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ITERATIONS OF THE ENTIRE TRANSCENDENTAL FUNCTIONS WITH REGULAR BEHAVIOR OF THE MINIMUM OF THE MODULUS

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Abstract. In the paper the Fatou set of an entire transcendental function is considered, i.e. the largest open set of the complex plane, where the family of iterations of the given function forms a normal family according to Montel. The entire function is assumed to be of an infinite lower order. The pair of conditions on the indices of the series providing that every component of the Fatou set is bounded is found. This pair of conditions is optimal in a certain sense and is stronger than the Fejér gap condition. The result under stronger sufficient conditions was proved earlier by Yu. Wang and Zh. Rakhmatullina.

Keywords: entire functions, Fejér gaps, iterations of functions, Fatou set

1. Definitions and basic properties

Let f be a nonlinear entire function of the complex variable z. Let us define the natural iterations of the function f by:

$$f^0(z) = z, \;\; f^1(z) = f(z), \; \dots, \; f^{k+1}(z) = f(f^k(z)) \quad (k = 1, 2, \dots). \tag{1}$$

Definition 1. *The class N of analytical functions in the domain D of the complex plane $\mathbb{C}$ of a function is called normal in D (according to Montel), if the subsequence $\{f_{k_p}\}$ can be singled out of any sequence $\{f_k\}$ of functions of N with the property, when either $\{f_{k_p}(z)\}$, or $\left\{\dfrac{1}{f_{k_p}(z)}\right\}$ converge everywhere in D, and uniformly on every compact subset of the domain D* [1]*. It is said in this case, that the sequence $\{f_{k_p}\}$ converges locally uniformly in D* [2].

Definition 2. *The Fatou set $\mathcal{F}(f)$ (or the normality set) of the function $f(z)$ is the largest open set of the complex plane where the family of iterations $\{f^k\}$ defined by the formula* (1) *forms a normal family (according to Montel). The complement of the Fatou set is called the Julia set $\mathcal{J}(f) = \mathbb{C} \backslash \mathcal{F}(f)$.*

Fatou and Julia sets of the entire function f possess the following properties:

Property 1. *The Fatou set of an entire function is open and the Julia set is closed.*

Property 2. *The sets $\mathcal{F}(f)$ and $\mathcal{J}(f)$ are completely invariant under f (i.e. each of these sets coincides both with its image, and with the complete counter image)* [3], [4]*:*

$$1^\circ.\; f^{-1}(\mathcal{F}(f)) = f(\mathcal{F}(f)) = \mathcal{F}(f); \qquad 2^\circ.\; f^{-1}(\mathcal{J}(f)) = f(\mathcal{J}(f)) = \mathcal{J}(f).$$

Property 3. *For any $k > 0$ the Fatou (Julia) set of the function f^k coincides with the the Fatou (Julia) set of the function f itself* [3], [4]*:*

$$3^\circ.\; \mathcal{F}(f^k) = \mathcal{F}(f); \qquad 4^\circ.\; \mathcal{J}(f^k) = \mathcal{J}(f).$$

Let us remind one more definition. *The component K* of the set D is any largest connected subset [5]. The following statement holds true: every set D can be uniquely presented as a finite or infinite unification of its components.

If f is a polynomial of degree at least two, the set $\mathcal{F}(f)$ contains the component $K = \{z\colon f^k(z) \to \infty\}$, which is unbounded and completely invariant [4]. For instance, the Julia set of the function $f(z) = z^2$ is a unit circumference: $\mathcal{J}(f) = \{z\colon |z| = 1\}$; and the Fatou set consists of two components: bounded one $\{z\colon |z| < 1\}$ and unbounded one $\{z\colon |z| > 1\}$.

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If f is a transcendental entire function, the set $\mathcal{J}(f)$ is always unbounded, and the set $\mathcal{F}(f)$ can contain either infinitely many unbounded components, or exactly one component, or none at all [4]. And we have the following

Property 4. *Any unbounded component of the set $\mathcal{F}(f)$ of the entire transcendental function f is simply connected* [6].

