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THE EXACT BOUNDS OF THE TYPES OF ENTIRE FUNCTIONS OF ORDER $\rho \in (0;1)$ LESS THAN UNITY WITH THE ZEROS LOCATED ON THE RAY

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Abstract. This paper is a detailed account of the author's report made during VI Ufa international conference Complex analysis and differential equations", devoted to the 70-th anniversary of Corresponding member of RAS V.V. Napalkov. Sharp lower estimates of an entire function type of a finite order with respect to such well-known characteristics of the distribution of its zeros as the density (conventional and average), step and lacunarity index. The solution of one new extremal problem is also given.

Keywords: type of an entire function, the upper and lower (average) density of zeros, step and lacunarity index of a sequence of zeros.

1. Introdiction

The success of the research of many analysis tasks depends on the accuracy of the integer function growth characteristic, due to the distribution of its zeros on the plane. To such tasks we can refer, for instance, the task of finding the radius of the exponents system completeness, numerical aspects of analytic continuation of Taylor and Dirichlet series sums beyond the border of the convergence area and others. That is why clearing-up of degree of the integer function growth influences the behavior of its zeros, and contrariwise, defines one of the most important areas of development of the integer functions theory.

The topic of the paper is a detailed discussion of lower estimates of an entire function type of a finite order with respect to such well-known characteristics of the distribution of its zeros as conventional and average density, step and lacunarity index. We shall cite the characteristics right away.

Let $f(z)$ be an entire function. The value of the type $f(z)$ with the order ρ is defined by the equality $\sigma_\rho(f) = \varlimsup\limits_{R\to+\infty} R^{-\rho} \ln \max\limits_{|z|=R} |f(z)|$.

Let further $\Lambda = \{\lambda_n\}_{n=1}^{\infty}$ be a sequence of complex numbers, sorted by the module increase, $0 < |\lambda_n| \nearrow +\infty$, $n_\Lambda(t) = \sum\limits_{|\lambda_n|\leqslant t} 1$ be a counting function of this sequence and $N_\Lambda(r) = \int\limits_0^r \frac{n_\Lambda(t)}{t}dt$ be its average counting function (a Nevanlinna function).

The upper density Λ with the index ρ (ρ-density) is called

$$\overline{\Delta}_\rho(\Lambda) = \varlimsup_{r\to+\infty} \frac{n_\Lambda(r)}{r^\rho},$$

and the upper average ρ-density is the value

$$\overline{\Delta}^*_\rho(\Lambda) = \varlimsup_{r\to+\infty} \frac{N_\Lambda(r)}{r^\rho}.$$

Replacement of these upper limits equalities to lower results in definitions of lower and average ρ-densities $\underline{\Delta}_\rho(\Lambda)$ and $\underline{\Delta}^*_\rho(\Lambda)$.

Let us call the ρ-step of the sequence Λ the characteristic

$$h_\rho(\Lambda) := \varliminf_{n\to+\infty} (|\lambda_{n+1}|^\rho - |\lambda_n|^\rho),$$

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and let the lacunarity index be the value

$$l(\Lambda) := \overline{\lim_{n\to+\infty}} \frac{|\lambda_{n+1}|}{|\lambda_n|}.$$

As it was written above, we are interested in lower estimates of an entire function type $f(z)$ by means of introduced characteristics of its zero set $\Lambda = \Lambda_f$. With this we consider, that $0 \notin \Lambda_f$, since this condition does not change any of the asymptotic characteristics, studied here.

There is well-known inequality [1], set by Georges Valiron for any $\rho > 0$ and $\Lambda \subset \mathbb{C}$:

$$\sigma_\rho(f) \geq \frac{1}{\rho e}\overline{\Delta}_\rho(\Lambda). \tag{1}$$

Constructed by B.Y. Levin in [2] the entire function of the order $\rho > 0$

$$f(z) = \prod_{n=1}^{\infty}\left(1 - \left(\frac{z}{2^{\frac{2^n}{\rho}}}\right)^{2^{2^n}}\right) \tag{2}$$

fulfills the equality in the correlation (1).

