ISSN 2304-0122 Ufa Mathematical Journal. Volume 4. 1 (2012). Pp. 58-64.

UDC 517.5

ON THE DISTRIBUTION OF INDICATORS OF UNCONDITIONAL EXPONENTIAL BASES IN SPACES WITH A POWER WEIGHT

K.P. ISAEV, K.V. TRUNOV

Abstract. In the present paper we consider the existence of unconditional exponential bases in a space of locally integrable functions on a bounded interval of the real number line I satisfying

$$\|f\| := \sqrt{\int_I |f(t)|^2 e^{-2h(t)}\, dt} < \infty,$$

where $h(t)$ is a convex function on this interval. The lower estimate was obtained for the frequency of indicators of unconditional bases of exponentials when $I = (-1;1)$, $h(t) = -\alpha \ln(1-|t|)$, $\alpha > 0$.

Keywords: series of exponents, unconditional bases, Riesz bases, power weights, Hilbert space.

1. Introduction

Let I be a limited interval of a real axis, $h(t)$ —a convex function on this interval and $L^2(I,h)$ — a space of locally integrated functions on I, satisfying the condition

$$||f|| := \sqrt{\int_I |f(t)|^2 e^{-2h(t)}\, dt} < \infty.$$

It is the Hilbert space with a scalar product

$$(f,g) = \int_I f(t)\overline{g}(t) e^{-2h(t)}\, dt.$$

The systems of element $\{e_k,\ k = 1,2,...\}$ in the Hilbert space is called an unconditional base (see [2]), if it is total and there are numbers $c, C > 0$, such that for any group of numbers $c_1, c_2, ..., c_n$ the following correlation holds true

$$c\sum_{k=1}^{n} |c_k|^2 ||e_k||^2 \le ||\sum_{k=1}^{n} c_k e_k||^2 \le C\sum_{k=1}^{n} |c_k|^2 ||e_k||^2.$$

It is known (see [3],[4]), that if the system $\{e_k,\ k = 1,2,...\}$ is an unconditional base, then any element of the space H can be only presented in the form of the following row

$$x = \sum_{k=1}^{\infty} x_k e_k,$$

and

$$c\sum_{k=1}^{\infty} |x_k|^2 ||e_k||^2 \le ||x||^2 \le C\sum_{k=1}^{\infty} |x_k|^2 ||e_k||^2.$$

In the paper [12] there was introduced the following characteristic for continuous functions on the plane u, measuring deviation of the given function from harmonic functions. For the continuous

© Isaev K.P., Trunov K.V. 2012.

Submitted on 20 December 2011.

The work is supported by RFBR (grant 10-01-00233-a).

function u, for $z \in \mathbb{C}$ and the positive number p we will define the supremum of all such $r > 0$ by $\tau(u, z, p)$, when the following condition holds true

$$\inf\{ \sup_{w \in B(z,r)} |u(w) - h(w)|,\ h \text{ harmonic in } B(z, r)\} \leq p.$$

Let us define the circle of the radius r in the point z by $B(z, r)$. It results right from the definition, that if $\tau(u, z_0, p) = \infty$ for some point z_0, then $\tau(u, z, p) \equiv \infty$.

It results from Lemma 1.1 in paper [6], that in the case, when u is a continuous subharmonic function, the value $\tau = \tau(u, \lambda, p)$ can be defined by the condition: if $H(z)$ is the lowest harmonic majorant of the function u in the circle $B(\lambda, \tau)$, then

$$\max_{z \in \overline{B}(\lambda,\tau)} (H(z) - u(z)) = 2p. \tag{1}$$

The following theorem (Theorem 2.1) was proved in paper [5]

Theorem 1. *If the system $\{e^{z_j t}\}_{j=1}^{\infty}$ is an unconditional base in the space $L_2(I, h)$, then there is an entire function L with simple zeros in the points z_j, $j = 1, 2, ...$, for which the following correlation holds true*

$$\frac{1}{P} K(z) \leq \sum_{j=1}^{\infty} \frac{|L(z)|^2 K(z_j)}{|L'(z_j)|^2 |z - z_j|^2} \leq P K(z),\ z \in \mathbb{C}, \tag{2}$$

where P is some positive constant and $K(z) = ||e^{zt}||^2$.

