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ABOUT THE UNIMPROVABILITY OF THE LIMITING EMBEDDING THEOREM FOR DIFFERENT METRICS IN THE LORENTZ SPACES WITH HERMITE'S WEIGHT

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Abstract. In this article we obtained inequality of different metrics in the Lorentz spaces with Hermit's weight for multiple algebraic polynomials. On this basis we established a sufficient condition of embedding of different metrics in the Lorenz spaces with Hermite's weight. Its unimprobality is shown in terms of the "extreme function". Let $f \in L_{p,\theta}(\mathbb{R}_n;\rho_n), 1 \leqslant p < +\infty, 1 \leqslant \theta \leqslant +\infty$. The sequense $\{l_k\}_{k=0}^{+\infty} \subset \mathbb{N}$ is such that $l_0 = 1$ and $l_{k+1} \cdot l_k^{-1} > a_0 > 1$, $\forall k \in \mathbb{Z}^+$. $f(\bar{x}) = \sum\limits_{k=0}^{+\infty} \Delta_{l_k,\ldots,l_k}(f;\bar{x})$ is some presentation of the functions in the metric $L_{p,\theta}(\mathbb{R}_n;\rho_n)$, where $\Delta_{l_0,\ldots,l_0}(f;\bar{x}) = T_{1,\ldots,1}, \Delta_{l_k,\ldots,l_k}(f;\bar{x}) = T_{l_k,\ldots l_k}(\bar{x}) - T_{l_{k-1},\ldots l_{k-1}}(\bar{x}), \forall k \in \mathbb{N}$. Here

$$T_{l_k,\ldots l_k}(\bar{x}) = \sum_{m_1=0}^{l_k-1} \ldots \sum_{m_n=0}^{l_k-1} a_{m_1,\ldots,m_n} \prod_{i=1}^{n} x_i^{m_i} -$$

are algebraic polynomials for all $k \in \mathbb{Z}^+$.

1^0. If the series

$$A(f)_{p\theta} = \sum_{k=0}^{+\infty} l_k^{\tau\left(\frac{n}{2p}-\frac{n}{2q}\right)} \left\|\Delta_{l_k,\ldots,l_k(f)}\right\|_{L_{p,\theta}(\mathbb{R}_n;\rho_n)}^{\tau}$$

converge under some q and τ: $p < q < +\infty$, $0 < \tau < +\infty$, then $f \in L_{q,\tau}(\mathbb{R}_n;\rho_n)$ and we have the inequality

$$\|f\|_{L_{q,\tau}(\mathbb{R}_n;\rho_n)} \leqslant C_{pq\theta\tau n} \times (A(f)_{p\theta})^{\frac{1}{\tau}}.$$

2^0. The condition 1^0 is unimprovable in the sense that there exists a function $f_0 \in L_{p,\theta}(\mathbb{R}_n;\rho_n)$ and $A(f_0)_{p\theta}$ diverges for it and $f_0 \notin L_{q,\tau}(\mathbb{R}_n;\rho_n)$.

At the same time, the function $f_0 \in L_{q-\varepsilon,\tau}(\mathbb{R}_n;\rho_n)$ for all $\varepsilon > 0 : p < (q-\varepsilon) < q$.

Keywords: Lorentz's space, Hermitte's weight, nonincreasing rearrangement, inequality of different metrics, theorem in embedding, non improving.

1. Introduction

The embedding theorem for various metrics in the Lebesgue spaces $L_p[0,2\pi]$, $1 \leqslant p < +\infty$ first appeared in 1958 in terms of inequalities in various metrics between trigonometric best approximations in the work [1] of A.A. Konyushkov.

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Theorem A. Let $f \in L_p[0, 2\pi), 1 \leqslant p < +\infty$.

If the series $\sum\limits_{k=1}^{+\infty} k^{\frac{1}{p}-\frac{1}{q}-1} E_k(f)_p$ converges for some q: $p < q \leqslant +\infty$, then $f \in L_q[0, 2\pi)$ and the inequality:

$$\|f\|_q \leqslant C_{pq} \left\{ \|f\|_p + \sum_{k=1}^{+\infty} k^{\frac{1}{p}-\frac{1}{q}-1} E_k(f)_p \right\}$$

holds. Here $C_{pq} > 0$ depends only on the mentioned parameters.

Later P.L. Ul'yanov in 1968 improved the Konyushkov theorem cited here in terms continuity modules [2], and in 1970 in terms of trigonometric best approximations [3]. Namely, in [3] the following statement is established.

Theorem B. Let $1 \leqslant p < q < +\infty$ and the function $f \in L_p[0, 2\pi)$. Then, the inequality $\|f\|_q \leqslant C_{pq} \left\{ \|f\|_p + \left[\sum\limits_{k=1}^{+\infty} k^{\frac{q}{p}-2} E_k(f)_p \right]^{\frac{1}{q}} \right\}$ holds.

Here C_{pq} depends only on the indicated parameters.

P.L. Ul'yanov demonstrated unimprovability of the embedding theorem, which he established in terms of continuity moduli in terms of the class H_p^ω. The unimprovability of Theorem B was established by V.I. Kolyada [4] in terms of the class $E_p(\lambda)$. Classes H_p^ω and $E_p(\lambda)$, where the unimprovability of P.L. Ul'yanov sufficient embedding conditions are indicated, are sufficiently narrow classes that are determined by a given majorant on the continuity module and on trigonometric best approximation of the functions $f \in L_p[0, 2\pi)$. Since the set of functions from $L_p[0, 2\pi)$, satisfying the sufficient embedding condition of P.L. Ual'yanov, is significantly wider than the indicated classes, we believe it is natural to demonstrate the unimprovability of the sufficient embedding condition of various metrics by means of the "extreme function". Namely, to construct a test function $f_0 \in L_p[0, 2\pi)$, $1 \leqslant p < q < +\infty$ such that it does not satisfy the condition of Theorem B and $f_0 \notin L_q[0, 2\pi)$, whereas for any indefinitely small $\varepsilon > 0$, $f_0 \in L_{q-\varepsilon}[0, 2\pi)$. Since the works by P.L. Ul'yanov appeared, this topic have been developing in various directions. In the present paper, we prove the B type theorem in the Lorenz space with Hermitte's weight $L_{p\theta}(\mathbb{R}_n; \rho_n)$. This space is quite a wide class of functions with elements possibly tending to infinity, quicker than any algebraic polynomial of many variables when $|\overline{x}| = \left\{ \sum\limits_{k=1}^{n} x_k^2 \right\}^{\frac{1}{2}} \to +\infty$. We also demonstrate the unimprovability of the theorem we established by means of the extreme function principle.