In his book [3] R.P. Boas gives an estimate of the entire function type f in case, when not only the upper but also the lower ρ-density of its zeros is known $\Lambda = \Lambda_f$:

$$\sigma_\rho(f) \geq \exp\left\{\underline{\Delta}_\rho(\Lambda)/\overline{\Delta}_\rho(\Lambda)\right\}\frac{1}{\rho e}\overline{\Delta}_\rho(\Lambda). \tag{3}$$

It is apparent that $\underline{\Delta}_\rho(\Lambda) = 0$ (3) results in (1).

If we apply the upper average ρ-density and Jensen formula

$$N_\Lambda(r) = \frac{1}{2\pi}\int_0^{2\pi} \ln|f(re^{i\theta})|d\theta,$$

we will obtain a simpler estimate

$$\sigma_\rho(f) \ \geq \overline{\Delta}^*_\rho(\Lambda), \tag{4}$$

which will amplify (1) due to the known inequality

$$\overline{\Delta}^*_\rho(\Lambda) \geq \frac{1}{e\rho}\overline{\Delta}_\rho(\Lambda).$$

An example of an infinite product (2) results in equality not only of the estimate (1), but also of the estimate (4). The calculation of lower ρ-densities in this example gives $\underline{\Delta}_\rho(\Lambda) = \underline{\Delta}^*_\rho(\Lambda) = 0$. But it is not clear whether the condition $\underline{\Delta}^*_\rho(\Lambda) > 0$ is able to amplify the estimate (4) like (3) improves (1), if $\underline{\Delta}_\rho(\Lambda) > 0$. Therewith precision of the estimate (3) was unknown until recently. In his overview A.U. Popov constructed examples of entire functions with equalities in correlations (3) and (4). In addition, he showed, that taking into account the lower average ρ-density cannot amplify (4) unlike the case with usual ρ-densities. Therefore, whatever the value can be $\underline{\Delta}^* \in [0; \overline{\Delta}^*]$, there is an entire function with zero set $\Lambda = \Lambda_f \subset \mathbb{C}$ of given average ρ-densities $\overline{\Delta}^*_\rho(\Lambda) = \overline{\Delta}^*$ and $\underline{\Delta}^*_\rho(\Lambda) = \underline{\Delta}^*$, for which the equality sign holds true in (4). This information is presented in the paper with A.U. Popov kind permission.

The accuracy of the classical estimates (1), (3) and (4)proved, allows to consider the above results as the solution of the following extremal sums from I to IV:

I. For fixed numbers $\beta > 0,\ \rho > 0$ to find

$$S_{\mathbb{C}}(\beta;\rho) := \inf\left\{\sigma_\rho(f):\ \Lambda_f = \Lambda \subset \mathbb{C},\ \overline{\Delta}_\rho(\Lambda) = \beta\right\}.$$

II. For fixed numbers $\beta > 0,\ \alpha \in [0;\beta],\ \rho > 0$ to find

$$S_{\mathbb{C}}(\alpha\,, \beta\,;\rho) := \inf\left\{\sigma_\rho(f):\ \Lambda_f = \Lambda \subset \mathbb{C},\ \underline{\Delta}_\rho(\Lambda) \geq \alpha,\ \overline{\Delta}_\rho(\Lambda) = \beta\right\}.$$

III. For fixed numbers $\beta^* > 0,\ \rho > 0$ to calculate

$$S^*_{\mathbb{C}}(\beta^*;\rho) := \inf\left\{\sigma_\rho(f):\ \Lambda_f = \Lambda \subset \mathbb{C},\ \overline{\Delta}^*_\rho(\Lambda) = \beta^*\right\}.$$

IV. For fixed numbers $\beta^* > 0,\ \alpha^* \in [0;\beta^*],\ \rho > 0$ to calculate

$$S^*_{\mathbb{C}}(\alpha^*, \beta^*; \rho) := \inf\left\{\sigma_\rho(f):\ \Lambda_f = \Lambda \subset \mathbb{C},\ \underline{\Delta}^*_\rho(\Lambda) \geq \alpha^*,\ \overline{\Delta}^*_\rho(\Lambda) = \beta^*\right\}.$$

Due to the written above, we know the least possible value of the type in every extremal sum from I to IV. Namely,

$$S_{\mathbb{C}}(\beta;\rho) = \frac{1}{e\rho}\beta, \tag{5}$$

$$S_{\mathbb{C}}(\alpha, \beta; \rho) = e^{\alpha/\beta}\frac{1}{e\rho}\beta, \tag{6}$$

$$S^*_{\mathbb{C}}(\beta^*;\rho) = S^*_{\mathbb{C}}(\alpha^*, \beta^*; \rho) = \beta^*. \tag{7}$$

In connection with the applications a special interest is given to the case of an entire function zeroes located on the ray, which results in the next extremal sums.