The function $\ln K(z)$ is subharmonic and continuous on all the plane.

In the continuation of the paper we will define the function $\tau(\ln K(w), z, \ln(5P))$ by $\tau(z)$, where P is a constant from the correlation (2). Hence,

$$\inf_h \{ \sup_{z \in \overline{B}(\lambda,\tau(\lambda))} |\ln K(z) - h(z)|,\ \text{h is harmonic in } B(\lambda, \tau(\lambda))\} = \ln(5P),$$

The following theorem was proved in [7] (see Theorem 3, Theorem 4 and its corollary).

Theorem 2. *Let the system $\{\exp(tz_i),\ i = 1, 2, ...\}$, make an unconditional base in the space $L_2(I, h)$. Then*

1) in any circle $B(z, 2\tau(z))$ there is at least one index z_i.

2) suppose $b = \frac{1}{20P^{\frac{3}{2}}}$. Then for any i, j, $i \neq j$, the following inequality holds true

$$|z_i - z_j| \geq 2b \max(\tau(z_i), \tau(z_j)).$$

The first statement of this theorem limits the frequency of indexes z_k below, and the second — above. On the basis of these multidirectional estimates Theorem 5 was proved in paper [7], and being applied to the situation considered in the paper, it can be formulated the following way.

Theorem 3. *Let $h(t)$ be a convex function on the interval $I = (-1, 1)$ and*

$$\widetilde{h}(x) = \sup_{t \in I} (xt - h(t))$$

be a function, conjugate to it by Jung. Let us assume, that $\widetilde{h} \in C^2(|x| > const)$ and for any positive number c the function $s(x) = \frac{1}{\sqrt{\widetilde{h}''(x)}}$ satisfies the condition

$$\left(\min_{y \in B(x, cs(x))} \widetilde{h}''(y) \right) \left(\max_{y \in B(x, cs(x))} \widetilde{h}''(y) \right)^{-1} \asymp 1,\ |x| \longrightarrow \infty. \tag{3}$$

Then there are no unconditional bases from exponents in the space $L_2(I, h)$.

The estimate of the function growth results from condition (3)

$$\lim_{|x| \longrightarrow \infty} \frac{|x| - \widetilde{h}(x)}{\ln |x|} = +\infty,$$

which is equivalent to the correlation

$$\lim_{|t| \longrightarrow 1} \frac{h(t)}{-\ln(1 - |t|)} = +\infty,$$

or for any $\alpha > 0$

$$(1-|t|)^\alpha = O(e^{h(t)}),\ |t| \longrightarrow 1.$$

In this paper we consider a problem of unconditional bases from exponents in spaces with not more than power weights, i.e. according to the condition, that for some $\alpha > 0$

$$e^{h(t)} = O((1-|t|)^\alpha),\ |t| \longrightarrow 1.$$

As model spaces we will consider the spaces $L_2(I, h)$ when $I = (-1; 1)$, $h(t) = -\alpha \ln(1-|t|)$ for $\alpha > 0$, which we will define by $L_2(\alpha)$.

We are going to prove the following, more precise estimate of unconditional bases frequency below.

Theorem 4. *Let the system $\{e^{z_k t}\}$ make an unconditional base in the space $L_2(\alpha)$. Then there are numbers $\delta_1 = \delta_1(\alpha) \in (0, 1)$ and $\delta_2 = \delta_2(\alpha) > 0$, $M = M(\alpha) > 0$, such that in case of sufficiently large $|x_0|$ for any y_0 in every rectangle $Q = \{z = x + iy :\ \delta_1 x_0 \le x \le \delta_2 x_0,\ |y - y_0| \le M|x_0|\}$ and $-Q$ there is at least one index z_k.*

The fact, that this estimate is more precise as to p.1 of Theorem 2, results from the awareness, that the value $\tau(z)$ in these spaces is comparable with $|Re\, z|$. When $\alpha > \frac{1}{2}$, the statement of this theorem is proved in paper [13] some other way.