2. Definition and auxiliary assumptions

Let us assume that $1 \leqslant p < +\infty$, $0 < \theta \leqslant +\infty$ and $f(\bar{x})$ is a function measurable in the Lebesgue sense on $\mathbb{R}_n$; $\rho_n(\bar{x}) = e^{-\frac{|\bar{x}|^2}{2}}$, $\bar{x} \in \mathbb{R}_n$; $|\bar{x}| = \left(\sum\limits_{k=1}^{n} x_k^2 \right)^{\frac{1}{2}}$, $d\bar{x} = dx_1, \dots\ dx_n$.

Denote by $F(|f\rho_n|; t)$ a nonincreasing rearrangement of the functions $|f(\bar{x})\rho_n(\bar{x})|$ on $\mathbb{R}_n$, $t \in [0; +\infty)$.

We consider that $f \in L_{p,\theta}(\mathbb{R}_n; \rho_n)$, [5], if the following value is finite:

$$\|f\|_{L_{p,\theta}(\mathbb{R}_n;\rho_n)} = \left\{ \frac{\theta}{p} \int_0^{+\infty} t^{\frac{\theta}{p}-1} (F(|f\rho_n|; t))^\theta dt \right\}^{\frac{1}{\theta}}, \text{for} \quad 0 < \theta < +\infty,$$

$$\|f\|_{L_{p\infty}(\mathbb{R}_n;\rho_n)} = \sup_{t>0} \left\{ t^{\frac{1}{p}} F(|f\rho_n|; t) \right\}, \text{for} \quad \theta = +\infty.$$

Let

$$P_{m_1,\dots,m_n}(\bar{x}) = \sum_{k_1=0}^{m_1-1} \dots \sum_{k_n=0}^{m_n-1} a_{k_1,\dots,k_n} \prod_{i=1}^{n} x_i^{k_i}$$

be an algebraic polynomial of the order $(m_{k_i} - 1)$ with respect to the variable $x_i, m_{k_i} \in \mathbb{N}, i = 1, \dots, n$.

Let us introduce the notation $\Delta_{1,...,1}(\overline{x}) = P_{1,...,1},\ P_{1,...,1} \in \mathbb{R}$ and

$$\Delta_{m_k,...,m_k}(\overline{x}) = P_{m_k,...,m_k}(\overline{x}) - P_{m_{k-1},...,m_{k-1}}(\overline{x}), k \in \mathbb{N}.$$

Lemma 1. Let $0 < p < q \leqslant +\infty,\ 0 < \theta \leqslant +\infty, 0 < \tau \leqslant +\infty$. For an algebraic polynomial $P_{m_1,...,m_n}(\bar{x})$, the following inequalities of various metrics hold:

$$\max_{\bar{x}\in\mathbb{R}_n} |P_{\bar{m}}(\bar{x})\rho_n(\bar{x})| \leqslant C_{pn} \prod_{k=1}^{n} m_k^{\frac{1}{2p}} \|P_{\bar{m}}\|_{L_{p\theta}(\mathbb{R}_n;\rho_n)},$$

$$\|P_{\overline{m}}\|_{L_{q,\tau}(\mathbb{R}_n;\rho_n)} \leqslant A_{pqn} \prod_{k=1}^{n} m_k^{\frac{1}{2p}-\frac{1}{2q}} \|P_{\overline{m}}\|_{L_{p,\theta}(\mathbb{R}_n;\rho_n)},$$

where the factors $C_{pn} > 0$, $A_{pqn} > 0$ depend only on the above parameters and $\overline{m} = (m_1, ..., m_n)$.

Proof. Since $\rho_n(\bar{x}) = e^{-\frac{|\bar{x}|^2}{2}}$, then $\lim\limits_{|\bar{x}|\to+\infty} |P_{\bar{m}}(\bar{x})\rho_n(\bar{x})| = 0$. Therefore, $M = \max\limits_{\bar{x}\in\mathbb{R}_n} |P_{\bar{m}}(\bar{x})\rho_n(\bar{x})|$ is reached at some point $\bar{x}_0 = (x_1^0, ..., x_n^0)$ with finite coordinates: $|P_{\bar{m}}(\bar{x}_0)\rho_n(x_0)| = M$.

Let $\bar{x} \in \mathbb{R}_n$, then $|\Delta\bar{x}_0| = \left(\sum\limits_{k=1}^{n} (x_k - x_k^0)\right)^{\frac{1}{2}}$.

$$|P_{\bar{m}}(\bar{x})\rho_n(\bar{x})| \geq |P_{\bar{m}}(\bar{x}_0)\rho_n(\bar{x})| - |(P_{\bar{m}}(\bar{x}_0) - P_{\bar{m}}(\bar{x}))\,\rho_n(\bar{x})|\,. \tag{1}$$

Since $\rho_n(\bar{x}) \neq 0$, $\forall\bar{x} \in \mathbb{R}_n$, then

$$|(P_{\bar{m}}(\bar{x}_0) - P_{\bar{m}}(\bar{x}))\,\rho_n(\bar{x})| = \left|\left[\left(\sum_{k=1}^{n} \frac{\partial P_{\bar{m}}(\bar{x}_0)}{\partial x_k}\Delta x_k^0\right) + o\left(\Delta x_k^0\right)\right] \times\right.$$

$$\left.\times\ \rho_n(\bar{x}_0)\cdot\frac{\rho_n(\bar{x})}{\rho_n(\bar{x}_0)}\right| \leqslant$$

$$\leqslant \sum_{k=1}^{n}\left|\frac{\partial P_{\bar{m}}(\bar{x}_0)}{\partial x_k}\rho_n(\bar{x}_0)\right| \cdot \frac{\rho_n(\bar{x})}{\rho_n(\bar{x}_0)}\,|\Delta\bar{x}_0| + o\left(\Delta\bar{x}_0\right)\rho_n(\bar{x}). \tag{2}$$

Here $\Delta x_k^0 = x_k - x_k^0$, $k = 1, ..., n$.

Let us enumerate the necessary properties of functions $\rho_n(\bar{x})$:

a) $0 \leqslant \rho_n(\bar{x}) \leqslant 1$, $\forall\bar{x} \in \mathbb{R}_n$;

b) $\rho_n(\bar{x}_0) \neq 0$;

c) $\rho_n(\bar{x}) \in C(\mathbb{R}_n)$ and $\left.\frac{\rho_n(\bar{x})}{\rho_n(\bar{x}_0)}\right|_{\bar{x}=\bar{x}_0} = 1$.

Hence, $\forall\varepsilon > 0\ \exists\delta_\varepsilon > 0$ such that $\forall\bar{x} \in U_{\delta_\varepsilon}(\bar{x}_0) = \{\bar{x} \in \mathbb{R}_n : |\bar{x} - \bar{x}_0| < \delta_\varepsilon\}$ the inequalities $(1-\varepsilon) < \frac{\rho_n(\bar{x})}{\rho_n(\bar{x}_0)} < (1+\varepsilon)$ hold. Let us assume that $\varepsilon = \frac{1}{2}$ then,

$$|P_{\bar{m}}(\bar{x}_0)\rho_n(\bar{x})| \geq |P_{\bar{m}}(\bar{x}_0)\rho_n(\bar{x}_0)| \cdot \frac{1}{2} = \frac{M}{2}, \quad \forall\bar{x} \in U_{\frac{1}{2}}(\bar{x}_0). \tag{3}$$

Let $0 < \delta' \leqslant \delta_{\frac{1}{2}}$. According to [5],

$$\left|\frac{\partial P_{\bar{m}}(\bar{x}_0)}{\partial x_k}\rho_n(\bar{x}_0)\right| \leqslant C\sqrt{m_k}\,|P_{\bar{m}}(\bar{x}_0)\rho_n(\bar{x}_0)|\,, \quad k = 1, ..., n$$

as for a polynomial of the variable x_k when the remaining variables are fixed.