I$^+$. For fixed numbers $\beta > 0,\ \rho > 0$ to find

$$S_{\mathbb{R}_+}(\beta;\rho) := \inf\left\{\sigma_\rho(f):\ \Lambda_f = \Lambda \subset \mathbb{R}_+,\ \overline{\Delta}_\rho(\Lambda) = \beta\right\}.$$

II$^+$. For fixed numbers $\beta > 0,\ \alpha \in [0;\beta],\ \rho > 0$ to find

$$S_{\mathbb{R}_+}(\alpha, \beta; \rho) := \inf\left\{\sigma_\rho(f):\ \Lambda_f = \Lambda \subset \mathbb{R}_+,\ \underline{\Delta}_\rho(\Lambda) \geq \alpha,\ \overline{\Delta}_\rho(\Lambda) = \beta\right\}.$$

III$^+$. For fixed numbers $\beta^* > 0,\ \rho > 0$ to calculate

$$S^*_{\mathbb{R}_+}(\beta^*;\rho) := \inf\left\{\sigma_\rho(f):\ \Lambda_f = \Lambda \subset \mathbb{R}_+,\ \overline{\Delta}^*_\rho(\Lambda) = \beta^*\right\}.$$

IV$^+$. For fixed numbers $\beta^* > 0,\ \alpha^* \in [0;\beta^*],\ \rho > 0$ to calculate

$$S^*_{\mathbb{R}_+}(\alpha^*, \beta^*; \rho) := \inf\left\{\sigma_\rho(f):\ \Lambda_f = \Lambda \subset \mathbb{R}_+,\ \underline{\Delta}^*_\rho(\Lambda) \geq \alpha^*,\ \overline{\Delta}^*_\rho(\Lambda) = \beta^*\right\}.$$

By now these sums are solved for the values $\rho \in (0;1)$. This example is considered below.

The sum I$^+$ was set up and solved by A.U. Popov in 2005 (see [4]):

$$S_{\mathbb{R}_+}(\beta;\rho) = \beta\, C(\rho), \text{ where } C(\rho) = \max_{a>0}\frac{\ln(1+a)}{a^\rho}.$$

The sum formulated by him II$^+$ was solved by V.B. Sherstukov in 2009 [5]:

$$S_{\mathbb{R}_+}(\alpha, \beta; \rho) = \frac{\pi\alpha}{\sin\pi\rho} + \max_{a>0}\int\limits_{a(\frac{\alpha}{\beta})^{1/\rho}}^{a}\frac{\beta a^{-\rho} - \alpha\tau^{-\rho}}{\tau+1}d\tau.$$

The sum III$^+$ was solved in the recent G.G. Braichev paper [6]:

$$S^*_{\mathbb{R}_+}(\beta^*;\rho) = C(\rho)\rho e\beta^*,$$

where $C(\rho)$ is the function of A.U.Popov from sum I.

The solution of sum IV$^+$ was first found for quite a wide class of entire functions with discretely measured zeros, determined by the condition of the limit existence $\lim\limits_{n\to\infty}\dfrac{N(|\lambda_n|)}{|\lambda_n|^\rho}$ from the paper [7]. The complete solution of this sum is given in [6]:

$$S^*_{\mathbb{R}_+}(\alpha^*, \beta^*; \rho) = \rho\left(\frac{\pi\alpha^*}{\sin\pi\rho} + \max_{b>0}\int\limits_{ba_1^{1/\rho}}^{ba_2^{1/\rho}}\frac{\beta^* b^{-\rho} - \alpha^*\tau^{-\rho}}{\tau+1}d\tau\right).$$

Here a_1 and a_2 are roots of the equation

$$a\ln\frac{e}{a}=\frac{\alpha^*}{\beta^*},\quad 0\leqslant a_1\leqslant 1\leqslant a_2\leqslant e. \tag{8}$$

Like in the extremal sums from I to IV, in the sums from I^+ to IV^+ precise lower bounds are obtained on some rather hard constructed sequences of entire functions zeros.