2. Preparatory statements

The system of exponents $\{e^{\lambda t}\}$, $\lambda \in \mathbb{C}$ is total in the space $L_2(I, h)$, therefore, the transform of Fourier-Laplace functionals $L : S \longrightarrow \widehat{S}(\lambda)$, defined by the formula

$$\widehat{S}(\lambda) = S(e^{\lambda t}),\ \lambda \in \mathbb{C},$$

sets mutually single-valued correlation between the conjugated space $L_2^*(I, h)$ and some linear manifold of entire functions $\widehat{L}_2(I, h)$. In this linear manifold we will consider an induced structure of the Hilbert space. Namely, if functionals $S_1, S_2 \in L_2^*(I, h)$ are generated by the functions $f_1, f_2 \in L_2(I, h)$, then we suppose

$$(\widehat{S}_1(\lambda), \widehat{S}_2(\lambda))_{\widehat{L}_2^*(I,h)} = (f_1, f_2)_{L_2(I,h)}.$$

It is easy to assure, that the function

$$K(\lambda, z) = \int_I e^{\lambda t + \bar{z} t - 2h(t)} dt,\ \lambda, z \in \mathbb{C}$$

is a reproducing kernel in the space $\widehat{L}_2(I, h)$, i.e.

$$(F(\lambda), K(\lambda, z)) = F(z), F \in \widehat{L}_2(I, h).$$

It was proved in paper [14], that in the space $\widehat{L}_2(I, h)$ the following equivalent norm can be introduced

$$||F||^2 = \int_{\mathbb{R}} \int_{\mathbb{R}} |F(x+iy)|^2 e^{-2\widetilde{h}(x)} \rho_{\widetilde{h}}(x) d\widetilde{h}'(x) dy, \tag{4}$$

where

$$\widetilde{h}(x) = \sup_{t \in I}(xt - h(t)),\ x \in \mathbb{R},$$

is conjugated by Jung to the function $h(t)$, and the number $\rho = \rho_{\widetilde{h}}(x)$ is defined as a supremum for all $t > 0$, for which

$$\int_{x-t}^{x+t} |\widetilde{h}'_+(y) - \widetilde{h}'_+(x)| dy \le 1.$$

It is shown in paper [5], that the norm of the space $\widehat{L}_2(I, h)$ can be also presented as

$$||F||^2 = \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|F(x+iy)|^2}{K(x)} d\widetilde{h}'(x) dy, \tag{5}$$

where $K(z) = K(z, z)$.

Whereas the Fourier-Laplace transform sets an isomorphism of the space $L_2^*(I, h)$ and $\widehat{L}_2(I, h)$, then the unconditional base of the exponent system $\{e^{t z_k}\}$ in the space $L_2(I, h)$ is equivalent to the statement, that the set of indexes $\{z_k\}$ is the set of uniqueness for the space $\widehat{L}_2(I,h)$, and for any function $F \in \widehat{L}_2(I,h)$ the following correlation holds true

$$\frac{1}{P}\sum_{k=1}^{\infty}\frac{|F(z_k)|^2}{K(z_k)} \leq ||F||^2 \leq P\sum_{k=1}^{\infty}\frac{|F(z_k)|^2}{K(z_k)}, \tag{6}$$

where P is some positive constant.

Let us calculate the introduced above characteristics for the space $L_2(\alpha)$.