Then, the inequality (2) can be extended as follows:

$$|(P_{\bar{m}}(\bar{x}_0) - P_{\bar{m}}(\bar{x}))\,\rho_n(\bar{x})| \leqslant \sum_{k=1}^{n} C\sqrt{m_k}\,|P_{\bar{m}}(\bar{x}_0)|\,\frac{3}{2}|\Delta\bar{x}_0| +$$

$$+o(|\Delta\bar{x}_0|) \leqslant \frac{3}{2}C\cdot M\sum_{k=1}^{n}\sqrt{m_k}\cdot\delta' + o(|\Delta\bar{x}_0|).$$

Let us assume that $\delta' = \min\left\{\delta_{\frac{1}{2}}, \frac{1}{9C\cdot\sum_{k=1}^{n}\sqrt{m_k}}\right\}$ then,

$$|(P_{\bar{m}}(\bar{x}_0) - P_{\bar{m}}(\bar{x}))\,\rho_n(\bar{x})| < \frac{3}{2}C\cdot M\sum_{k=1}^{n}\sqrt{m_k}\cdot\frac{1}{9C\cdot\sum_{k=1}^{n}\sqrt{m_k}} + o(|\Delta\bar{x}_0|) =$$

$$= \frac{M}{6} + o(|\Delta\bar{x}_0|).$$

Since the addend $o(|\Delta\bar{x}_0|)$ is an infinitely small value when $|\Delta\bar{x}_0| \to 0$, there is a number $\delta_0 > 0$: $0 < \delta_0 < \delta'$ such that $o(|\Delta\bar{x}_0|) \leqslant \frac{M}{12}$, $\forall\bar{x} \in U_{\delta_0}(\bar{x}_0$. Thus, $\forall\bar{x} \in U_{\delta_0}(\bar{x}_0)$:

$$|(P_{\bar{m}}(\bar{x}_0) - P_{\bar{m}}(\bar{x}))\,\rho_n(\bar{x})| < \frac{M}{4}. \tag{4}$$

Then, the inequalities (1), (3), (4) $\forall\bar{x} \in U_{\delta_0}(\bar{x}_0)$ provide: $|P_{\bar{m}}(\bar{x})\rho_n(\bar{x})| \geq \frac{M}{4}$.

Hence, the nonincreasing rearrangement of functions $|P_{\bar{m}}(\bar{x})\rho_n(\bar{x})|$ on the interval $\Delta = [0, mes\,(U_{\delta_0}(\bar{x}_0))]$ has the estimate

$$\frac{M}{4} \leqslant F\left(|P_{\bar{m}}\rho_n|\,; t\right) \leqslant M,$$

where $mes\,(U_{\delta_0}(\bar{x}_0)) = \frac{\pi^{n/2}}{\Gamma\left(\frac{n}{2}+1\right)}\cdot\delta_0^n$.

Let $\alpha_n \in (0, 1]$ be such a number that

$$0 < \frac{\alpha_n}{9C\sum_{k=1}^{n}\sqrt{m_k}} \leqslant \frac{\pi^{n/2}}{\Gamma\left(\frac{n}{2}+1\right)}\cdot\delta_0^n.$$

Then, $\Delta' = \left[0, \frac{\alpha_n}{9C\sum_{k=1}^{n}\sqrt{m_k}}\right] \subset \left[0, \frac{\pi^{n/2}}{\Gamma\left(\frac{n}{2}+1\right)}\cdot\delta_0^n\right]$. Therefore, $\forall t \in \Delta'$ we have:

$$M = 4\cdot\frac{M}{4}\cdot\left(\frac{9C\sum_{k=1}^{n}\sqrt{m_k}}{\alpha_n}\right)^{\frac{1}{p}}\cdot\left\{\frac{\theta}{p}\int_{\Delta'} t^{\frac{\theta}{p}-1}dt\right\}^{\frac{1}{\theta}} \leqslant$$

$$\leqslant 4\cdot\left(9C\alpha_n^{-1}\right)^{\frac{1}{p}}\cdot\left(2\prod_{k=1}^{n}\sqrt{m_k}\right)^{\frac{1}{p}}\left\{\frac{\theta}{p}\int_{\Delta'} t^{\frac{\theta}{p}-1}\left(F\left(|P_{\bar{m}}\rho_n|\,; t\right)\right)^{\theta}dt\right\}^{\frac{1}{\theta}} \leqslant$$

$$\leqslant 4\cdot\left(18C\alpha_n^{-1}\right)^{\frac{1}{p}}\prod_{k=1}^{n}m_k^{\frac{1}{2p}}\left\{\frac{\theta}{p}\int_{0}^{+\infty} t^{\frac{\theta}{p}-1}\left(F\left(|P_{\bar{m}}\rho_n|\,; t\right)\right)^{\theta}dt\right\}^{\frac{1}{\theta}}.$$

Thus,

$$\max_{\bar{x}\in\mathbb{R}_n}|P_{\bar{m}}(\bar{x})\rho_n(\bar{x})| \leqslant C_{pn}\prod_{k=1}^{n}m_k^{\frac{1}{2p}}\|P_{\bar{m}}\|_{L_{p\theta}(\mathbb{R}_n;\rho_n)}, 0 < p < +\infty, 0 < \theta \leqslant +\infty. \tag{5}$$

We could write $\theta = +\infty$ here because the constant involved in the inequality is independent of θ therefore, we can turn to the limit when $\theta \to +\infty$.