2. Review of the results

Fatou began to study the iterations of entire functions in 1926 [7]. Almost 40 years later, I. Baker in his papers [6], [8]–[12] obtained results, which greatly influenced the development of the topic. Baker proved the following

Theorem 1 ([13]). *If for transcendental entire function f there is an unbounded invariant component of the Fatou set, then the growth of f must exceed the order* $1/2$, *minimal type.*

It is shown in [13], that when positive values of parameter a are quite large, the Fatou set $\mathcal{F}(g)$ of the function

$$g(z) = \frac{1}{\sigma}\left(\frac{\sin\sqrt{\sigma z}}{\sqrt{\sigma z}} + \sigma z + a\right), \qquad \sigma > 0,$$

(of the order $\rho = 1/2$ and the type σ) contains an unbounded invariant component. There is another example of this, it is the function $F(z) = \cos\sqrt{\sigma^2 z + \frac{9}{4}\pi^2}$, $0 < \sigma < \sqrt{3\pi}$, of the order $\frac{1}{2}$ and type σ, the Fatou set $\mathcal{F}(F)$ of which also contains an unbounded invariant component [13], [14].

In 1981 Baker raised a question [13] of whether every component of the set $\mathcal{F}(f)$ must be bounded if the entire transcendental function f is of sufficiently small growth. Due to Theorem 1 and the examples given, it is natural to consider the Baker problem in the class of entire transcendental functions of the order $\rho < 1/2$.

Baker himself ... Hinkkanen ... under which the set F(f) does not contain unbounded components in the given class of functions f. ...

Baker himself [13], and later Stallard [15], Anderson and Hinkkanen [16] obtained different sufficient conditions, under which the set $\mathcal{F}(f)$ does not contain unbounded components in the given class of functions f. These conditions are as follows:

1) Baker, 1981, [13]: when $r \to \infty$

$$\ln M(r, f) = O\{(\ln r)^p\} \qquad (1 < p < 3),$$

where $M(r, f) = \max\limits_{|z|=r} |f(z)|$;

2) Stallard, 1993, [15]: there exists $\varepsilon \in (0, 1)$, that when $r \geqslant r_0$

$$\ln\ln M(r, f) < \frac{(\ln r)^{1/2}}{(\ln\ln r)^{\varepsilon}};$$

3) Stallard, 1993, [15]: for the entire function f of the order $\rho < 1/2$

$$\lim_{r\to\infty} \frac{\ln M(2r, f)}{\ln M(r, f)} = c,$$

where $c = c(f)$ is a finite constant, which depends only on f (this theorem is sharp: the function $g(z)$ from the example given above [13] has the order $\rho = 1/2$, Stallard condition is true with the constant $c = \sqrt{2}$ for it (when $\sigma = 1$). However, $\mathcal{F}(f)$ contains an unbounded component);

4) Anderson and Hinkkanen, 1998, [16]: for the entire function f of the order $\rho < 1/2$ there exists $c > 0$, that with $x \geqslant x_0$

$$\frac{\varphi'(x)}{\varphi(x)} \geqslant \frac{1 + c}{x}, \qquad \varphi(x) = \ln M(e^x, f).$$

The research of the class of the entire transcendental functions of the kind

$$f(z) = \sum_{n=1}^{\infty} a_n z^{p_n}, \qquad p_n \in \mathbb{N},\ \ 0 < p_n \uparrow \infty. \tag{2}$$

is of special interest, as existence of gaps in such functions specifies a number of additional properties, allowing to consider the components of the set $\mathcal{F}(f)$ in case of any finite and even infinite increase order.

The research of the Fatou set $\mathcal{F}(f)$ of functions of the kind (2) is closely related to a number of classical problems. During the XX century there has appeared a lot of papers, concerning Picard, Borel and asymptotic values, Julia lines, the problem of connection between maximum and minimum modulus, and also distribution of values of entire functions with different gap conditions. Let us refer to the results of studying the Fatou set of functions of the kind (2).