Lemma 1. *If*

$$K_\alpha(z) = ||e^{zt}||^2_{L_2(\alpha)} = K_\alpha(z,z) = \int_{-1}^{1} e^{2Rezt}(1-|t|)^{2\alpha}dt,$$
$$h_\alpha(t) = -\alpha\ln(1-|t|),\ \rho_\alpha(x) = \rho_{\widetilde{h}_\alpha}(x),\ \tau_\alpha(z,p) = (\ln K_\alpha(w), z, p),$$

then

$$\widetilde{h}_\alpha(x) = |x| - \alpha\ln|x| + a_\alpha,\ |x| \geq X(\alpha),$$
$$\tau_\alpha(z,p) \asymp |Rez|+1,\ |Rez| \longrightarrow \infty,\ \rho_\alpha(x) = \sqrt{1-e^{-\frac{1}{2\alpha+1}}x},\ x > X(\alpha),$$
$$\ln K_\alpha(x) = 2|x| - (2\alpha+1)\ln|x| + b_\alpha + o(1),\ |x| \longrightarrow \infty,$$

where

$$b_\alpha = \ln\frac{1}{2^{2\alpha+1}}\int_0^\infty e^{-y}y^{2\alpha}dy.$$

Whereas the functions $\widetilde{h}_\alpha(x), \rho_\alpha(x)$ are positive and contiguous, then, in particular, the following correlations hold true

$$e^{\widetilde{h}_\alpha(x)} \asymp e^{|x|-\alpha\ln(|x|+1)},\ x \in \mathbb{R},$$
$$\widetilde{h}''_\alpha(x) \asymp (|x|+1)^{-2},\ x \in \mathbb{R},$$

$$\rho_\alpha(x) \asymp (|x|+1),\ x \in \mathbb{R},$$
$$K_\alpha(x) \asymp e^{2|x|-(2\alpha+1)\ln(|x|+1)},\ x \in \mathbb{R}.$$

Proof. The function $K(x)$ is even, therefore, we will make calculations for $x > 0$. The asymptotic representation for $\ln K_\alpha(x)$ results from the correlation

$$\int_{-1}^{1} e^{2xt}(1-|t|)^{2\alpha}dt = \int_{-1}^{0} e^{2xt}(1+t)^{2\alpha}dt + e^{2x}\int_0^1 e^{-2x(1-t)}(1-t)^{2\alpha}dt =$$
$$= O(1) + \frac{e^{2x}}{(2x)^{2\alpha+1}}\int_0^{2x} e^{-y}y^{2\alpha}dy = \frac{e^{2x+b_\alpha}}{x^{2\alpha+1}}(1+o(1)),\ x \to \infty.$$

The function $\widetilde{h}_\alpha(x)$ for large x is calculated by the definition. Expressions for τ_α, ρ_α were calculated in paper [13]. □

Lemma 2. *For $\delta_1, \delta_2, M > 0$ and $x_0 \in \mathbb{R}_+$ via $Q(x_0,\delta_1,\delta_2,M)$ we will define the rectangle*

$$Q = \{x+iy:\ \delta_1 x_0 \leq x \leq \delta_2 x_0,\ |y| \leq M x_0\}.$$

Then for any $\varepsilon > 0$ we can find quite a low number of $\delta_1 = \delta_1(\varepsilon) > 0$, and quite large numbers of $\delta_2 = \delta_2(\varepsilon) > 0$, $M = M(\delta_1,\delta_2,\varepsilon) > 0$ so, that when $x_0 > X(\delta_1,\delta_2,\varepsilon)$ the following correlation will hold true

$$\int_{\mathbb{C}\setminus Q(x_0,\delta_1,\delta_2,M)} |K_\alpha(x+iy,x_0)|^2 e^{-2\widetilde{h}_\alpha(x)}\rho_\alpha(x)d\widetilde{h}'_\alpha(x)dy \leq \varepsilon K_\alpha(x_0,x_0).$$

Proof. Let us take positive x_0 and present the X-axis in the form of gaps integration

$$I_1 = \{x:\ x > \delta_2 x_0\},\ I_2 = \{x:\ -\delta_1 x_0 \leq x < \delta_1 x_0\},$$
$$I_3 = \{x:\ -2x_0 < x < -\delta_1 x_0\},\ I_4 = \{x:\ x \leq -2x_0\},$$
$$I = \{x:\ \delta_1 x_0 \leq x \leq \delta_2 x_0\}.$$