Now let $0 < q < +\infty$, $0 < \tau < +\infty$ and $a_n = \left(\prod_{k=1}^{n}\sqrt{m_k}\right)^{-1}$.

$$\|P_{\bar{m}}\|_{L_{q\tau}(\mathbb{R}_n;\rho_n)} = \frac{\tau}{q}\int_{0}^{a_n} t^{\frac{\tau}{q}-1}\left(F\left(|P_{\bar{m}}\rho_n|\,; t\right)\right)^{\tau}dt +$$

$$+\frac{\tau}{q}\int_{a_n}^{+\infty} t^{\frac{\tau}{q}-1}\left(F\left(|P_{\bar{m}}\rho_n|\,; t\right)\right)^{\tau}dt = J_1 + J_2. \tag{6}$$

$$J_1 \leqslant M^{\tau}\frac{\tau}{q}\int_{0}^{a_n} t^{\frac{\tau}{q}-1}dt = M^{\tau}\left(\prod_{k=1}^{n}m_k\right)^{-\frac{\tau}{2q}} \leqslant (5) \leqslant$$

$$\leqslant C_{pn}^{\tau}\left(\prod_{k=1}^{n}m_k\right)^{\frac{\tau}{2p}-\frac{\tau}{2q}}\|P_{\bar{m}}\|_{L_{p\theta}(\mathbb{R}_n;\rho_n)}^{\tau}. \tag{7}$$

Furthermore, for any $t > 0$:

$$t^{\frac{1}{p}} F\left(|P_{\bar{m}}\rho_n|\,;t\right) = F\left(|P_{\bar{m}}\rho_n|\,;t\right)\left\{\frac{\theta}{p}\int_0^t u^{\frac{\theta}{p}-1} du\right\}^{\frac{1}{\theta}} \leqslant$$

$$\leqslant \left\{\frac{\theta}{p}\int_0^t u^{\frac{\theta}{p}-1}\left(F\left(|P_{\bar{m}}\rho_n|\,;t\right)\right)^{\theta} du\right\}^{\frac{1}{\theta}} \leqslant \|P_{\bar{m}}\|_{L_{p\theta}(\mathbb{R}_n;\rho_n)}. \quad (8)$$

$$J_2 = \left(\sup_{t\geq 0} t^{\frac{1}{p}} F\left(|P_{\bar{m}}\rho_n|\,;t\right)\right)^{\tau} \cdot \frac{\tau}{q}\int_{a_n}^{+\infty} t^{\frac{\tau}{q}-\frac{\tau}{p}-1} dt \leqslant (8) \leqslant$$

$$\leqslant C_{pq}^{\tau} \cdot \|P_{\bar{m}}\|_{L_{p\theta}(\mathbb{R}_n;\rho_n)}^{\tau} \cdot a_n^{-\left(\frac{\tau}{p}-\frac{\tau}{q}\right)} =$$

$$= C_{pq}^{\tau} \cdot \left(\sum_{k=1}^{n} \sqrt{m_k}\right)^{\frac{\tau}{p}-\frac{\tau}{q}} \|P_{\bar{m}}\|_{L_{p\theta}(\mathbb{R}_n;\rho_n)}^{\tau} \leqslant$$

$$\leqslant D_{pqn}^{\tau} \cdot \left(\prod_{k=1}^{n} \sqrt{m_k}\right)^{\frac{\tau}{p}-\frac{\tau}{q}} \|P_{\bar{m}}\|_{L_{p\theta}(\mathbb{R}_n;\rho_n)}^{\tau}. \quad (9)$$

Then, (6), (7), (9) entail that

$$\|P_{\bar{m}}\|_{L_{q\tau}(\mathbb{R}_n;\rho_n)} \leqslant A_{pqn}\prod_{k=1}^{n} m_k^{\frac{1}{2p}-\frac{1}{2q}} \|P_{\bar{m}}\|_{L_{p\theta}(\mathbb{R}_n;\rho_n)},$$

$0 < p < q < +\infty, 0 < \theta \leqslant +\infty, 0 < \tau < +\infty$.

Here, as well as in the case (5), we pass to the limit when $\tau \to +\infty$.

Lemma 2[6]**.** Let $f \in L(\Omega), \Omega \subset \mathbb{R}_n$ and $\alpha \in [0, \mu(\Omega)]$. Then,

$$\sup_{E\subset\Omega}\ \sup_{\mu(E)=\alpha}\left\{\int_E |f(\overline{x})|d\overline{x}\right\} = \int_0^{\alpha} F(|f|;t)dt.$$

Lemma 3[7]**.** Let the sequence $\{\mu(l)\}_{l=0}^{+\infty}$ be such that $\mu(0) = 1$, $\frac{\mu(l+1)}{\mu(l)} \geq \alpha > 1$, $\forall l \in \mathbb{Z}^+$ then, the inequalities

$$\sum_{l=0}^{+\infty} \mu(l)^r \left(\sum_{k=0}^{l} a_k\right)^q \leqslant c_1 \sum_{l=0}^{+\infty} \mu(l)^r a_l^q, r < 0;$$

$$\sum_{l=0}^{+\infty} \mu(l)^r \left(\sum_{k=l}^{+\infty} a_k\right)^q \leqslant c_2 \sum_{l=0}^{+\infty} \mu(l)^r a_l^q, r > 0,$$

where $c_i > 0, i = 1, 2,$ depend only on the parameters α, r, q for the numbers $q > 0$ and $\{a_k\}_{k=0}^{+\infty}, a_k \geq 0$.

Lemma 4. Let $1 < p < +\infty$, $1 \leqslant \theta \leqslant +\infty$. There is a sequence of nonnegative algebraic polynomials $\{P_m^*(x)\}_{m=1}^{+\infty}, x \in \mathbb{R}_1$ of the power not higher than $(m-1)$ such that $C_p' m^{-\frac{1}{2p}} \leqslant \|P_m^*\|_{L_{p,\theta}(\mathbb{R};\rho)} \leqslant C_p'' m^{-\frac{1}{2p}}$, $m \in \mathbb{N}$. Here $\rho(x) = e^{-\frac{x^2}{2}}$, $x \in \mathbb{R}$ and $C_p' > 0, C_p'' > 0$ depend only on the indicated parameters.

Proof. The sequence of nonnegative algebraic polynomials $\{P_m^*(x)\}_{m=1}^{+\infty}$ was constructed in [8] such that $P_m^*(0) = 1$, $A_r' m^{-\frac{1}{2r}} \leqslant \|P_m^*\|_{L_r(\mathbb{R};\rho)} \leqslant A_r'' m^{-\frac{1}{2r}}$, $1 \leqslant r < +\infty$. Let $1 \leqslant r < p < +\infty$ then, by virtue of Lemma 1

$$\|P_m^*\|_{L_{p\theta}(\mathbb{R};\rho)} \leqslant A_{pr} m^{\frac{1}{2r}-\frac{1}{2p}} \|P_m^*\|_{L_r(\mathbb{R};\rho)} \leqslant B_{pr} m^{-\frac{1}{2p}}.$$

If $1 < p < q < +\infty$ then,

$$\|P_m^*\|_{L_{p\theta}(\mathbb{R};\rho)} \geq A_{pq}^{-1} m^{\frac{1}{2q}-\frac{1}{2p}} \|P_m^*\|_{L_q(\mathbb{R};\rho)} \geq C_{pq} m^{-\frac{1}{2p}}.$$

