{1/(1-\alpha)}$	$g=0$
4.6	$1,3,4,6$	4.6	3.1	$f=Ct^{-\alpha}u$	$g=Ct^{-\alpha}v$
4.7	$1,3,4,5$	5.3	3.2	$f=Ct^{-\alpha}u$	$g=Ct^{-\alpha}v$
4.8	$2,\delta 1+3,4,\beta 1+6$	5.1	3.1	$f=0$	$g=0$
	$\delta=-\beta$			$f=Ct^{-\alpha-\alpha/\lambda}v^{\alpha\beta/\lambda}$	$g=Ct^{-\alpha-\alpha/\lambda}uv^{(\beta+1)/\lambda}$
	$\delta=\beta-2(\beta+1)/\alpha$			$f=Ct^{-\alpha-\alpha/\lambda}v^{(\beta+1)/\lambda}$	$g=0, \quad \lambda=\beta+1-\alpha\beta$
4.9	$2,3,4,5$	6.1	3.2	$f=Ct^{-2\alpha}u$	$g=Ct^{-2\alpha}v$
4.10	$k1+6,3,4,5$	5.3	4.1	$f=Ct^{-\alpha}u$	$g=Ct^{-\alpha}v$
4.11	$\pm 2+6,3,4,5$	5.4_0	4.1	$f=Ct^{-2\alpha}u$	$g=Ct^{-2\alpha}v$
5.1	$1,2,3,4,6$	$=$	3.1	$f=0$	$g=0$
5.2	$1,2,3,4,5$	6.1	3.2	$f=0$	$g=0$
5.3	$1,3,4,5,6$	$=$	4.1	$f=Ct^{-\alpha}u$	$g=Ct^{-\alpha}v$
5.4	$\beta 1+6,2,3,4,5$	6.1	4.1	$f=0$	$g=0$
	$(\beta=0)$			$f=Ct^{-2\alpha}u$	$g=Ct^{-2\alpha}v$
6.1	$1,2,3,4,5,6$	$=$	4.1	$f=0$	$g=0$

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