Then the supplement to the rectangle $Q(x_0,\delta_1,\delta_2,M)$ will be expanded into two half-planes $Q_1 = I_1 \times \mathbb{R}$ and $Q_4 = I_4 \times \mathbb{R}$, two vertical strips $Q_2 = I_2 \times \mathbb{R}$ and $Q_3 = I_3 \times \mathbb{R}$ and two semi-strips

$Q_+ = I\times\{y>Mx_0\}, Q_- = I\times\{y<-Mx_0\}$. Note, that the function $K_\alpha(x+iy,x_0)$ is the Fourier function transform $e^{(x+x_0)t-2h_\alpha(t)}$ with fixed x and, according to the Plancherel theorem

$$\int_{-\infty}^{\infty}|K_\alpha(x+iy,x_0)|^2dy = 2\pi\int_{-1}^{1}e^{2(x+x_0)t-4h_\alpha(t)}dt.$$

As it was proved in paper [15], for any convex function $u(t)$ the following correlation holds true

$$\int_{-1}^{1}e^{yt-u(t)}dt \asymp \frac{e^{\widetilde{u}(y)}}{\rho_{\widetilde{u}}(y)},\ y\in\mathbb{R}.$$

Hence, according to Lemma 1 we have

$$\int_{-\infty}^{\infty}|K_\alpha(x+iy,x_0)|^2dy \asymp \frac{e^{4\widetilde{h}_\alpha(\frac{x+x_0}{2})}}{\rho_\alpha(\frac{x+x_0}{2})} \asymp \frac{e^{4\widetilde{h}_\alpha(\frac{x+x_0}{2})}}{(|x+x_0|+1)},$$

therefore,

$$\int_{-\infty}^{\infty}|K_\alpha(x+iy,x_0)|^2e^{-2\widetilde{h}_\alpha(x)}\rho_\alpha(x)\,dy \prec \frac{x_0(|x|+1)e^{4\widetilde{h}_\alpha(\frac{x+x_0}{2})-2\widetilde{h}_\alpha(x)-2\widetilde{h}_\alpha(x_0)}}{|x+x_0|+1}K_\alpha(x_0).$$

In the half-plane Q_1 we obtain the estimate when $\delta_2\longrightarrow\infty$ is uniform on x_0

$$\int_{Q_1}|K_\alpha(x+iy,x_0)|^2e^{-2\widetilde{h}_\alpha(x)}\rho_\alpha(x)dyd\widetilde{h}'_\alpha(x) \prec K_\alpha(x_0)\int_{\delta_2}^{\infty}\frac{dy}{(y+1)y} = o(K_\alpha(x_0)).$$

If $\delta_1\le\frac12$ and $|x|\le\delta_1x_0$, then we have

$$\frac{x_0(|x|+1)e^{4\widetilde{h}_\alpha(\frac{x+x_0}{2})-2\widetilde{h}_\alpha(x)-2\widetilde{h}_\alpha(x_0)}}{|x+x_0|+1} \prec \frac{x_0(x_0+1)^{2\alpha}(|x|+1)^{2\alpha+1}}{(|x+x_0|+1)^{4\alpha+1}} \prec \frac{(|x|+1)^{2\alpha+1}}{(|x+x_0|+1)^{2\alpha}},$$

therefore,in the strip Q_2 we deal with $\delta_1\longrightarrow 0$, which is uniform on $x_0>1$

$$\int_{Q_2}|K_\alpha(x+iy,x_0)|^2e^{-2\widetilde{h}_\alpha(x)}\rho_\alpha(x)dyd\widetilde{h}'_\alpha(x) \prec \frac{K_\alpha(x_0)}{x_0^{2\alpha}}\int_{-\delta_1x_0}^{\delta_1x_0}(|x|+1)^{2\alpha-1}dx = o(K_\alpha(x_0)).$$