Lemma 5. Let us assume that $1 \leqslant p < q < r \leqslant +\infty$, $1 \leqslant \theta \leqslant +\infty, 1 \leqslant \tau \leqslant +\infty$ and the sequence of positive numbers $\{\mu(l)\}$ satisfying the condition $\mu(0) = 1$ is given

$$\frac{\mu(l+1)}{\mu(l)} \geq \alpha > 1, \forall l \in \mathbb{Z}^+$$

and

$$\psi(\overline{x}) = \sum_{l=0}^{+\infty} \psi_l(\overline{x})$$

in the sense of $L^{loc}(\mathbb{R}_n)$, where $\psi_l(x) \in L_{p\theta}(\mathbb{R}_n;\rho_n) \bigcap L_{r\theta}(\mathbb{R}_n;\rho_n)$. Then, the following inequality holds

$$\|\psi\|_{L_{q\tau}(\mathbb{R}_n;\rho_n)} \leqslant$$

$$\leqslant C_{pq\tau\theta rn} \left\{ \sum_{l=0}^{+\infty} \left[\mu(l)^{\tau\left(\frac{1}{r}-\frac{1}{q}\right)} \|\psi_l\|^{\tau}_{L_{r\theta}(\mathbb{R}_n;\rho_n)} + \mu(l)^{\tau\left(\frac{1}{p}-\frac{1}{q}\right)} \|\psi_l\|^{\tau}_{L_{p\theta}(\mathbb{R}_n;\rho_n)} \right] \right\}.$$

Here $C_{pq\tau\theta rn} > 0$ depends only on the above parameters.

Proof. Applying the Hölder inequality, we obtain:

$$\phi(y) = \int_0^y F(|\psi\rho_n|;t)dt = \int_0^y y^{\frac{1}{p}+\frac{1}{p'}-\frac{1}{\theta}-\frac{1}{\theta'}} F(|\psi\rho_n|;t)dt \leqslant$$

$$\leqslant \left\{ \int_0^y y^{\frac{\theta}{p}-1} (F(|\psi\rho_n|;t))^{\theta} dt \right\}^{\frac{1}{\theta}} \cdot \left\{ \int_0^y t^{\frac{\theta'}{p'}-1} dt \right\}^{\frac{1}{\theta'}} =$$

$$= C_{p\theta} y^{1-\frac{1}{p}} \left\{ \int_0^{+\infty} y^{\frac{\theta}{p}-1} (F(|\psi\rho_n|;t))^{\theta} dt \right\}^{\frac{1}{\theta}} \leqslant C'_{p\theta} y^{1-\frac{1}{p}} \sum_{l=0}^{+\infty} \|\psi_l\|_{L_{p\theta}(\mathbb{R}_n;\rho_n)}. \quad (10)$$

Likewise, in view of Lemma 2, by means of the Hölder inequality $\forall k \in \mathbb{N}$ we obtain:

$$\phi(y) = \int_0^y F(|\psi\rho_n|;t)du = \sup_{E\subset\mathbb{R}_n} \sup_{\mu(E)=y} \int_E |\sum_{l=0}^{+\infty} \psi_l(\overline{x})\rho_n(\overline{x})| d\overline{x} \leqslant$$

$$\leqslant \sup_{E\subset\mathbb{R}_n} \sup_{\mu(E)=y} \int_E \left| \sum_{l=0}^{k} \psi_l(\overline{x})\rho_n(\overline{x}) \right| d\overline{x} + \sup_{E\subset\mathbb{R}_n} \sup_{\mu(E)=y} \int_E \left| \sum_{l=k+1}^{+\infty} \psi_l(\overline{x})\rho_n(\overline{x}) \right| d\overline{x} =$$

$$= \int_0^y F(|\sum_{l=0}^{k} \psi_l\rho_n|;t)dt + \int_0^y F(|\sum_{l=k+1}^{+\infty} \psi_l\rho_n|;t)dt \leqslant$$

$$\leqslant C_{r\theta} y^{1-\frac{1}{r}} \left\{ \int_0^{+\infty} t^{\frac{\theta}{r}-1} (F(|\sum_{l=0}^{k} \psi_l\rho_n|;t))^{\theta} dt \right\}^{\frac{1}{\theta}} +$$

$$+ C_{p\theta} y^{1-\frac{1}{p}} \left\{ \int_0^{+\infty} t^{\frac{\theta}{p}-1} (F(|\sum_{l=k+1}^{+\infty} \psi_l\rho_n|;t))^{\theta} dt \right\}^{\frac{1}{\theta}} \leqslant$$

$$\leqslant C'_{r\theta} y^{1-\frac{1}{r}} \sum_{l=0}^{k} \|\psi_l\|_{L_{r\theta}(\mathbb{R}_n;\rho_n)} + C'_{p\theta} y^{1-\frac{1}{p}} \sum_{l=k+1}^{+\infty} \|\psi_l\|_{L_{p\theta}(\mathbb{R}_n;\rho_n)}. \quad (11)$$

Further,

$$\|\psi\|^{\tau}_{L_{q\tau}(\mathbb{R}_n;\rho_n)} \leqslant$$

$$\leqslant C^{\tau}_{q\tau} \int_0^{+\infty} y^{\frac{\tau}{q}-1} \left[\frac{1}{y} \int_0^y F(|\psi\rho_n|;t)dt \right]^{\tau} dy = C^{\tau}_{q\tau} \int_0^{+\infty} y^{\frac{\tau}{q}-1} \left[\frac{1}{y}\phi(y) \right]^{\tau} dy \leqslant$$

$$\leqslant C^{\tau}_{q\tau} \left\{ \int_0^1 y^{\frac{\tau}{q}-1} \left[\frac{1}{y}\phi(y) \right]^{\tau} dy + \int_1^{+\infty} y^{\frac{\tau}{q}-1} \left[\frac{1}{y}\phi(y) \right]^{\tau} d\tau \right\} = I_1 + I_2.$$

In view of (11), let us estimate I_1:

$$I_1 \leqslant (C''_{rpq\theta})^\tau \sum_{k=0}^{+\infty} \left\{ \int\limits_{\frac{1}{\mu(k+1)}}^{\frac{1}{\mu(k)}} y^{\frac{\tau}{q}-1} \left[\frac{1}{y}\phi(y)\right]^\tau d\tau \right\} \leqslant$$

$$\leqslant (C''_{rpq\theta})^\tau \sum_{k=0}^{+\infty} \int\limits_{\frac{1}{\mu(k+1)}}^{\frac{1}{\mu(k)}} y^{\frac{\tau}{q}-\tau-1} \left[y^{1-\frac{1}{\tau}} \sum_{l=0}^{k} \|\psi_e\|_{L_{p\theta}(\mathbb{R}_n;\rho_n)} + \right.$$

$$\left. + y^{1-\frac{1}{p}} \sum_{l=k+1}^{+\infty} \|\psi_e\|_{L_{p\theta}(\mathbb{R}_n;\rho_n)} \right]^\tau \leqslant$$