They say, that the entire function of the form (2) has Fabry gaps, if $n = o(p_n)$ when $n \to \infty$, and has Fejer gaps, if

$$\sum_{n=1}^{\infty} \frac{1}{p_n} < \infty.$$

U. Wang in the paper [17] proved the following theorems:

Theorem 2 ([17]). *Let f be an entire function of the form (2),*

$$\rho_* = \varliminf_{r\to\infty} \frac{\ln\ln M(r,f)}{\ln r} \quad \text{and} \quad \rho = \varlimsup_{r\to\infty} \frac{\ln\ln M(r,f)}{\ln r}$$

being its lower order and order correspondingly. If $0 < \rho_* \leqslant \rho < \infty$, and the function f possesses Fabry gaps, then every component of the set $\mathcal{F}(f)$ is limited.

Theorem 3 ([17]). *Let the entire function f of the form (2) satisfy the condition: there exists such $T_0 > 1$ that*

$$\varliminf_{r\to\infty} \frac{\ln M(r^{T_0}, f)}{\ln M(r,f)} > T_0. \tag{3}$$

If with some $\eta > 0$

$$p_n > n \ln n (\ln\ln n)^{2+\eta} \qquad (n \geqslant n_0), \tag{4}$$

then every component of the set $\mathcal{F}(f)$ is bounded.

Note, that in Theorem 2 the condition (3) does not appear obviously. This theorem requires realization of conditions $0 < \rho_*$ and $\rho < \infty$. However, it is easy to check up, that the left part in the estimate (3) in this case is equal to $+\infty$. However, in Theorem 2 when $T_0 \geqslant q\,\frac{\rho}{\rho_*}$ $(q > 1)$ the condition (3) is fulfilled automatically.

Further, for any entire function f and any $T > 1$ it results from Hadamard three-circle theorem that [18]

$$\varliminf_{r\to\infty} \frac{\ln M(r^T)}{\ln M(r)} \geqslant T. \tag{5}$$

Theorem 3 considers the entire functions of arbitrary growth (the situations $\rho_* = 0$ and $\rho = \infty$ are possible), hence, unlike Theorem 2 it is necessary to postulate the fulfillment of a stronger estimate than (5).

As for the condition (4), using this condition Hayman showed in [19], that for any entire function f of the form (2) when $r \to \infty$ outside some set of zero logarithmic density there holds the asymptotic equality

$$\ln M(r) = (1 + o(1)) \ln m(r). \tag{6}$$

In the proof of Theorem 3 this estimate is used substantially. Thus, the Hayman condition (4) in Theorem 3 is stipulated by the estimate (6).

The condition (4) can be substituted by a weaker one. It is proved in paper [20] that

Theorem 4. *Let f be an entire transcendental function, given by the gap power series (2), for which with some $T_0 > 1$ the estimate (3) holds true. If $n = o(p_n)$ when $n \to \infty$ and*

$$\sum_{n=1}^{\infty} \frac{1}{p_n} \ln \frac{p_n}{n} < \infty, \tag{7}$$

then every component of the set $\mathcal{F}(f)$ is bounded.

The matter is that the equality (6) holds true even under the condition (7) [21]. In all other aspects proofs of Theorems 3 and 4 are almost the same.

The aim of this paper is to show that Theorem 3 holds true in the most common situation, namely, the condition (7) can be significantly weakened. It turns out that the latter condition can be replaced by a pair of optimal conditions, under which the estimate of the type (6) holds true for any function of the kind (2).

The given pair of conditions is necessary and sufficient for estimate of the form (6) and it consists of Fejer condition and some condition for concentration of points p_n in terms

$$\delta_n = \int_1^{p_n} \frac{\mu(p_n;t)}{t}dt,$$

where $\mu(p_n;t)$ is the number of points $p_k \neq p_n$ from the segment $\{x\colon |p_n - x| \leqslant t\}$.

It turns out, the Fejer gap condition is also necessary for every component of the set $\mathcal{F}(f)$ of any entire function f of the kind (2) to be bounded.

To prove the main result we need the following lemma, proved by Baker [13] with the application of Schottky theorem.

