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EQUIVALENCE GROUP ANALYSIS AND NONLINEAR SELF-ADJOINTNESS OF THE GENERALIZED KOMPANEETS EQUATION

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Abstract. Equivalence group analysis is applied to the Kompaneets equation. We compute the equivalence Lie algebra for the corresponding *generalized* Kompaneets equation. We also show that the generalized Kompaneets equation is nonlinearly self-adjoint.

The principle of an *a priori* use of symmetries gives a possibility to use the equivalence algebra in order to approximate the Kompaneets equation by an equation having a wider class of symmetries. Using an additional symmetry of the approximating equation and the nonlinear self-adjointness, one can construct new group invariant solutions and conservation laws.

Keywords: Kompaneets equation, Generalized Kompaneets equation, Equivalence algebra, Nonlinear self-adjointness, Invariant solution, Conservation law.

1. Preliminaries

In the present work we apply to the Kompaneets equation the following method of finding conservation laws by means of nonlinear self-adjointness described in detail in [4].

Consider a system of $\overline{m}$ differential equations

$$F_{\bar{\alpha}}\big(x, u, u_{(1)}, \ldots, u_{(s)}\big) = 0, \quad \bar{\alpha} = 1, \ldots, \overline{m}, \tag{1.1}$$

with m dependent variables $u = (u^1, \ldots, u^m)$ and n independent variables $x = (x^1, \ldots, x^n)$. Recall that the adjoint system to Eqs. (1.1) is defined by

$$F^*_\alpha\big(x, u, v, u_{(1)}, v_{(1)}, \ldots, u_{(s)}, v_{(s)}\big) \equiv \frac{\delta(v^{\bar{\beta}} F_{\bar{\beta}})}{\delta u^\alpha} = 0, \quad \alpha = 1, \ldots, m. \tag{1.2}$$

Definition 1. *The system (1.1) is said is said to be nonlinearly self-adjoint if the adjoint equations (1.2) are satisfied for all solutions u of the original system (1.1) upon a substitution*

$$v^{\bar{\alpha}} = \varphi^{\bar{\alpha}}(x, u), \quad \bar{\alpha} = 1, \ldots, \overline{m}, \tag{1.3}$$

such that

$$\varphi(x, u) \neq 0. \tag{1.4}$$

In other words, the following equations hold:

$$F^*_\alpha\big(x, u, \varphi(x, u), \ldots, u_{(s)}, \varphi_{(s)}\big) = \lambda^{\bar{\beta}}_\alpha F_{\bar{\beta}}\big(x, u, \ldots, u_{(s)}\big), \quad \alpha = 1, \ldots, m, \tag{1.5}$$

where $\lambda^{\bar{\beta}}_\alpha$ are undetermined coefficients, and $\varphi_{(\sigma)}$ are derivatives of (1.3),

$$\varphi_{(\sigma)} = \{D_{i_1} \cdots D_{i_\sigma}\big(\varphi^{\bar{\alpha}}(x, u)\big)\}, \quad \sigma = 1, \ldots, s.$$

Here v and φ are the $\overline{m}$-dimensional vectors

$$v = (v^1, \ldots, v^{\overline{m}}), \quad \varphi = (\varphi^1, \ldots, \varphi^{\overline{m}}),$$

and Eq. (1.4) means that not all components $\varphi^{\bar{\alpha}}(x, u)$ of φ vanish simultaneously.

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Construction of conserved vectors associated with symmetries of differential equations is based on the following theorem.

Theorem 1. *Consider a nonlinearly self-adjoint system of differential equations (1.1). Let its adjoint system (1.2) be satisfied for all solutions of the system of equations (1.1) upon a substitution*

$$v^\alpha = \varphi^\alpha(x,u), \quad \alpha = 1,\ldots,m. \tag{1.6}$$

Then any Lie point, contact or Lie-Bäcklund symmetry

$$X = \xi^i(x,u,u_{(1)},\ldots)\frac{\partial}{\partial x^i} + \eta^\alpha(x,u,u_{(1)},\ldots)\frac{\partial}{\partial u^\alpha}\,, \tag{1.7}$$

as well as a nonlocal symmetry of Eqs. (1.1) leads to a conservation law

$$D_i(C^i) = 0 \tag{1.8}$$

constructed by the following formula:

$$C^i = W^\alpha \left[\frac{\partial\mathcal{L}}{\partial u_i^\alpha} - D_j\left(\frac{\partial\mathcal{L}}{\partial u_{ij}^\alpha}\right) + D_jD_k\left(\frac{\partial\mathcal{L}}{\partial u_{ijk}^\alpha}\right) - \ldots\right] \tag{1.9}$$

$$+ D_j\left(W^\alpha\right)\left[\frac{\partial\mathcal{L}}{\partial u_{ij}^\alpha} - D_k\left(\frac{\partial\mathcal{L}}{\partial u_{ijk}^\alpha}\right) + \ldots\right] + D_jD_k\left(W^\alpha\right)\left[\frac{\partial\mathcal{L}}{\partial u_{ijk}^\alpha} - \ldots\right],$$

where

$$W^\alpha = \eta^\alpha - \xi^j u_j^\alpha. \tag{1.10}$$

Here $\mathcal{L}$ is given by the formula

$$\mathcal{L} = v^\beta F_\beta \tag{1.11}$$

and is called the formal Lagrangian for the system (1.1). In (1.9) the formal Lagrangian should be written in the symmetric form with respect to all mixed derivatives $u_{ij}^\alpha,\ u_{ijk}^\alpha,\ldots$, and the "non-physical variables" v^α should be eliminated via Eqs. (1.6).

2. The Kompaneets equation

2.1. Introduction. The equation

$$\frac{\partial n}{\partial t} = \frac{1}{x^2}\frac{\partial}{\partial x}\left[x^4\left(\frac{\partial n}{\partial x} + n + n^2\right)\right], \tag{2.1}$$

known as the Kompaneets equation or the *photon diffusion equation*, was derived independently by A.S. Kompaneets[1] [1] and R. Weymann [2]. They take as a starting point the kinetic equations for the distribution function of a photon gas and arrive at Eq. (2.1) under certain idealized conditions. This equation provides a mathematical model for describing the time development of the energy spectrum of a low energy homogeneous photon gas interacting with a rarefied electron gas via the Compton scattering. Here n is the density of the photon gas (photon number density), t is time and x is connected with the photon frequency ν by the formula

$$x = \frac{h\nu}{kT_e}\,, \tag{2.2}$$

where h is Planck's constant and kT_e is the *electron temperature* with the standard notation k for Boltzmann's constant. According to this notation, $h\nu$ has the meaning of the *photon energy.* The nonrelativistic approximation is used, i.e. it is assumed that the electron temperatures satisfy the condition $kT_e \ll mc^2$, where m is the electron mass and c is the light velocity. The term *low energy photon gas* means that $h\nu \ll mc^2$.

The Kompaneets equation (2.1) admits only the time translation group with the generator

$$X_1 = \frac{\partial}{\partial t}\,\cdot \tag{2.3}$$

[1]He mentions in his paper that the work has been done in 1950 and published in *Report N. 336* of the Institute of Chemical Physics of the USSR Acad. Sci.

Consequently, the only group invariant solution of the Kompaneets equation is the steady-state solution $n = n(x)$ defined by the Riccati equation

$$\frac{dn}{dx} + n^2 + n = \frac{C}{x^4} \cdot$$

It is shown in [3] that Lie group analysis provides more invariant solutions for certain approximations of Eq. (2.1).

The aim of the present paper (see also the preprints [4] and [5]) is to discuss possibilities provided by the equivalence group analysis applied to the so-called *generalized Kompaneets equation*.

2.2. Nonlinear self-adjointness. For unifying the notation, we denote the dependent variable n by u and write Eq. (2.1) in the form

$$u_t = \frac{1}{x^2} D_x\big[x^4(u_x + u + u^2)\big], \tag{2.4}$$

or in the expanded form

$$u_t = x^2 u_{xx} + (x^2 + 4x + 2x^2 u)u_x + 4x(u + u^2). \tag{2.5}$$

The formal Lagrangian (see [4]) for Eq. (2.5) is written

$$\mathcal{L} = v[-u_t + x^2 u_{xx} + (x^2 + 4x + 2x^2 u)u_x + 4x(u + u^2)].$$

Working out the variational derivative of this formal Lagrangian,

$$\frac{\delta \mathcal{L}}{\delta u} = D_t(v) + D_x^2(x^2 v) - D_x[(x^2 + 4x + 2x^2 u)v] + 2x^2 v u_x + 4x(1 + 2u)v,$$

we obtain the following adjoint equation (see [4], Section 1.3) to Eq. (2.4):

$$\frac{\delta \mathcal{L}}{\delta u} \equiv v_t + x^2 v_{xx} - x^2(1 + 2u)v_x + 2(x + 2xu - 1)v = 0. \tag{2.6}$$