$$\leqslant (C'''_{rpq\theta})^\tau \sum_{k=0}^{+\infty} \int\limits_{\frac{1}{\mu(k+1)}}^{\frac{1}{\mu(k)}} y^{\frac{\tau}{q}-\tau-1} \left\{ y^{\tau-\frac{\tau}{r}} \left(\sum_{l=0}^{k} \|\psi_e\|_{L_{r\theta}(\mathbb{R}_n;\rho_n)} \right)^\tau + \right.$$

$$\left. + y^{\tau-\frac{\tau}{p}} \left(\sum_{l=k+1}^{+\infty} \|\psi_e\|_{L_{r\theta}(\mathbb{R}_n;\rho_n)} \right)^\tau \right\} dy \leqslant$$

$$\leqslant (C^{IV}_{rpq\theta})^\tau \left\{ \sum_{k=0}^{+\infty} (\mu(k))^{\tau(\frac{1}{r}-\frac{1}{q})} \left(\sum_{l=0}^{k} \|\psi\|_{L_{r\theta}(\mathbb{R}_n;\rho_n)} \right)^\tau + \right.$$

$$\left. + \sum_{k=0}^{+\infty} (\mu(k+1))^{\tau(\frac{1}{p}-\frac{1}{q})} \left(\sum_{l=k+1}^{+\infty} \|\psi\|_{L_{r\theta}(\mathbb{R}_n;\rho_n)} \right)^\tau \right\} \leqslant (\text{Lemma } 3) \leqslant$$

$$\leqslant (C^{V}_{rpq\theta})^\tau \left\{ \sum_{k=0}^{+\infty} \left[\mu(k)^{\tau(\frac{1}{r}-\frac{1}{q})} \|\psi_k\|^\tau_{L_{r\theta}(\mathbb{R}_n;\rho_n)} + \mu(k)^{\tau(\frac{1}{p}-\frac{1}{q})} \|\psi_k\|^\tau_{L_{p\theta}(\mathbb{R}_n;\rho_n)} \right] \right\}.$$

We estimate the addend I_2 by means of (10):

$$I_2 \leqslant (b'_{rpq\theta})^\tau \int\limits_1^{+\infty} y^{\frac{\tau}{q}-1} \left[\frac{1}{y} \cdot y^{1-\frac{1}{p}} \sum_{l=0}^{+\infty} \|\psi_e\|_{L_{p\theta}(\mathbb{R}_n;\rho_n)} \right]^\tau dy =$$

$$= (1 \leqslant p < q < +\infty) = (b'_{rpq\theta})^\tau \left(\sum_{l=0}^{+\infty} \|\psi_e\|_{L_{p\theta}(\mathbb{R}_n;\rho_n)} \right)^\tau.$$

The conditions, imposed on the sequence of numbers $\{\mu(k)\}$, give us a possibility to make the following calculations:

$$\sum_{l=0}^{+\infty} \|\psi_e\|_{L_{p\theta}(\mathbb{R}_n;\rho_n)} \leqslant \left\{ \sum_{l=0}^{+\infty} (\mu(l))^{\tau(\frac{1}{p}-\frac{1}{q})} \|\psi_e\|^\tau_{L_{p\theta}(\mathbb{R}_n;\rho_n)} \right\}^{\frac{1}{\tau}} \times$$

$$\times \left\{ \sum_{l=0}^{+\infty} (\mu(l))^{-\tau'(\frac{1}{p}-\frac{1}{q})} \right\}^{\frac{1}{\tau'}} = C_{pq\tau} \left\{ \sum_{l=0}^{+\infty} (\mu(l))^{\tau(\frac{1}{p}-\frac{1}{q})} \|\psi_e\|^\tau_{L_{p\theta}(\mathbb{R}_n;\rho_n)} \right\}^{\frac{1}{\tau}}.$$

3. Main results

This section provides a limiting embedding theorem for different metrics in the Lorenz spaces with Hermite's weight and demonstrates the unimprovability of conditions of the given theorem.

Theorem 1. $1 \leqslant p < +\infty$, $1 \leqslant \theta \leqslant +\infty$ and the sequence $\{l_k\}_{k=0}^{+\infty} \subset \mathbb{Z}^+$ is such that $l_0 = 1$, $l_{k+1} \cdot l_k^{-1} \geq a_0 > 1$, $\forall k \in \mathbb{Z}^+$. Let $f \in L_{p\theta}(\mathbb{R}_n; \rho_n)$ and the sequence of algebrac polynomials $\left\{P_{l_k,\ldots,l_k(\bar{x})}\right\}_{k=0}^{+\infty}$ be such that the representation

$$f(\bar{x}) = \sum_{k=0}^{+\infty} \Delta_{l_k,\ldots,l_k}(\bar{x})$$

holds in the metrics of the space $L_{p\theta}(\mathbb{R}_n; \rho_n)$. If the series

$$\sum_{k=0}^{+\infty} l_k^{\tau\left(\frac{n}{2p}-\frac{n}{2q}\right)} \left\|\Delta_{l_k,\ldots,l_k^{(f)}}\right\|^{\tau}_{L_{p,\theta}(\mathbb{R}_n;\rho_n)}$$

converges for some q and τ: $p < q < +\infty$, $1 \leqslant \tau < +\infty$ then, $f \in L_{q,\tau}(\mathbb{R}_n; \rho_n)$ and the inequality

$$\|f\|_{L_{q,\tau}(\mathbb{R}_n;\rho_n)} \leqslant C_{pq\theta\tau n} \left[\sum_{k=0}^{+\infty} l_k^{\tau\left(\frac{n}{2p}-\frac{n}{2q}\right)} \left\|\Delta_{l_k,\ldots,l_k^{(f)}}\right\|^{\tau}_{L_{p,\theta}(\mathbb{R}_n;\rho_n)}\right]^{\frac{1}{\tau}}$$

holds.

Proof. Let us introduce the notation $b_k = l_k^{\frac{n}{2}}$, $k \in \mathbb{Z}^+$. Obviously, $b_0 = 1$ and $\frac{b_{k+1}}{b_k} \geq a_0^{\frac{n}{2}} > 1$, $\forall k \in \mathbb{Z}^+$. Let us apply Lemma 5 to the expansion $f(\bar{x}) = \sum\limits_{k=0}^{+\infty} \Delta_{l_k,\ldots,l_k}$ when $r = +\infty$. Then,

$$\|f\|^{\tau}_{L_{q\tau}(\mathbb{R}_n;\rho_n)} \leqslant$$
$$\leqslant C_{pq\tau\theta n} \left\{\sum_{k=0}^{+\infty} \left[l_k^{-\frac{\tau n}{2q}} \|\Delta_{l_k,\ldots,l_k}\|^{\tau}_{L_{\infty,\theta}(\mathbb{R}_n;\rho_n)} + l_k^{\frac{\tau n}{2}(\frac{1}{p}-\frac{1}{q})} \|\Delta_{l_k,\ldots,l_k}\|^{\tau}_{L_{p\theta}(\mathbb{R}_n;\rho_n)}\right]\right\}.$$