Lemma 1 ([13])**.** *Let analytical in the domain D function g of the family G omit the values* 0 *and* 1*. If D_0 is a compact connected subset in D on which $|g(z)| \geqslant 1$ for any $g \in G$, then there exist such constants U, V, depending only on D_0 and D, that for every function $g \in G$, z, z' from D_0 the following estimate holds true*

$$|g(z')| < U|g(z)|^V.$$

We will also need the following theorem from [22] (the assertion is given in the context of power series of the form (2) and in some simplified way).

Theorem 5 ([22])**.** *Let the following conditions hold true:*

$$\textit{1)}\ \sum_{n=1}^{\infty} \frac{1}{p_n} < \infty; \qquad \textit{2)}\ \int_1^{\infty} \frac{c(t)}{t^2}dt < \infty, \tag{8}$$

where

$$c(t) = \max_{p_n \leqslant t} q_n, \qquad q_n = -\ln|q'(p_n)|, \qquad q(z) = \prod_{k=1}^{\infty}\left(1 - \frac{z^2}{p_k^2}\right).$$

Then there exists such set $E \subset [0,\infty)$ of a finite logarithmic measure, that for any circle $C(r) = \{z\colon |z| = r\}$ there exists a modified ¡¡circle¿¿ $C^*(r)$ with the properties:

1. *If* mes e *is the arc length $e \subset C(r)$, then*
$$\lim_{r\to\infty} \frac{\text{mes}\,[C(r)\cap C^*(r)]}{r} = 2\pi;$$
2. $\lim\limits_{r\to\infty} \max\limits_{z\in C^*(r)} \ln\dfrac{r}{|z|} = 0$*;*
3. *when $r \to \infty$ outside E*
$$\ln M(r,f) = (1+o(1))\ln m^*(r),$$
where $m^*(r) = \min\limits_{z\in C^*(r)} |f(z)|$.

For Theorem 5 to hold true for any function f, the conditions (8) are also necessary [23].

The main result of the paper is

Theorem 6. *Let f be an entire transcendental function, given by gap power series* (2)*, for which with some $T_0 > 1$ the estimate* (3) *holds true. If the pair of conditions* (8) *holds true, then every component of the set $\mathcal{F}(f)$ is limited.*

It is shown, that for any entire function f of the form (2), for which

$$\sum_{n=1}^{\infty} \frac{1}{p_n} = \infty,$$

there exists a component of the set $\mathcal{F}(f)$, containing the ray $[0,\infty)$.

3. Proof of Theorem 6

According to the condition (3) with some $T_1 > T_0 > 1$ the following estimate holds true

$$\frac{\ln M(r^{T_0}, f)}{\ln M(r, f)} \geqslant T_1, \qquad r \geqslant x_0. \tag{9}$$

Let $T_0 < T < T_1$, and $q > 1$ be such that $qT < T_1$. Then there exists $\varepsilon > 0$ such that $(1-\varepsilon)T_1 \geqslant qT$.

According to Theorem 5 by the chosen this way $\varepsilon > 0$ there exists the set $E \subset [0, \infty)$ of finite logarithmic measure, that

$$m^*(r) > M(r, f)^{1-\varepsilon} \tag{10}$$

when $r \in [0, \infty) \setminus E$, $m^*(r)$ is minimum of the modulus f of the function on the curve, close to the circle $|z| = r$ (in the sense of Theorem 5).

Further, the function f is a transcendental one, therefore $M(r, f)$ increases faster then any degree of r^N. Let $R_1 > 0$ be such number that

$$M(r, f) > 2r^{qT} \quad \text{with} \quad r \geqslant R_1.$$

Considering this, let us regard the sequence $\{R_n\}$, where $R_{n+1} = M(R_n, f)$ $(n \geqslant 1)$. It is definite, that $R_n \uparrow \infty$ when $n \to \infty$, and $J_n \subset I_n$, where

$$J_n = [R_n^T, \tfrac{1}{2}R_n^{qT}], \qquad I_n = [R_n, R_{n+1}] \qquad (q > 1,\ T > 1).$$