According to [4], Eq. (2.4) is nonlinearly self-adjoint if there exists a substitution

$$v = \varphi(t, x, u) \neq 0$$

such that

$$\frac{\delta \mathcal{L}}{\delta u}\bigg|_{v=\varphi(t,x,u)} = \lambda[-u_t + x^2 u_{xx} + (x^2 + 4x + 2x^2 u)u_x + 4x(u + u^2)], \tag{2.7}$$

where λ is an undetermined variable coefficient. We have:

$$\begin{aligned} v_t &= D_t[\varphi(t, x, u)] = \varphi_u u_t + \varphi_t, \\ v_x &= D_x[\varphi(t, x, u)] = \varphi_u u_x + \varphi_x, \\ v_{xx} &= D_x(v_x) = \varphi_u u_{xx} + \varphi_{uu} u_x^2 + 2\varphi_{xu} u_x + \varphi_{xx}. \end{aligned} \tag{2.8}$$

Inserting (2.8) in the expression for the variational derivative given by (2.6) and singling out the terms containing u_t and u_{xx} in Eq. (2.7), we obtain the following equation:

$$\varphi_u[u_t + x^2 u_{xx}] = \lambda[-u_t + x^2 u_{xx}].$$

Since this equation should be satisfied identically in u_t and u_{xx}, it yields

$$\lambda = \varphi_u = 0.$$

Hence $\varphi = \varphi(t, x)$ and Eq. (2.7) becomes:

$$\varphi_t + x^2 \varphi_{xx} - x^2(1 + 2u)\varphi_x + 2(x + 2xu - 1)\varphi = 0. \tag{2.9}$$

This equation should be satisfied identically in t, x and u. Therefore we nullify the coefficient for u and obtain

$$x\varphi_x - 2\varphi = 0,$$

whence

$$\varphi(t, x) = c(t)x^2.$$

Substitution in Eq. (2.9) yields $c'(t) = 0$. Hence, $v = \varphi(t, x) = Cx^2$ with an arbitrary constant C. Since $\lambda = 0$ in (2.7) and the adjoint equation (2.6) is linear and homogeneous in v, one can let $C = 1$. Thus, we have demonstrated the following statement.

Proposition 1. *The adjoint equation (2.6) has the solution*

$$v = x^2 \tag{2.10}$$

for any solution u of Eq. (2.4). In other words, the Kompaneets equation (2.4) is nonlinearly self-adjoint with the substitution (1.3) given by (2.10).

2.3. Simple proof of nonlinear self-adjointness. We can provide a simple proof of the nonlinear self-adjointness of the Kompaneets equation by using Theorem 8.1 from [4]. Recall this theorem in the case of a system containing a single differential equation

$$F\big(t, x, u, u_t, u_x, u_{xx}\big) = 0. \tag{2.11}$$

In this case, the above Theorem 8.1 states that if Eq. (2.11) has a nontrivial conservation law of the form

$$D_t(C^1) + D_x(C^2) = \mu(t, x, u)\, F\big(t, x, u, u_t, u_x, u_{xx}\big) \tag{2.12}$$

then Eq. (2.11) is nonlinearly self-adjoint, and the substitution (1.3) is given by

$$v = \mu(t, x, u). \tag{2.13}$$

Let us return to Eq. (2.4). In this case we have

$$F\big(t, x, u, u_t, u_x, u_{xx}\big) = u_t - \frac{1}{x^2}\, D_x\big[x^4(u_x + u + u^2)\big]. \tag{2.14}$$

Furthermore, Eq. (2.4) can be written as a conservation equation with the conserved vector

$$C^1 = x^2\, u, \quad C^2 = -x^4(u_x + u + u^2).$$

With this vector, the conservation law (2.12) is written

$$D_t(C^1) + D_x(C^2) = x^2\ F\big(t, x, u, u_t, u_x, u_{xx}\big), \tag{2.15}$$

where F is given by Eq. (2.14). Hence $\mu(t, x, u) = x^2$, and (2.13) yields the substitution (2.10) thus proving Proposition 1.