By means of the inequality of different metrics, provided in Lemma 1, the given expression can be extended as follows :

$$\|f\|^{\tau}_{L_{q\tau}(\mathbb{R}_n;\rho_n)} \leqslant C'_{pq\tau\theta n} \left\{\sum_{k=0}^{+\infty} l_k^{\tau\left(\frac{n}{2p}-\frac{n}{2q}\right)} \|\Delta_{l_k,\ldots,l_k}\|^{\tau}_{L_{p\theta}(\mathbb{R}_n;\rho_n)}\right\}.$$

Theorem 2. Let $1 \leqslant p < q < +\infty$, $1 < \theta < +\infty$, $1 \leqslant \tau \leqslant +\infty$ and $f \in L_{p\theta}(\mathbb{R}_n; \rho_n)$, $\{l_k\}_{k=0}^{+\infty} \subset \mathbb{Z}^+$: $l_0 = 1$, $l_{k+1} \cdot l_k^{-1} \geq a_0 > 1$. Let us assume that the sequence of multiple algebraic polynomials $\left\{T_{l_k,\ldots,l_k(\bar{x})}\right\}_{k=0}^{+\infty}$ is such that the equality

$$f(\bar{x}) = \sum_{k=0}^{+\infty} \Delta_{l_k,\ldots,l_k}(\bar{x})$$

holds in the metrics $L_{p\theta}(\mathbb{R}_n; \rho_n)$.

Then, the inequality

$$\|f\|_{L_{p\theta}(\mathbb{R}_n;\rho_n)} \geq A_{pq\theta\tau n} \left\{\sum_{k=0}^{+\infty} l_k^{\theta\left(\frac{n}{2q}-\frac{n}{2p}\right)} \|\Delta_{l_k,\ldots,l_k}\|^{\theta}_{L_{q\tau}(\mathbb{R}_n;\rho_n)}\right\}^{\frac{1}{\theta}}$$

holds. Here $A_{pq\theta\tau n} > 0$ depends only on the above parameters.

Proof. Let $p+p' = pp'$, $\theta+\theta' = \theta\theta'$ and $g \in L_{p'\theta'}(\mathbb{R}_n; \rho_n)$, and the sequence of algebraic polynomials $\{\phi_{l_m,\ldots,l_m}\}_{m=0}^{+\infty}$ be a sequence of polynomials of the best approximation in the metrics $L_{p'\theta'}(\mathbb{R}_n; \rho_n)$ for it:

$$g(\bar{x}) \sim \phi_{1,\ldots,1} + \sum_{m=1}^{+\infty} \left(\phi_{l_m,\ldots,l_m}(\bar{x}) - \phi_{l_{m-1},\ldots,l_{m-1}}(\bar{x})\right) = \sum_{m=0}^{+\infty} \Delta_{l_m,\ldots,l_m}(g; \bar{x}).$$

Since

$$\int\limits_{R_n} f(\overline{x})g(\overline{x})\rho_n^2(\overline{x})d\overline{x} \leqslant \|f\|_{L_{p\theta}(\mathbb{R}_n;\rho_n)} \cdot \|g\|_{L_{p'\theta'}(\mathbb{R}_n;\rho_n)}$$

then,

$$\|f\|_{L_{p\theta}(\mathbb{R}_n;\rho_n)} \geq \sup\Bigg\{\int\limits_{R_n} f(\overline{x})g(\overline{x})\rho_n^2(\overline{x})d\overline{x}$$

$$\Bigg| \sup \text{taken with respect to all} \quad g \in L_{p'\theta'}(\mathbb{R}_n;\rho_n) \quad \text{such that} \quad \|g\|_{L_{p'\theta'}(\mathbb{R}_n;\rho_n)} \leqslant 1 \Bigg\} =$$

$$= \sup\left\{\int\limits_{R_n}\left(\sum_{m=0}^{+\infty}\Delta_{l_k,\ldots,l_k}(f;\overline{x})\right)\cdot\left(\sum_{m=0}^{+\infty}\Delta_{l_m,\ldots,l_m}(g;\overline{x})\right)\rho_n^2(\overline{x})dx\right.$$

$$\Bigg| \sup \text{taken with respect to all} \quad g \in L_{p'\theta'}(\mathbb{R}_n;\rho_n) \quad \text{such that} \quad \|g\|_{L_{p'\theta'}(\mathbb{R}_n;\rho_n)} \leqslant 1$$

$$\left.\text{and} \int\limits_{R_n} \Delta_{l_k,\ldots,l_k}(f;\overline{x})\cdot\Delta_{l_m,\ldots,l_m}(g;\overline{x})\rho_n^2(\overline{x})d\overline{x} = 0, k \neq m\right\} =$$

$$= \frac{1}{C_{q'p'\theta'\tau'n}} \sup\left\{\frac{1}{\lambda_1}\cdot C_{q'p'\theta'\tau'n}\cdot\pi^{\frac{n}{2}}\cdot T_{1,\ldots,1}\cdot\phi_{1,\ldots,1}\cdot\lambda_1 + C_{q'p'\theta'\tau'n}\sum_{k=1}^{+\infty}\lambda_k\times\right.$$

$$\times\int\limits_{R_n}\Delta_{l_k,\ldots,l_k}(f;\overline{x})\cdot\Delta_{l_k,\ldots,l_k}(g;\overline{x})\frac{1}{\lambda_{l_k}}\rho_n^2(\overline{x})d\overline{x}\Bigg| \sup \text{taken with respect to all possible g},$$

$$\{\lambda_{l_k}\}_{k=0}^{+\infty} : \texttt{and)}\, C_{q'p'\tau'\theta'n}\pi^{\frac{n}{2}}\,|\phi_{1,\ldots,1}| \leqslant \lambda_1;$$

$$\texttt{b)}\, C_{q'p'\tau'\theta'n}\|\Delta_{l_k,\ldots,l_k}(g)\|_{L_{q'\tau'}(\mathbb{R}_n;\rho_n)} \leqslant \lambda_{l_k}, \forall k \in N;$$

$$\left.\texttt{in)}\left[\sum_{k=0}^{+\infty} l_k^{\theta'(\frac{n}{2q'}-\frac{n}{2p'})}\lambda_{l_k}^{\theta'}\right]^{\frac{1}{\theta'}} \leqslant 1\right\} =$$

$$= C^{-1}_{q'p'\tau'\theta'n}\sup\left\{\sum_{k=1}^{+\infty}\lambda_k\|\Delta_{l_k,\ldots,k}(f)\|_{L_{q\tau}(\mathbb{R}_n;\rho_n)}\right| \sup \texttt{taken with respect to}$$

$$\left.\texttt{all possible}\,\{\lambda_{l_k}\}_{k=0}^{+\infty} : \left\{\sum_{k=0}^{+\infty} l_n^{\theta'(\frac{n}{2q'}-\frac{n}{2p'})}\lambda_{l_k}^{\theta'}\right\}^{\frac{1}{\theta'}} \leqslant 1\right\} =$$