We should point out at the essential distinctions of the extremal sums for the samples $\Lambda_f\subset\mathbb{C}$ and $\Lambda_f\subset\mathbb{R}_+$. It can be seen from the given formulae, that the dependance of the extremal sums type on parameters is more difficult is sums for functions with zeros on the ray, than with zeros on all the plane. That is why in the papers cited above there are given double-sided estimates of corresponding values by elementary functions and some values studied earlier.

Values comparison in the extremal types of sums I, I^+ and III, III^+ with the consideration of the obtained [4] inequality $C(\rho)>\dfrac{1}{\rho e}$, $\rho\in(0;1)$, allows to conclude, that $S_{\mathbb{R}_+}(\beta;\rho)>S_{\mathbb{C}}(\beta;\rho)$ and $S^*_{\mathbb{R}_+}(\beta^*;\rho)>S^*_{\mathbb{C}}(\beta^*;\rho)$.

As for sums II and IV^+, we will cite only some asymptotic under $\rho\to+0$ functions, connected with these sums. Thus, for instance, the following equalities hold true

$$S_{\mathbb{R}_+}(\alpha\,,\beta\,;\rho)=e^{\alpha/\beta}\frac{1}{e\rho}+O\left(e^{-\frac{1}{\rho}(1-\alpha/\beta)}\right),\ \alpha\leqslant 0.2\,\beta;$$
$$S^*_{\mathbb{R}_+}(\alpha^*\,,\beta^*\,;\rho)=\beta^*+O\left(\rho a_2^{-1/\rho}\right),\ \rho\to+0,$$

where $a_2=a_2(\frac{\alpha^*}{\beta^*})$ is still the major root of equation (8).

The correlations given demonstrate significant influence of the entire function roots arguments on the lowest possible value of its ρ-type and show, that this influence decreases with the decrease of ρ, and disappears in the limit with $\rho\to+0$.

In the theory of trigonometric series, Dirichlet series there very often used notions, similar to the step of sequence and its lacunarity index (see, for instance, [8], [9]). We will show the way these notions influence on the value of extremal type of entire functions.

Let us formulate the following extremal sums.

V^+. For fixed numbers $\beta>0$, $h\in\left[0;\beta^{-1}\right]$, $\rho>0$ to calculate

$$\hat{S}_{\mathbb{R}_+}(\beta,h;\rho):=\inf\left\{\sigma_\rho(f):\ \Lambda_f=\Lambda\subset\mathbb{R}_+,\ \overline{\Delta}_\rho(\Lambda)=\beta,h_\rho(\Lambda)\geq h\right\}.$$

VI^+. For fixed numbers $\beta>0$, $\alpha\in[0;\beta]$, $h\in\left[0;\beta^{-1}\right]$, $\rho>0$ to calculate

$$\hat{S}_{\mathbb{R}_+}(\alpha\,,\beta\,,h;\rho):=\inf\left\{\sigma_\rho(f):\Lambda_f=\Lambda\subset\mathbb{R}_+,\ \underline{\Delta}_\rho(\Lambda)\geq\alpha,\ \overline{\Delta}_\rho(\Lambda)=\beta,h_\rho(\Lambda)\geq h\right\}.$$

The limit on the parameter h in these sums is natural, as it is the result of the easily checked inequality $h_\rho(\Lambda)\overline{\Delta}_\rho(\Lambda)\leqslant 1$, connecting the ρ-step with the upper ρ-density sequence $\Lambda\subset\mathbb{C}$ (for $\rho=1$ and $\Lambda\subset\mathbb{R}_+$ this correlation is cited in [9]).