3. Generalized Kompaneets equation

3.1. The generalized model. In the original derivation of Eq. (2.1), the following more general equation with undetermined functions $f(n)$ and $h(x)$ appears incidentally (see [1], Eqs. (9), (10) and their discussion):

$$\frac{\partial n}{\partial t} = \frac{1}{h(x)}\, \frac{\partial}{\partial x}\left[h^2(x)\left(\frac{\partial n}{\partial x} + f(n)\right)\right]. \tag{3.1}$$

Then, using a physical reasoning, Kompaneets takes $f(n) = n(1 + n)$ and $h(x) = x^2$. This choice restricts the symmetry properties of the model significantly. Namely, Equation (2.1) has only the time-translational symmetry with the generator (2.3), see Section 2.1.

The generalized model (3.1) can be used for extensions of symmetry properties using the *theorem on projections* (N.H. Ibragimov, 1986; see Paper 3 in [6]) and the *principle of an a priori use of symmetries* [7]. In this way, exact solutions known for particular approximations to the Kompaneets equation can be obtained. Moreover, this approach may also lead to new approximations of solutions and conservation laws when one has to take into account various perturbations of the idealized situation assumed in the Kompaneets model (2.1).

3.2. Nonlinear self-adjointness. Let us write the generalized model (3.1) as follows:

$$u_t = \frac{1}{h(x)}\, D_x\big\{h^2(x)[u_x + f(u)]\big\}, \quad h'(x) \neq 0. \tag{3.2}$$

Its expanded form is

$$u_t = h(x)\big(u_{xx} + f'(u)u_x\big) + 2h'(x)\big(u_x + f(u)\big). \tag{3.3}$$

The formal Lagrangian for Eq. (3.3) is

$$\mathcal{L} = v\,\big[- u_t + h(x)\big(u_{xx} + f'(u)u_x\big) + 2h'(x)\big(u_x + f(u)\big)\big], \tag{3.4}$$

where v is a new dependent variable. Using this formal Lagrangian we obtain the following adjoint equation to the generalized Kompaneets equation (3.2):

$$\frac{\delta \mathcal{L}}{\delta u} \equiv v_t + h(x) v_{xx} - h(x) f'(u) v_x + [h'(x) f'(u) - h''(x)] v = 0. \tag{3.5}$$

Proceeding as in Section 2.3, one can demonstrate the following statement on the nonlinear self-adjointness of Eq. (3.2).

Proposition 2. *The generalized Kompaneets equation (3.2) is nonlinearly self-adjoint with the substitution (1.3) given by*

$$v = h(x). \tag{3.6}$$

4. Equivalence group generators

For calculating the equivalence Lie algebra, we write Eq. (3.3) in the form

$$u_t = h\big[u_{xx} + f_u u_x\big] + 2h_x\big[u_x + f\big], \tag{4.1}$$

$$f_t = f_x = 0, \quad h_t = h_u = 0. \tag{4.2}$$

The equivalence generators

$$Y = \xi^1 \frac{\partial}{\partial t} + \xi^2 \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial u} + \mu^1 \frac{\partial}{\partial f} + \mu^2 \frac{\partial}{\partial h} \tag{4.3}$$

are obtained from the invariance condition of Eqs. (4.1). The coefficients ξ^i and η of the operator (4.3) depend on the variables t, x, u, whereas the coefficients μ^α depend on t, x, u, f, h.

The prolongation of the operator (4.3) to the derivatives of u, f, and h involved in Eqs. (4.1)-(4.2) is written

$$\begin{aligned} \widetilde{Y} =& \xi^1 \frac{\partial}{\partial t} + \xi^2 \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial u} + \mu^1 \frac{\partial}{\partial f} + \mu^2 \frac{\partial}{\partial h} + \zeta_1 \frac{\partial}{\partial u_t} + \zeta_2 \frac{\partial}{\partial u_x} + \zeta_{22} \frac{\partial}{\partial u_{xx}} \\ &+ \omega_1^1 \frac{\partial}{\partial f_t} + \omega_2^1 \frac{\partial}{\partial f_x} + \omega_0^1 \frac{\partial}{\partial f_u} + \omega_1^2 \frac{\partial}{\partial h_t} + \omega_2^2 \frac{\partial}{\partial h_x} + \omega_0^2 \frac{\partial}{\partial h_u}\,\cdot \end{aligned} \tag{4.4}$$