Whereas when $n \to \infty$

$$\ln\text{-mes}\, J_n = \ln \frac{R_n^{qT}}{2R_n^T} = -\ln 2 + (q-1)T \ln R_n \to \infty,$$

and

$$\sum_{n=1}^{\infty} \ln\text{-mes}\,(E \cap J_n) < \infty,$$

then

$$\lim_{n\to\infty} \ln\text{-mes}\,(E \cap J_n) = 0.$$

It means, that when $n \geqslant n_1$ every segment J_n contains a point ρ_n, which does not belong to E. Hence, if we take into account the estimates (9), (10), we will obtain

$$m^*(\rho_n) > M(\rho_n, f)^{1-\varepsilon} \geqslant [M(R_n^T, f)]^{1-\varepsilon} > M(R_n, f)^{(1-\varepsilon)T_1}, \qquad n \geqslant n_1.$$

Whereas $(1-\varepsilon)T_1 \geqslant qT$, then when $n \geqslant n_1$

$$m^*(\rho_n) > M(R_n, f)^{qT} = R_{n+1}^{qT}, \tag{11}$$

where $q > 1$, $T > 1$.

Our task is to show, that every component of the set $\mathcal{F}(f)$ is bounded. Let us assume the opposite. Let $\mathcal{F}(f)$ contain an unbounded component D. Then, according to the property 4 it is simply connected.

Then we will apply some Baker ideas. Whereas D is a component of $\mathcal{F}(f)$, and it is unbounded, then there is a number $n_2 \geqslant n_1$, such that $D \cap A_n \neq \varnothing$ with all $n \geqslant n_2$, where $A_n = \{z\colon |z| = R_n\}$.

Let us introduce in the consideration also circles

$$C_n = \{z\colon |z| = \rho_n\}, \qquad B_n = \{z\colon |z| = R_n^{qT}\} \qquad (q > 1,\ T > 1).$$

Let us remind, that $R_n^T \leqslant \rho_n \leqslant \frac{1}{2}R_n^{qT}$, $R_n < R_n^T < R_n^{qT} < R_{n+1}$.

Assume $n \geqslant n_2$. Whereas the set D is connected, and $D \cap A_n \neq \varnothing$, then there is a curve γ in D, connecting some point $a_n \in A_n$ with some point $b_{n+1} \in B_{n+1}$ (Fig. 1). Whereas $m^*(\rho_n) \to \infty$ when $n \to \infty$ (it is definite from the estimate (11)), then $f(D)$ is an unbounded connected subset $\mathcal{F}(f)$, containing continuum $f(\gamma)$. Here

$$m^*(\rho_n) = \min_{z \in C_n^*} |f(z)|,$$

where C_n^* is a "circle" close to C_n (in the sense of Theorem 5). According to Theorem 5

$$\frac{\rho_n}{|z|} \to 1 \tag{12}$$

uniformly by z from C_n^* when $n \to \infty$.