The solution for the sum V^+ with $\rho\in(0;1)$ is found in the paper [10]:

$$\hat{S}_{\mathbb{R}_+}(\beta\,,h;\rho)=\frac{1}{h}\sup_{a>0}\left\{a^{-\rho}\ln\frac{1+a}{\left(1+as^{-1/\rho}\right)^s}+\int\limits_a^{as^{-1/\rho}}\frac{\tau^{-\rho}}{\tau+1}d\tau\right\}$$
$$=\sup_{a>0}\left\{\beta a^{-\rho}\ln(1+a)+\frac{1}{h}\int\limits_a^{as^{-1/\rho}}\frac{\tau^{-\rho}-sa^{-\rho}}{\tau+1}d\tau\right\},$$

where $s=1-\beta h$. From the latest form of the solution it is easy to find the inequality which holds true when $h>0$:

$$\hat{S}_{\mathbb{R}_+}(\beta\,,h;\rho)>S_{\mathbb{R}_+}(\beta;\rho)=\beta\max_{a>0}\frac{\ln(1+a)}{a^\rho}=\beta\,C(\rho).$$

However, the case $h=0$, of the sum V^+, considered as a limited one, gives the equality $\hat{S}_{\mathbb{R}_+}(\beta\,,0;\rho)=S_{\mathbb{R}_+}(\beta;\rho)$, which is apparent from the sum setting.

Another borderline example, when $h = \beta^{-1}$, is again taken as a limited one, results in the equality $\hat{S}_{\mathbb{R}_+}(\beta\,,\beta^{-1};\rho) = \dfrac{\pi\beta}{\sin\pi\rho}$. Hence, according to the known estimate $\sigma_\rho(f) \leqslant \dfrac{\pi\,\overline{\Delta}_\rho(\Lambda_f)}{\sin\pi\rho}$, $\rho\in(0;1)$, we obtain a pure" equality $\sigma_\rho(f) = \dfrac{\pi\beta}{\sin\pi\rho}$ for any entire function f, where $\Lambda_f\subset\mathbb{R}_+$, $\overline{\Delta}_\rho(\Lambda_f)=\beta$ and $h_\rho(\Lambda_f)=\beta^{-1}$. It was considered before, that such an equality is possible only for the entire function f with a measurable sequence of positive zeros $\Lambda_f=(\lambda_n)_{n=1}^\infty$, i.e. such, having the limit $\lim\limits_{n\to\infty}\dfrac{n}{\lambda_n^\rho}=\beta$. But, of course, the condition $h_\rho(\Lambda_f) = 1/\,\overline{\Delta}_\rho(\Lambda_f)$ does not mean measurability of the sequence Λ_f (examples of such sequences can be found in the thesis [11]).

We have acquainted the reader with the results of extremal sums for entire functions with zeros on the ray, obtained recently or having been already published or are expecting to be published. The solution of the sum VI$^+$ was obtained by the author's follower O.V. Sherstukova quite recently:

$$\hat{S}_{\mathbb{R}_+}(\alpha\,,\beta\,,h;\rho) = \frac{\pi\alpha}{\sin\pi\rho} + \sup_{a>0}\left\{\int\limits_{a\left(\frac{\alpha}{\beta}\right)^{1/\rho}}^{a\nu^{1/\rho}} \frac{\beta a^{-\rho}-\alpha\tau^{-\rho}}{\tau+1}d\tau + \frac{1}{h}\int\limits_{a}^{a\nu^{1/\rho}} \frac{\tau^{-\rho}-a^{-\rho}}{\tau+1}d\tau\right\} =$$

$$= \frac{\pi\alpha}{\sin\pi\rho} + \sup_{a>0}\left\{\int\limits_{a\left(\frac{\alpha}{\beta}\right)^{1/\rho}}^{a} \frac{\beta a^{-\rho}-\alpha\tau^{-\rho}}{\tau+1}d\tau + \frac{s}{h}\int\limits_{a}^{a\nu^{1/\rho}} \frac{\nu\tau^{-\rho}-a^{-\rho}}{\tau+1}d\tau\right\},$$

where $\nu = \dfrac{1-\alpha h}{1-\beta h}$, $s = 1-\beta h$.

It is again of use to compare solutions of the extremal sums II$^+$ and VI$^+$:
$\hat{S}_{\mathbb{R}_+}(\alpha\,,\beta\,,h;\rho) > S_{\mathbb{R}_+}(\alpha\,,\beta\,,\rho)$ when $h>0$, $\alpha<\beta$.

The complete proof of the result is supposed to be published in this issue of the journal.