The invariance of the system (4.1)-(4.2) requires that the following determining equations should be satisfied on the manifold given by Eqs. (4.1)-(4.2):

$$\widetilde{Y}\big(- u_t + h\big[u_{xx} + f_u u_x\big] + 2h_x\big[u_x + f\big)\big] = 0, \tag{4.5}$$

$$\widetilde{Y} f_t = 0, \quad \widetilde{Y} f_x = 0, \quad \widetilde{Y} h_t = 0, \quad \widetilde{Y} h_u = 0. \tag{4.6}$$

The coefficients $\zeta_1,\ \zeta_2,\ \zeta_{22}$ of the prolonged operator (4.4) are calculated by the usual prolongation formulae:

$$\begin{aligned} \zeta_1 &= D_t(\eta) - u_t D_t(\xi^1) - u_x D_t(\xi^2), \\ \zeta_2 &= D_x(\eta) - u_t D_x(\xi^1) - u_x D_x(\xi^2), \\ \zeta_{22} &= D_x^2(\eta) - u_t D_x^2(\xi^1) - u_x D_x^2(\xi^2) - 2u_{tx} D_x(\xi^1) - 2u_{xx} D_x(\xi^2). \end{aligned} \tag{4.7}$$

In calculating ω_i^α the independent variables are t, x, u and the dependent variables are f and h. We consider the operators of total differentiations $\widetilde{D}_t,\ \widetilde{D}_x,\ \widetilde{D}_u$ with respect to the independent variables t, x, u. Taking into account Eqs. (4.2), one can write these operators in the form

$$\begin{aligned} \widetilde{D}_t &= \frac{\partial}{\partial t}, \\ \widetilde{D}_x &= \frac{\partial}{\partial x} + h_x \frac{\partial}{\partial h}\,, \\ \widetilde{D}_u &= \frac{\partial}{\partial u} + f_u \frac{\partial}{\partial f}\,\cdot \end{aligned} \tag{4.8}$$

The coefficients ω_i^α of the prolonged operator (4.4) are given by

$$\omega_i^\alpha = \widetilde{D}_i(\mu^\alpha) - f_u^\alpha \widetilde{D}_i(\eta), \quad i = 1, 2, 0, \tag{4.9}$$

where $\widetilde{D}_1 = \widetilde{D}_t,\ \widetilde{D}_2 = \widetilde{D}_x,\ \widetilde{D}_0 = \widetilde{D}_u$.

Let us begin the analysis of the determining equation by investigating Eqs. (4.6). They are written

$$\omega_1^1 = 0, \quad \omega_2^1 = 0, \quad \omega_1^2 = 0, \quad \omega_0^2 = 0. \tag{4.10}$$

The equations (4.9) and (4.8) give

$$\begin{aligned}
\omega_1^1 &= \widetilde{D}_t(\mu^1) - f_u \widetilde{D}_t(\eta) = \frac{\partial \mu^1}{\partial t} - f_u \frac{\partial \eta}{\partial t}\,,\\
\omega_2^1 &= \widetilde{D}_x(\mu^1) - f_u \widetilde{D}_x(\eta) = \frac{\partial \mu^1}{\partial x} + h_x \frac{\partial \mu^1}{\partial h} - f_u \frac{\partial \eta}{\partial x}\,,\\
\omega_1^2 &= \widetilde{D}_t(\mu^2) - h_x \widetilde{D}_t(\xi^2) = \frac{\partial \mu^2}{\partial t} - h_x \frac{\partial \xi^2}{\partial t}\,,\\
\omega_0^2 &= \widetilde{D}_u(\mu^2) - h_x \widetilde{D}_u(\xi^2) = \frac{\partial \mu^2}{\partial u} + f_u \frac{\partial \mu^2}{\partial f} - h_x \frac{\partial \xi^2}{\partial u}\,.
\end{aligned} \tag{4.11}$$