For the fixed $\delta_1\le\frac12$ for $-2x_0\le x\le-\delta_1x_0$ we obtain

$$\frac{x_0(|x|+1)e^{4\widetilde{h}_\alpha(\frac{x+x_0}{2})-2\widetilde{h}_\alpha(x)-2\widetilde{h}_\alpha(x_0)}}{|x+x_0|+1} \prec e^{-2\delta_1x_0}(x_0+1)^{4\alpha+2},$$

hence, in the strip Q_3 with $x_0\longrightarrow\infty$ we have

$$\int_{Q_3}|K_\alpha(x+iy,x_0)|^2e^{-2\widetilde{h}_\alpha(x)}\rho_\alpha(x)dyd\widetilde{h}'_\alpha(x) \prec K_\alpha(x_0)e^{-2\delta_1x_0}(x_0+1)^{4\alpha+3} = o(K_\alpha(x_0)).$$

With the fixed $\delta_1\le\frac12$ for $x\le-2x_0$ we have

$$\frac{x_0(|x|+1)e^{4\widetilde{h}_\alpha(\frac{x+x_0}{2})-2\widetilde{h}_\alpha(x)-2\widetilde{h}_\alpha(x_0)}}{|x+x_0|+1} \prec e^{-4x_0}(x_0+1)^{2\alpha+1}(|x|+1)^{-2\alpha},$$

therefore, in the strip Q_4 with $x_0\longrightarrow\infty$ the following estimate holds true

$$\int_{Q_4}|K_\alpha(x+iy,x_0)|^2e^{-2\widetilde{h}_\alpha(x)}\rho_\alpha(x)dyd\widetilde{h}'_\alpha(x) \prec K_\alpha(x_0)e^{-4x_0}(x_0+1)^{2\alpha+1}\int_{2x_0}^{+\infty}(|x|+1)^{-2\alpha-2}\,dx = o(K_\alpha(x_0)).$$

If we choose δ_1 and δ_2 the right way, we can proceed to estimates of integrals on semi-strips $Q_\pm$. For this we will apply the following representation for the reproducing kernel with $z=x+iy\ne w=x_0+iy_0$

$$K_\alpha(z,w) = \int_{-1}^{1}e^{zt+\overline{w}t-2h_\alpha(t)}dt = \int_{-1}^{1}e^{2(xt-h_\alpha(t))}d\frac{e^{(\overline{w}-\overline{z})t}}{\overline{w}-\overline{z}} =$$

$$= \frac{2}{\overline{w}-\overline{z}}\int_{-1}^{1}e^{zt+\overline{w}t-2h_\alpha(t)}(x-h'_\alpha(t))dt.$$

According to the Cauchy-Bunyakovsky inequality, we obtain

$$|K_\alpha(z,w)|^2 \le \frac{4}{|w-z|^2}\int_{-1}^{1}e^{2xt-2h_\alpha(t)}|x-h'_\alpha(t)|dt\cdot\int_{-1}^{1}e^{2x_0t-2h_\alpha(t)}|x_0-h'_\alpha(t)|dt.$$

The function $h'_\alpha(t)$ we change the sign only in the point $t=0$, therefore

$$\int_{-1}^{1} e^{2xt-2h_\alpha(t)}|x-h'_\alpha(t)|dt \le \int_{-1}^{1} e^{2xt-2h_\alpha(t)}|x|dt - \frac{1}{2}\int_0^1 e^{2xt}de^{-2h_\alpha(t)} \le$$

$$\le |x|K_\alpha(x)+1+xK_\alpha(x) \le 3K_\alpha(x)|x|,$$

when $K_\alpha(x)\ge 1$. It results from the latter two estimates, that

$$|K_\alpha(z,w)|^2 \le \frac{36|x||x_0|}{|w-z|^2}K_\alpha(x)K_\alpha(x_0).$$

Hence, from the estimates in Lemma 1 we obtain

$$\int_{Q_+} |K_\alpha(x+iy,x_0)|^2 e^{-\widetilde{h}_\alpha(x)}\rho_\alpha dy d\widetilde{h}_\alpha(x) \le$$

$$\le 36K_\alpha(x_0)x_0 \int_I\int_{Mx_0}^{\infty} \frac{x}{((x-x_0)^2+y^2)(x+1)^2}dydx \prec \frac{1}{M}K_\alpha(x_0).$$

Therefore, choosing the number M large enough, we can consider the integral on the strip Q_+ sufficiently low. The same way we can estimate an integral on the semi-strip Q_-. □

Lemma 2 has been proved. It is easy to see, that the following lemma has been proved the same way.