In the conclusion of the paper we will give the solution of a new extremal sum, where the influence of the lacunarity index of zeros sequence on the value of the entire function type is taken into account. The sum consists in finding the lower possible ρ-type of the entire function $f(z)$, if the lacunarity index and sequence density of its zeros are given. Exactly, for the given numbers $\beta>0\,,l\ge 1\,,\rho>0$ it is necessary to calculate the values

$$\tilde{S}_{\mathbb{C}}(\beta\,,l;\rho) := \inf\left\{\sigma_\rho(f):\ \Lambda_f=\Lambda\subset\mathbb{C},\ \overline{\Delta}_\rho(\Lambda)=\beta,\ l(\Lambda)=l\right\}$$

and

$$\tilde{S}_{\mathbb{R}_+}(\beta\,,l;\rho) := \inf\left\{\sigma_\rho(f):\ \Lambda_f=\Lambda\subset\mathbb{R}_+,\ \overline{\Delta}_\rho(\Lambda)=\beta,\ l(\Lambda)=l\right\}.$$

Here, as before, for $\Lambda_f=\{\lambda_n\}$ holds true $l:=l(\Lambda)=\varlimsup\limits_{n\to+\infty}\dfrac{|\lambda_{n+1}|}{|\lambda_n|}$.

We are intended to apply the given above results and also some correlations, connecting the lacunarity index of the sequence with its upper and lower ρ-densities. Let us single out connections, which are of interest for us, refusing for time saving the dependence on $\Lambda_f=\Lambda$ and ρ in notations of densities and the lacunarity index, i.e. we will simply write $\overline{\Delta}$, $\underline{\Delta}$, $\overline{\Delta}^*$, $\underline{\Delta}^*$, l.

Let $\Lambda=\{\lambda_n\}_{n=1}^\infty\subset\mathbb{C}$ be written in the order of modules nondecreasing:

$$0<|\lambda_1|=\ldots=|\lambda_{n_1}|<|\lambda_{n_1+1}|=\ldots=|\lambda_{n_2}|<\ldots,\ |\lambda_n|\nearrow+\infty.$$

The counting function of this sequence is $n_\Lambda(t)=0$ when $t\in[0;|\lambda_1|)$ and $n_\Lambda(t)=n_k$ when $t\in[|\lambda_{n_k}|;|\lambda_{n_k+1}|)$, $k=1,2,\ldots$. Therefore

$$\overline{\Delta} = \varlimsup_{n\to\infty}\frac{n}{|\lambda_n|^\rho} = \varlimsup_{k\to\infty}\frac{n_k}{|\lambda_{n_k}|^\rho},\qquad \underline{\Delta} = \varliminf_{n\to\infty}\frac{n}{|\lambda_n|^\rho} = \varliminf_{k\to\infty}\frac{n_k}{|\lambda_{n_k+1}|^\rho}.$$

Hence, we easily obtain

$$\overline{\Delta} = \overline{\lim_{k\to\infty}} \frac{n_k}{|\lambda_{n_{k+1}}|^\rho} \frac{|\lambda_{n_{k+1}}|^\rho}{|\lambda_{n_k}|^\rho} \geq \underline{\Delta} \,\overline{\lim_{k\to\infty}} \frac{|\lambda_{n_{k+1}}|^\rho}{|\lambda_{n_k}|^\rho} = \underline{\Delta}\, l^\rho, \Rightarrow$$

$$\overline{\Delta} \geq \underline{\Delta}\, l^\rho. \tag{9}$$

By analogy we deduce $\overline{\Delta}^* \leqslant \widetilde{\Delta}\, l^\rho$, where $\widetilde{\Delta} = \widetilde{\Delta}_\rho(\Lambda) := \overline{\lim\limits_{n\to\infty}} \dfrac{N_\Lambda(|\lambda_n|)}{|\lambda_n|^\rho}$ is the upper average discrete ρ-density of the sequence Λ.