Substituting (4.11) in Eqs. (4.10), we obtain the system

$$\begin{aligned}
&\frac{\partial \mu^1}{\partial t} - f_u \frac{\partial \eta}{\partial t} = 0\,,\\
&\frac{\partial \mu^1}{\partial x} + h_x \frac{\partial \mu^1}{\partial h} - f_u \frac{\partial \eta}{\partial x} = 0\,,\\
&\frac{\partial \mu^2}{\partial t} - h_x \frac{\partial \xi^2}{\partial t} = 0\,,\\
&\frac{\partial \mu^2}{\partial u} + f_u \frac{\partial \mu^2}{\partial f} - h_x \frac{\partial \xi^2}{\partial u} = 0.
\end{aligned} \tag{4.12}$$

Since these equations should hold identically for all f_u and h_x, Eqs. (4.12) give rise to the following system:

$$\begin{aligned}
&\frac{\partial \mu^1}{\partial t} = 0, \quad \frac{\partial \eta}{\partial t} = 0\,,\\
&\frac{\partial \mu^1}{\partial x} = 0, \quad \frac{\partial \mu^1}{\partial h} = 0, \quad \frac{\partial \eta}{\partial x} = 0\,,\\
&\frac{\partial \mu^2}{\partial t} = 0, \quad \frac{\partial \xi^2}{\partial t} = 0\,,\\
&\frac{\partial \mu^2}{\partial u} = 0, \quad \frac{\partial \mu^2}{\partial f} = 0, \quad \frac{\partial \xi^2}{\partial u} = 0.
\end{aligned} \tag{4.13}$$

The general solution of Eqs. (4.13) is given by

$$\xi^1(t, x, u), \quad \xi^2(x), \quad \eta(u), \quad \mu^1(u, f), \quad \mu^2(h, x). \tag{4.14}$$

We turn now to Eq. (4.5). It is written

$$-\zeta_1 + h[\zeta_{22} + f_u \zeta_2 + \omega_0^1 u_x] + [u_{xx} + f_u u_x]\mu^2 + 2h_x[\zeta_2 + \mu^1] + 2[u_x + f]\omega_2^2 = 0. \tag{4.15}$$

Using the formula (4.9), we obtain the coefficients ω involved in Eq. (4.15):

$$\begin{aligned}\omega_0^1 &= \widetilde{D}_u(\mu^1) - f'\widetilde{D}_x(\eta) = \frac{\partial \mu^1}{\partial u} + f'\frac{\partial \mu^1}{\partial f} - f'\frac{\partial \eta}{\partial u}\,,\\ \omega_2^2 &= \widetilde{D}_x(\mu^2) - h'\widetilde{D}_x(\xi^2) = \frac{\partial \mu^2}{\partial x} + h'\frac{\partial \mu^2}{\partial h} - h'\frac{\partial \xi^2}{\partial x}\,.\end{aligned} \tag{4.16}$$

Then we insert the information (4.14) in the prolongation formulae (4.7) and obtain:

$$\begin{aligned}\zeta_1 &= \eta' u_t - u_t(\xi^1_t + u_t\xi^1_u)\,,\\ \zeta_2 &= \eta' u_x - u_t(\xi^1_x + u_x\xi^1_u) - u_x\xi^2_x\,,\\ \zeta_{22} &= \eta' u_{xx} + \eta'' u_x^2 - u_t(\xi^1_{xx} + 2u_x\xi^1_{xu} + \xi^1_{uu}u_x^2 + \xi^1_u u_{xx})\,,\\ &\quad - u_x\xi^2_{xx} - 2u_{tx}(\xi^1_x + u_x\xi^1_u) - 2u_{xx}\xi^2_x\,.\end{aligned} \tag{4.17}$$

We substitute the expressions (4.16)-(4.17) in the determining equation (4.15) and first nullify the term with u_{tx} :

$$-2u_{tx}(\xi^1_x + u_x\xi^1_u) = 0.$$

This yields the equations

$$\xi^1_x = \xi^1_u = 0.$$

Hence,

$$\xi^1 = \xi^1(t). \tag{4.18}$$

Now we substitute $\xi^1_x = \xi^1_u = 0$ in Eqs. (4.16) - (4.17) and obtain:

$$\omega^1_o = \mu^1_u + [\mu^1_f - \eta'(u)]f_u, \quad \omega^2_2 = \mu^2_x + [\mu^2_h - \xi^2_x]h_x\,; \tag{4.19}$$

$$\begin{aligned}\zeta_1 &= [\eta'(u) - \xi^1_t]u_t, \quad \zeta_2 = \eta'(u)u_x - \xi^2_x u_x,\\ \zeta_{22} &= [\eta'(u) - 2\xi^2_x]u_{xx} + \eta''(u)u_x^2 - \xi^2_{xx}u_x.\end{aligned} \tag{4.20}$$