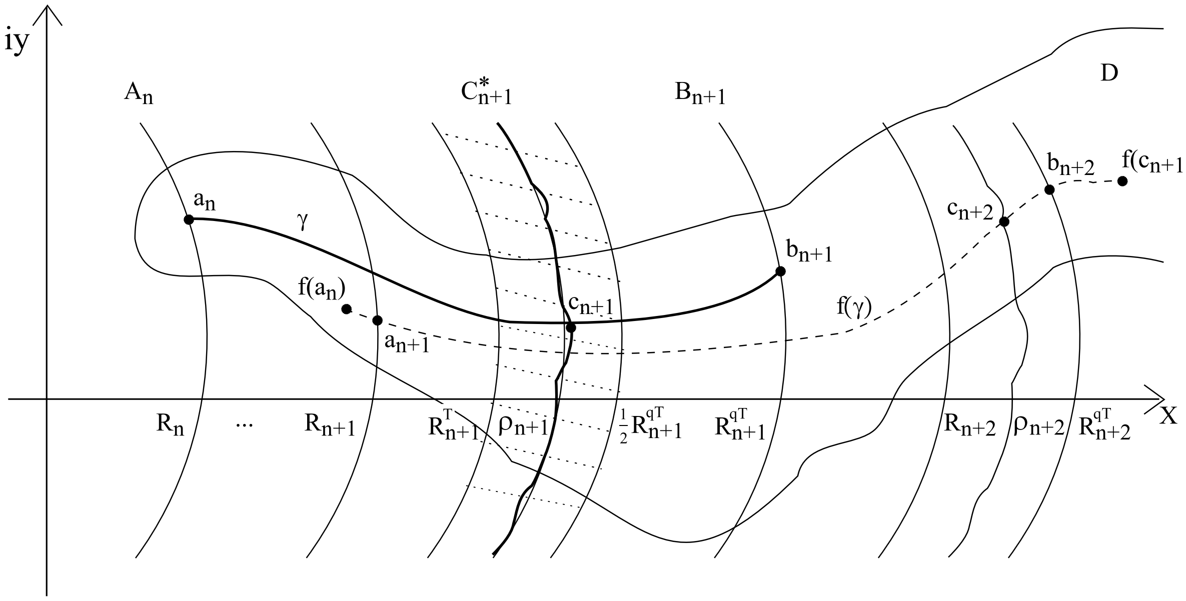

FIGURE 1. An unbounded component $D \subset \mathcal{F}(f)$

Let c_n be a point on the curve γ, trough which C_n^* passes. It is definite, that when $n \geqslant n_3 > n_2$ γ also contains some point c_{n+1} of a modified circle C_{n+1}^*, and $|c_{n+1}| < R_{n+1}^{qT}$ (it results from $R_{n+1}^T \leqslant \rho_{n+1} \leqslant \frac{1}{2} R_{n+1}^{qT}$, if we consider (12)).

Whereas a_n is a point of γ, $|a_n| = R_n$, then $|f(a_n)| \leqslant M(R_n, f) = R_{n+1}$. On the other hand, it results from (11), that

$$|f(c_{n+1})| \geqslant m^*(\rho_{n+1}) > R_{n+2}^{qT}.$$

Consequently, the curve $f(\gamma)$ contains an arc $\gamma^{(1)}$, connecting some point $a_{n+1}^{(1)} \in A_{n+1}$ with the point $b_{n+2}^{(1)}$ of the circle B_{n+2}. With all this $\gamma^{(1)}$ contains some point $c_{n+2}^{(1)}$ of a modified circle C_{n+2}^*. Continuing by induction, we will obtain, that $f^k(D)$ is an unbounded connected subset of $\mathcal{F}(f)$, containing an arc $\gamma^{(k)}$ on the curve $f^k(\gamma)$, which connects points $a_{n+k}^{(k)} \in A_{n+k}$ and $b_{n+k+1}^{(k)} \in B_{n+k+1}$ and contains a point $c_{n+k+1}^{(k)} \in C_{n+k+1}^*$, where n ($n \geqslant n_3$) is fixed, $k \geqslant 1$. Moreover,

$$\min_{z \in \gamma^{(k)}} |f^k(z)| = |f(z_k)| \geqslant R_{n+k},$$

where z_k is some point of γ.

The family $\{f^k\}$ is normal in D. Consequently, there is a subsequence $\{f^{k_p}\}$, which converges locally uniformly in D. Without loss of generality, we assume that $z_{k_p} \to z_0 \in \gamma$. Whereas $|f(z_{k_p})| \to \infty$ with $k_p \to \infty$, then the sequence $\{f^{k_p}\}$ converges to infinity uniformly on γ. Hence, for any $s > 0$ with $k_p \geqslant N(s) > n_3$

$$\min_{z \in \gamma} |f^{k_p}(z)| \geqslant s. \tag{13}$$

Let us consider the family of functions $G = \{g_{k_p}\}_{k_p \geqslant N}$, where

$$g_{k_p}(z) = \frac{f^{k_p}(z) - a}{b - a},$$

a, b are arbitrary, but fixed points from the Julia set $\mathcal{J}(f)$, such that $a \neq b$. We will choose the value N later.

Let us assure, that with some N the family of functions G satisfies the conditions of Lemma 1, if we take the defined above unbounded component of the set $\mathcal{F}(f)$ as the domain D and put $D_0 = \gamma$.