The correlations between usual and average sequences Λ are described by the inequalities (see [12, ch. 2, §2], [7]):

$$\rho a_1 \overline{\Delta}^* \leqslant \underline{\Delta} \leqslant \rho \widetilde{a}_1 \overline{\Delta}^*, \;\; \rho \widetilde{a}_2 \overline{\Delta}^* \leqslant \overline{\Delta} \leqslant \rho a_2 \overline{\Delta}^*. \tag{10}$$

Here, as before, a_1 and a_2 are roots of the equation $a \ln \dfrac{e}{a} = \dfrac{\underline{\Delta}^*}{\overline{\Delta}^*}$, whereas $\widetilde{a}_1$ and $\widetilde{a}_2$ are roots of the equation $a \ln \dfrac{e}{a} = \dfrac{\widetilde{\Delta}}{\overline{\Delta}^*}$, and $a_1 \leqslant \widetilde{a}_1 \leqslant 1 \leqslant \widetilde{a}_2 \leqslant a_2$.

If $\widetilde{\Delta} = \underline{\Delta}^*$, it is apparent, that $\widetilde{a}_1 = a_1$ and $\widetilde{a}_2 = a_2$. Therefore, it results from (10) for such equality sequences as

$$\underline{\Delta} = \rho a_1 \overline{\Delta}^*, \; \overline{\Delta} = \rho a_2 \overline{\Delta}^* . \tag{11}$$

As we are intended to demonstrate now, the property $\widetilde{\Delta} = \underline{\Delta}^*$ ensures equality in case (9) too. The discretely measurable sequences are the ones, satisfying the condition of the limit existence $\lim\limits_{n\to\infty} \dfrac{N(|\lambda_n|)}{|\lambda_n|^\rho} = \widetilde{\Delta}$ (i.e. $\widetilde{\Delta} = \underline{\Delta}^*$).

The class of discretely measurable sequences is, in fact, rather wide: for arbitrary numbers $\rho > 0$, $\beta > 0$ and $\alpha \in [0;\beta]$, as it is shown in [7], there are discretely measurable sequences Λ with densities $\overline{\Delta}_\rho(\Lambda) = \beta$ and $\underline{\Delta}_\rho(\Lambda) = \alpha$.

The following statement is ensured by the common results of the study [12, p.212, theorem 3].

Let $f(z)$ be an entire function of the finite order $\rho > 0$ with the discretely measurable sequence of zeros Λ and l be the lacunarity index of the sequence Λ. Then

$$\overline{\Delta}^* \leqslant \underline{\Delta}^* \frac{c^{\frac{1}{c-1}}}{e \ln c^{\frac{1}{c-1}}}, \quad \text{where} \quad c = l^\rho. \tag{12}$$

Relying on this result, let us proof the inequality, which is opposite to (9). For this purpose let us find parametric representation of the roots a_1, a_2 of the equation $a \ln \dfrac{e}{a} = \theta$. Having defined $\dfrac{a_2}{a_1} = q$, or $a_2 = q a_1$ $(q > 1)$, let us write

$$\theta = a_1 \ln \frac{e}{a_1} = a_2 \ln \frac{e}{a_2} = q a_1 \ln \frac{e}{q a_1} = q a_1 \left(\ln \frac{e}{a_1} - \ln q \right) = q\theta - q a_1 \ln q.$$

Hence, $a_1 = \theta \dfrac{q-1}{q \ln q}$, $a_2 = \theta \dfrac{q-1}{\ln q}$. Inserting the expression obtained for a_2 into the equality $\theta = a_2 \ln \dfrac{e}{a_2}$, we obtain $\theta = \theta \dfrac{q-1}{\ln q} \ln \dfrac{e \ln q}{\theta(q-1)}$, i.e. $\ln q^{\frac{1}{q-1}} = \ln \dfrac{e \ln q}{\theta(q-1)}$. Therefore, $q^{\frac{1}{q-1}} = \dfrac{e \ln q}{\theta(q-1)}$, i.e. $\theta = \dfrac{e \ln q}{q^{\frac{1}{q-1}}(q-1)} = e \dfrac{\ln q^{\frac{1}{q-1}}}{q^{\frac{1}{q-1}}}$. We finally approach to the correlations

$$\theta = e \frac{\ln q^{\frac{1}{q-1}}}{q^{\frac{1}{q-1}}}, \;\; a_1 = \theta \frac{q-1}{q \ln q} = e\, q^{\frac{q}{1-q}}, \;\; a_2 = \theta \frac{q-1}{\ln q} = e\, q^{\frac{1}{1-q}} . \tag{13}$$