Upon inserting these expressions in the determining equation (4.15), we obtain

$$\begin{aligned}& -\eta' u_t + h[\eta' u_{xx} + \eta'' u_x^2 - \xi^2_{xx}u_x - 2\xi^2_x u_{xx} + f'\eta' u_x -\\ & f'\xi^2_x u_x + \mu'_u u_x + (\mu'_f - \eta')f' u_x] + [u_{xx} + f' u_x]\mu^2 +\\ & 2h'[\mu' + \eta' u_x] + 2(u_x + f)\cdot[\mu^2_x + (\mu^2_h - \xi^2_x)h'] = 0.\end{aligned} \tag{4.21}$$

We replace u_t with its expression given by Eq. (4.1) and write the first term in Eq. (4.21) in the form

$$-\eta'[hu_{xx} + hf_u u_x + 2h_x u_x + 2h_x f].$$

Applying the usual procedure of solving determining equations to the resulting determining equation (4.21), we arrive at the following general solution of the determining equations (4.5)-(4.6):

$$\begin{aligned}\xi^1 &= C_1 + C_2 t, \quad \xi^2 = C_3 + C_4 x, \quad \eta = C_5 + C_6 u,\\ \mu^1 &= (C_6 - C_4)f, \quad \mu^2 = (2C_4 - C_2)h,\end{aligned} \tag{4.22}$$

where $C_1, C_2, C_3, C_4, C_5, C_6$ are arbitrary constants. The general solution of the determining equations provides the following equivalence generator for the generalized Kompaneets equation (3.3):

$$Y = C_1 Y_1 + C_2 Y_2 + \cdots + C_6 Y_6. \tag{4.23}$$

Thus, the equivalence Lie algebra of the generalized Kompaneets equation (3.3) is spanned by the operators

$$
\begin{aligned}
&Y_1 = \frac{\partial}{\partial t}, \quad Y_2 = t\frac{\partial}{\partial t} - h\frac{\partial}{\partial h},\\
&Y_3 = \frac{\partial}{\partial x}, \quad Y_4 = x\frac{\partial}{\partial x} - f\frac{\partial}{\partial f} + 2h\frac{\partial}{\partial h},\\
&Y_5 = \frac{\partial}{\partial u}, \quad Y_6 = u\frac{\partial}{\partial u} + f\frac{\partial}{\partial f}\cdot
\end{aligned}
\tag{4.24}
$$

5. Models with two symmetries

Consider the following projections X and Z of the equivalence generator (4.23):

$$
\begin{aligned}
&\mathrm{pr}_{(t,x,u)}(Y) = X \equiv \xi^1\frac{\partial}{\partial t} + \xi^2\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial u},\\
&\mathrm{pr}_{(x,u,f,h)}(Y) = Z \equiv \xi^2\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial u} + \mu^1\frac{\partial}{\partial f} + \mu^2\frac{\partial}{\partial h}\cdot
\end{aligned}
\tag{5.1}
$$

We will use the theorem on projections (see Paper 3 in [6]). In our case, the theorem states that if the equations

$$f = F(u), \quad h = H(x) \tag{5.2}$$

are invariant with respect to a group with the generator Z, then the corresponding equation (3.3) admits the group with the generator X.

Example 1. *To illustrate the method consider the simple example based on the operator Y_6 from (4.24). Eqs (5.1) give*

$$X = u\frac{\partial}{\partial u}, \quad Z = Y_6 = u\frac{\partial}{\partial u} + f\frac{\partial}{\partial f}\cdot \tag{5.3}$$

The invariance conditions for Eqs. (5.2) with respect to Z have the form

$$[Z(f - F(u))]_{f=F(u)} = 0, \quad [Z(h - H(x))]_{h=H(x)} = 0. \tag{5.4}$$

In our case they provide one equation:

$$F - u\frac{dF}{du} = 0,$$

whence, $F = ku$, $k =$const., and the function $h(x)$ is arbitrary. Hence, the equation

$$u_t = h(x)\big(u_{xx} + ku_x\big) + 2h'(x)\big(u_x + ku\big) \tag{5.5}$$

admits, along with (2.3), the additional operator (see (5.3))

$$X = u\frac{\partial}{\partial u}\cdot \tag{5.6}$$

Example 2. *Let us find the model based on the operator*

$$Y = Y_4 - Y_6 - \varepsilon Y_5 = x\frac{\partial}{\partial x} - (u + \varepsilon)\frac{\partial}{\partial u} - 2f\frac{\partial}{\partial f} + 2h\frac{\partial}{\partial h}\cdot \tag{5.7}$$

Its projection X defined by (5.1) is

$$X = x\frac{\partial}{\partial x} - (u + \varepsilon)\frac{\partial}{\partial u} \tag{5.8}$$

whereas the projection Z is identical to the operator (5.7).