Lemma 3. *For $\delta_1,\delta_2,M>0$ and $x_0\in\mathbb{R}_+$, $y_0\in\mathbb{R}$ via $Q(x_0,y_0,\delta_1,\delta_2,M)$ we will define the rectangle*

$$Q=\{x+iy:\ \delta_1x_0\le x\le\delta_2x_0,\ |y-y_0|\le Mx_0\}.$$

Then for any $\varepsilon>0$ we can find quite a low number of $\delta_1=\delta_1(\varepsilon)>0$, and quite large numbers of $\delta_2=\delta_2(\varepsilon)>0$, $M=M(\delta_1,\delta_2,\varepsilon)>0$ so, that when $x_0>X(\delta_1,\delta_2,\varepsilon)$, the following correlation will hold true

$$\int_{\mathbb{C}\setminus Q(x_0,y_0,\delta_1,\delta_2,M)} |K_\alpha(x+iy,x_0+iy_0)|^2 e^{-2\widetilde{h}_\alpha(x)}\rho_\alpha(x)d\widetilde{h}'_\alpha(x)dy \le \varepsilon K_\alpha(x_0,x_0).$$

3. Low estimate of frequency indexes. Proof of Theorem 4

Let the system $\{\exp(z_jt)\}$ make an unconditional base in the space $L_2(\alpha)$. Then, as it was already pointed out in section 2, the system $K_\alpha(z,z_j)$ makes an unconditional base in the space $\widehat{L}_2(\alpha)$, i.e. for some P correlation (6) holds true. In this correlation we can define the norm by formula (5). Let us take sufficiently low positive ε, the degree of infinitesimality we will define later. By this number ε, according to Lemma 3, we will find numbers $\delta_1\in(0,\frac{1}{2})$, δ_2 and M, for which the statement of Lemma 3 holds true.

Assume, that for some $x_0\in\mathbb{R}_+, y_0$ in the rectangle $Q:=Q(x_0,y_0,\delta_1+\frac{1}{4},\delta_2+\frac{1}{4},M+\frac{1}{4})$ there are no indexes z_j.

According to Lemma 1, values $\tau_\alpha(z)$ and $\rho_\alpha(z)$ are comparable with $|\mathrm{Re}z|+1$. Considering item 1 of Theorem 2, we can assert, that there is a number $\sigma>0$, such that circles $B_j=B(z_j,\sigma(|\mathrm{Re}z_j|+1))$ do not cross pairwise and lie outside the rectangle Q. According to the definition of the value τ_α in every circle B_j there is a harmonic function H_j, which stands by from the function $\ln K_\alpha$ for not more than $\ln(5P)$. According to the properties of subharmonic functions, for any entire function F the following inequality holds true

$$|F(z_j)|^2e^{-2H_j(z_j)} \le \frac{1}{\pi\sigma^2(|\mathrm{Re}z_j|+1)^2}\int_{B_j}|F(z)|^2e^{-2H_j(z)}dm(z),$$

where $dm(z)$ is the Lebesgue planar measure. Whereas in the circle B_j $|\mathrm{Re}z_j|+1\asymp|\mathrm{Re}z|+1$, then

$$|F(z_j)|^2e^{-2H_j(z_j)} \prec \int_{B_j}\frac{|F(x+iy)|^2}{K(x)(|x|+1)^2}dxdy \prec \int_{B_j}\frac{|F(x+iy)|^2}{K(x)}d\widetilde{h}'(x)dy.$$