Considering that $\theta := \dfrac{\underline{\Delta}^*}{\overline{\Delta}^*} \in (0;1)$ and comparing (12) with (13), we obtain

$$\theta = e\,\frac{\ln q^{\frac{1}{q-1}}}{q^{\frac{1}{q-1}}} \geq e\,\frac{\ln c^{\frac{1}{c-1}}}{c^{\frac{1}{c-1}}}\,,\ c = l^\rho.$$

Due to the function decreasing $F(x) := e\dfrac{\ln x^{\frac{1}{x-1}}}{x^{\frac{1}{x-1}}}$ on $(1;+\infty)$ means, that $c \geq q$. But from (10) we obtain $\dfrac{\overline{\Delta}}{\underline{\Delta}} \leqslant \dfrac{a_2}{a_1} = q \leqslant c = l^\rho$. Therefore, $\overline{\Delta} \leqslant \underline{\Delta}\, l^\rho$, that together with (9) leads to the equality needed $\overline{\Delta} = \underline{\Delta}\, l^\rho$ (and also $q = l^\rho$). Applying this and also equalities (11) and (13), we can write for the discretely measurable sequences, that

$$\underline{\Delta} = l^{-\rho}\,\overline{\Delta},$$

$$\overline{\Delta}^* = \frac{\overline{\Delta}}{\rho a_2} = \frac{\overline{\Delta}}{\rho e}\, q^{\frac{1}{q-1}},$$

$$\underline{\Delta}^* = \theta\,\overline{\Delta}^* = \frac{\overline{\Delta}}{\rho}\,\frac{\ln q}{q-1}.$$

It is convenient to collect all the information obtained into a separate statement.

Proposition. *Let Λ be a discretely measurable sequences of complex numbers with the densities $\overline{\Delta} = \beta$, $\underline{\Delta} = \alpha$, $\underline{\Delta}^* = \alpha^*$, $\overline{\Delta}^* = \beta^*$ and l be the lacunarity index of the sequence Λ. Then the following equalities hold true*

$$\alpha = l^{-\rho}\beta,$$

$$\beta^* = \frac{\beta}{\rho e}\, q^{\frac{1}{q-1}},\ \alpha^* = \frac{\beta}{\rho}\,\frac{\ln q}{q-1},$$

$$a_1 = e\, q^{\frac{q}{1-q}},\ a_2 = e\, q^{\frac{1}{1-q}},\ \textit{where}\ \ q = l^\rho.$$

Relying on the proposition and extremal sums II and IV$^+$ solved, we obtain the following result.

Theorem. *Assume $\beta > 0$, $l > 1$, $\rho \in (0;1)$. Let, further, $\tilde{S}^*_{\mathbb{R}_+}(\beta\,,l;\rho)$ and $\tilde{S}^*_{\mathbb{C}}(\beta\,,l;\rho)$ be precise lower bounds $\tilde{S}_{\mathbb{R}_+}(\beta\,,l;\rho)$ and $\tilde{S}_{\mathbb{C}}(\beta\,,l;\rho)$ correspondingly, taken by discretely measurable sequences. Hence,*

$$\tilde{S}^*_{\mathbb{R}_+}(\beta\,,l;\rho) = \beta L \left\{ \frac{\pi}{\sin \pi\rho} + \sup_{a>0} \int\limits_{al^{-1}}^{a} \frac{L^{-1}a^{-\rho} - \tau^{-\rho}}{\tau+1} d\tau \right\},\ \textit{where}\ \ L = \frac{\ln l^\rho}{l^\rho - 1}\,,$$

$$\tilde{S}^*_{\mathbb{C}}(\beta\,,l;\rho) \geq \frac{\beta}{\rho e}\,\exp\left\{l^{-\rho}\right\}\,.$$

Remark. The question about precision of the last estimate remains unanswered. As for the first statement of the theorem, the choice of IV$^+$ instead of II$^+$ as a reference sum, is determined by the fact that the extremal sequence in the sum II$^+$ is not a discretely measurable one. Therefore, as it is demonstrated in [7], application on this sum would result in trivially inexact conclusion.

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