The invariance conditions (5.4) are written

$$-2F + (u + \epsilon)\frac{dF}{du} = 0, \quad 2H - x\frac{dH}{dx} = 0$$

and yield

$$f = C(u + \epsilon)^2, \quad h = Kx^2.$$

Let us take $C = 1$, $K = 1$ for the sake of simplicity. Thus, the theorem on projections guarantees that the equation

$$u_t = x^2[u_{xx} + 2(u+\varepsilon)u_x] + 4x[u_x + (u+\varepsilon)^2] \tag{5.9}$$

admits the two-dimensional Lie algebra spanned by

$$X_1 = \frac{\partial}{\partial t}, \quad X_2 = x\frac{\partial}{\partial x} - (u+\varepsilon)\frac{\partial}{\partial u} \cdot \tag{5.10}$$

Let us construct the invariant solution with respect to X_2. The invariants for X_2 are determined by the equation

$$X_2 J(t,x,u) = 0,$$

which is written

$$x\frac{\partial J}{\partial x} - (u+\varepsilon)\frac{\partial J}{\partial u} = 0.$$

One solution is

$$J_1 = t.$$

By solving the characteristic equation

$$\frac{dx}{x} + \frac{du}{u+\varepsilon} = 0$$

we obtain the second solution

$$J_2 = x(u+\varepsilon).$$

The invariant solution is obtained by letting

$$J_2 = \phi(J_1).$$

In other words, we let

$$x(u+\varepsilon) = \phi(t)$$

or

$$u = -\varepsilon + \frac{\phi(t)}{x} \cdot \tag{5.11}$$

Substituting (5.11) into (5.9), we obtain

$$\phi' = 2(\phi^2 - \phi),$$

whence

$$\phi = \frac{1}{1 - Ce^{2t}} \cdot$$

Substitution of ϕ in Eq. (5.11) yields

$$u = -\varepsilon + \frac{1}{x(1 - Ce^{2t})} \cdot \tag{5.12}$$

6. Conservation laws

Example 3. *Let us construct the conservation law*

$$D_t(C^1) + D_x(C^2) = 0$$

for the generalized Kompaneets equation (3.2) using the symmetry (2.3). In this case $W = -u_t$, the formal Lagrangian is given by (3.4), and the first component of the vector (1.9), due to the substitution (3.6), has the form

$$C^1 = W\frac{\partial \mathcal{L}}{\partial u_t} = -h(x)u_t = -D_x\big\{h^2(x)[u_x + f(u)]\big\}.$$

According to the general theory we can transfer the terms of the form $D_x(\ldots)$ to the component C^2 of the conserved vector. As a result, we obtain the trivial vector $C^1 = C^2 = 0$. Hence, the time translation symmetry (2.3) leads to a trivial conserved vector.

Example 4. *Let us construct the conserved vector for Eq. (5.5) using its additional symmetry (5.6),*

$$X = u\frac{\partial}{\partial u}\cdot$$

In this case $W = u$, *the formal Lagrangian is given by*

$$\mathcal{L} = v[-u_t + h(x)\big(u_{xx} + ku_x\big) + 2h'(x)\big(u_x + ku\big)]$$

and the first component of the vector (1.9), due to the substitution (3.6), is written

$$C^1 = h(x)u.$$

Calculating the second component of the vector (1.9), we arrive at the conservation law

$$D_t[h(x)u] - D_x\big\{h^2(x)[u_x + ku]\big\} = 0. \tag{6.1}$$

Remark 1. *The conservation law (6.1) is valid for Eq. (3.2) with an arbitrary function* $f(u)$:

$$D_t[h(x)u] - D_x\big\{h^2(x)[u_x + f(u)]\big\} = 0. \tag{6.2}$$

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