Whereas according to the properties 2, 3 for all $k \geqslant 1$, for any $a, b \in \mathcal{J}(f)$ with $z \in D \subset \mathcal{F}(f)$, iterations $f^k(z)$ omit the values a, b, then the functions $g_{k_p}(z)$ omit the values 0 and 1 in D with all $p \geqslant 1$. Moreover, if we choose in (13) $s_0 = s_{(a,b)} \overset{\text{def}}{=} |a| + |b - a|$, we will obtain, that with

$k_p \geqslant N(s_0) > n_3$

$$|g_{k_p}(z)| = \frac{|f^{k_p}(z) - a|}{|b-a|} \geqslant \frac{\big||f^{k_p}(z)| - |a|\big|}{|b-a|} \geqslant 1, \qquad z \in \gamma.$$

Thus, the family of functions G satisfies the conditions of Lemma 1 with $N = N(s_0)$. Therefore, there exist constants U, V, depending only on γ and D, that

$$|g_{k_p}(z')| < U|g_{k_p}(z)|^V \tag{14}$$

for all $z, z' \in \gamma$.

It can be checked, that for all $z \in \gamma$

$$A|f^{k_p}(z)| \leqslant |g_{k_p}| \leqslant B|f^{k_p}(z)|,$$

where

$$A = \frac{1}{s_0}, \quad B = \frac{|a| + s_0}{s_0|b-a|}, \quad s_0 = |a| + |b-a|. \tag{15}$$

Consequently, for all $z, z' \in \gamma$ with $k_p \geqslant N$

$$|f^{k_p}(z')| < U^*|f^{k_p}(z)|^V, \qquad U^* = \frac{UB^V}{A}.$$

Let $k_p \geqslant N$, z, z' be such points of γ that:

$$1) \quad f^{k_p}(z) = a^{(k_p)}_{n+k_p}, \qquad a^{(k_p)}_{n+k_p} \in A_{n+k_p};$$

$$2) \quad f^{k_p}(z') = c^{(k_p)}_{n+k_p+1}, \qquad c^{(k_p)}_{n+k_p+1} \in C^*_{n+k_p+1}.$$

Then with $k_p \geqslant N$

$$M(R_{n+k_p}, f) = R_{n+k_p+1} < |c^{(k_p)}_{n+k_p+1}| = |f^{k_p}(z')| < U^*|f^{k_p}(z)|^V = U^*|a^{(k_p)}_{n+k_p}|^V = U^* R^V_{n+k_p},$$

that contradicts the fact, that f is a transcendental function, as $R_{n+k_p} \to \infty$ with $k_p \to \infty$.

The theorem has been proved.

4. On considerable Fejer condition

The condition *1)* from (8) is also necessary necessary for the truth of Theorem 6. Actually, for any sequence $\{p_n\}$, such that

$$\sum_{n=1}^{\infty} \frac{1}{p_n} = \infty,$$

there is an entire function

$$f(z) = \sum_{n=1}^{\infty} a_n z^{p_n}$$

with real coefficients, bounded on the real axis [24], [25]. Hence, there is an open connected set $D \supset \mathbb{R}$, on which the function f is also bounded. Let $|f(z)| \leqslant 1$ with $z \in D$. Whereas all iterations $f^k(x)$ for real x are real, then $|f^k(x)| \leqslant 1$, $x \in \mathbb{R}$. It is obvious,

$$a = \sum_{n=1}^{\infty} |a_n| < \infty.$$

Let K be any compact from D. Then for $z \in K$ we have:

$$|f(z)| \leqslant 1,$$

$$|f^2(z)| = \Big|\sum_{n=1}^{\infty} a_n[f(z)]^{p_n}\Big| \leqslant \sum_{n=1}^{\infty} |a_n||f(z)|^{p_n} \leqslant a,$$

$$\dots \qquad \dots \qquad \dots$$

$$|f^k(z)| = \Big|\sum_{n=1}^{\infty} a_n[f^{k-1}(z)]^{p_n}\Big| \leqslant a.$$

Therefore, the family $\{f^k\}$ is normal in D. This implies, that $\mathbb{R} \subset D \subset \mathcal{F}(f)$. It means, that the set $\mathcal{F}(f)$ contains an unbounded component.

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