Let us summarize these estimates by all j:

$$\sum_j \frac{|F(z_j)|^2}{K(z_j)} \prec \int_{\mathbb{C}\setminus Q}\frac{|F(x+iy)|^2}{K(x)}d\widetilde{h}'(x)dy.$$

We will apply this estimate to the function $F(z) = K_\alpha(z, x_0 + iy_0)$. According to Lemma 3, we will obtain, that due to the choice of the rectangle size Q the following inequality holds true

$$\sum_j \frac{|F(z_j)|^2}{K(z_j)} \leq \varepsilon K_\alpha(x_0, x_0) = \varepsilon ||K_\alpha(z, x_0 + iy_0)||^2.$$

If $\varepsilon < \frac{1}{P}$, then it contradicts condition (6).

Theorem 4 has been proved.

BIBLIOGRAPHY

1. Bari N.K. *On bases in the Hilbert space* // DAS. 1946. V. 54. P. 383–386. In Russian.
2. Nikolsky N.K., Pavlov B.S., Hrushev S.V. *Unconditional bases of exponents and reproducing kernels. I.* // Preprint LOMI. P. 8–80. In Russian.
3. Gohberg I.Ts., Krein M.G. *Introductiion in the theory of linear non-selfconjugated operators in the Hilbert space.* Moscow, Nauka. 1965. 448 p. In Russian.
4. Nikolsky N.K. *Lectures on operator shift.* Moscow, Nauka. 1980. In Russian.
5. Bashmakov R.A. *Systems of exponents in power Hilbert spaces on R* // Candidate's thesis. Institute of Mathematica with computer centre RAS. 2006. In Russian.
6. Yulmukhametov R.S. *Asymptotic appriximation of subharmonic functions* // Sib. math. journal. 1985. V. 26. No 4. P. 159–175. In Russian.
7. Isaev K.P., Yulmukhametov R.S. *On unconditional bases of exponents in the Hilbert spaces* // Ufa math.journal. V. 3, No 1. 2011. P. 3–15. In Russian.
8. Levin B.Ya.*Interpolation by entire exponential functions* // Math. physics and funct. analysis. PTIST AS USSR. 1969. Issue. 1. P. 136–146. In Russian.
9. Levin B.Ya., Lubarsky U.I. *Interpolation by entire functions of special classes and exponents decomposition associated with it* // Izv. AS USSR. Ser. math. 1975. V. 39. No 3. P. 657–702. In Russian.
10. Isaev K.P. *Riesz bases of exponents in Bergman spaces on convex polygons* // Ufa math.journal. V. 2, No 1. 2010. P. 1-–86. In Russian.
11. A. Borichev, Yu.Lyubarskii *Riesz bases of reproducing kernels in Fock type spaces* // Journal of the Institute of Mathematics of Jussieu 9 (2010). P. 449–461.
12. Bashmakov R.A., Putintseva A.A., Yulmukhametov R.S. *Entire functions of a sine type and their application* // Algebra and analysis. 22:5. 2010. P. 49-–68. In Russian.
13. K.P. Isaev, R.S. Yulmukhametov *Lower estimate of frequency of indicators of unconditional exponential bases in spaces with a power weight* // Eurasian Mathematical Journal [submitted for printing].
14. Lutsenko B.I., Yulmukhametov R.S. *Generalization of the Paley–Wiener theorem on power spaces* //Math. notes. 1990. V. 48, No 5. P. 80-–87. In Russian.
15. Napalkov V.V., Bashmakov R.A., Yulmukhametov R.S. *Asymptotic behaviour of Laplace integrals and geometric characteristics of convex functions* // DAS. 2007. V. 413, No 1. P. 20–22. In Russian.

Konstantin Petrovich Isaev,
Institute of mathematica with computer centre RAS,
112, Chernyshevsky str.,
Ufa, Russia, 450008
E-mail: `orbit81@list.ru`

Trunov Kirill Vladimirovich,
Institute of mathematica with computer centre RAS,
112, Chernyshevsky str.,
Ufa, Russia, 450008
E-mail: `trounovkv@mail.ru`