$$= C^{-1}_{q'p'\tau'\theta'n}\sup\left\{\sum_{k=0}^{+\infty} l_k^{(\frac{n}{2q'}-\frac{n}{2p'})}\cdot\lambda_{l_k}\|\Delta_{l_k,\ldots,k}(f)\|_{L_{q\tau}(\mathbb{R}_n;\rho_n)}\cdot l_k^{(\frac{n}{2q}-\frac{n}{2p})}\right| \sup \texttt{taken}$$

$$\left.\texttt{with respect to all possibe}\,\{\lambda_{l_k}\}_{k=0}^{+\infty} : \left\{\sum_{k=0}^{+\infty} l_k^{\theta'(\frac{n}{2q'}-\frac{n}{2p'})}\lambda_{l_k}^{\theta'}\right\}^{\frac{1}{\theta'}} \leqslant 1\right\} =$$

$$= C^{-1}_{q'p'\tau'\theta'n}\left\{\sum_{k=0}^{+\infty} l_k^{\theta(\frac{n}{2q}-\frac{n}{2p})}\|\Delta_{l_k,\ldots,k}(f)\|^{\theta}_{L_{q\tau}(\mathbb{R}_n;\rho_n)}\right\}^{\frac{1}{\theta}},$$

which was to be proved.

On the third unit of inequalities, we took into account the validity of the inequality:

$$\|g\|_{L_{p'\theta'}(\mathbb{R}_n;\rho_n)} \leqslant C_{q'p'\tau'\theta'n}\left[|\phi_{1,\ldots,1}|^{\theta'} + \sum_{k=1}^{+\infty} l_k^{\theta'(\frac{n}{2q'}-\frac{n}{2p'})}\|\Delta_{l_k,\ldots,k}\|_{L_{q'\tau'}(\mathbb{R}_n;\rho_n)}\right]^{\frac{1}{\theta'}} \leqslant$$

$$\leqslant \left[\sum_{k=0}^{+\infty} l_k^{\theta'(\frac{n}{2q'}-\frac{n}{2p'})} \lambda_{l_k}^{\theta'}\right]^{\frac{1}{\theta'}} \leqslant 1.$$

Theorem 3. Let $1 < p < q < +\infty$, $1 \leqslant \theta \leqslant +\infty$, $1 \leqslant \tau < +\infty$ and $\{l_k\}_{k=0}^{+\infty} \subset \mathbb{Z}^+$ be such that $l_0 = 1$, $l_{k+1} \cdot l_k^{-1} \geq a_0 > 1, \forall k \in \mathbb{Z}^+$. Theorem 1 is unimprovable in the sense that there is a function $f_0 \in L_{p,\theta}(\mathbb{R}_n; \rho_n)$ for which the series

$$\sum_{k=0}^{+\infty} l_k^{\tau\left(\frac{n}{2p}-\frac{n}{2q}\right)} \|\Delta_{l_k,\dots,l_k}(f_0)\|^{\tau}_{L_{p,\theta}(\mathbb{R}_n;\rho_n)}$$

diverges and $f_0 \notin L_{q,\tau}(\mathbb{R}_n; \rho_n)$, but for any positive number $\varepsilon > 0$: $p < (q-\varepsilon) < q$ the function $f_0 \in L_{q-\varepsilon,\tau}(\mathbb{R}_n; \rho_n)$.

Proof. Consider the series

$$\sum_{k=0}^{+\infty} l_k^{\frac{n}{2q}} \prod_{i=1}^{n} P^*_{l_k}(x_i),$$

where the polynomials $P^*_{l_k}(x_i)$ are from Lemma 4.

By means of Lemma 4, we obtain

$$\left\|\sum_{k=M}^{N} l_k^{\frac{n}{2q}} \prod_{i=1}^{n} P^*_{l_k}(x_i)\right\|_{L_{p,\theta}(\mathbb{R}_n;\rho_n)} \leqslant$$

$$\leqslant \sum_{k=M}^{N} l_k^{\frac{n}{2q}} \|P^*_{l_k}\|^n_{L_{p,\theta}(\mathbb{R}_n;\rho_n)} \leqslant (C''_{p\theta})^n \sum_{k=M}^{N} l_k^{-(\frac{n}{2p}-\frac{n}{2q})} \longrightarrow 0,$$

and $min(N, M) \longrightarrow +\infty$.

Whence, there is a function $f_0 \in L_{p\theta}(\mathbb{R}_n; \rho_n)$ such that the equality

$$f_0(\overline{x}) = \sum_{k=0}^{+\infty} l_k^{\frac{n}{2q}} \prod_{i=1}^{n} P^*_{l_k}(x_i)$$

holds in the meaning of convergence of the space $L_{p\theta}(\mathbb{R}_n; \rho_n), 1 < p < +\infty, 1 \leqslant \theta \leqslant +\infty$.

If we introduce the notation $T_{l_m,\dots,l_m}(\overline{x}) = \sum_{k=0}^{m} l_k^{\frac{n}{2q}} \prod_{i=1}^{n} P^*_{l_k}(x_i)$ then,

$$\Delta_{l_\nu,\dots,l_\nu}(f_0; \overline{x}) = l_\nu^{\frac{n}{2q}} \prod_{i=1}^{n} P_{l_\nu}(x_i), \nu \in Z^+.$$

The following chain of inequalities holds by virtue of Lemma 4:

$$\sum_{k=0}^{+\infty} l_k^{\tau\left(\frac{n}{2p}-\frac{n}{2q}\right)} \|\Delta_{l_k,\dots,l_k}(f_0)\|^{\tau}_{L_{p,\theta}(\mathbb{R}_n;\rho_n)} = \sum_{k=0}^{+\infty} l_k^{\tau\left(\frac{n}{2p}-\frac{n}{2q}\right)} \cdot l_k^{\frac{n}{2q}} \|P^*_{l_k}\|^{n\tau}_{L_{p\theta}(\mathbb{R}_n;\rho_n)} \geq$$

$$\geq (C'_{p\theta})^{\tau n} \sum_{k=0}^{N} l_k^{\tau(\frac{n}{2p}-\frac{n}{2q})} \cdot l_k^{-\tau(\frac{n}{2p}-\frac{n}{2q})} = (C'_{p\theta})^{\tau n}(N+1) \to +\infty,$$

when $N \to +\infty$. Thus, the series in left-hand side of the given relations diverges on the function $f_0 \in L_{p\theta}(\mathbb{R}_n; \rho_n)$. According to Theorem 2, for the same function one has:

$$\left\|\sum_{k=0}^{M} l_k^{\frac{n}{2q}} \prod_{i=1}^{n} P^*_{l_k}(\cdot)\right\|_{L_{q\tau}(\mathbb{R}_n;\rho_n)} \geq C_{q\tau\theta n} \left\{\sum_{k=0}^{M} l_k^{\tau(\frac{n}{4q}-\frac{n}{2q})} l_k^{\frac{n\tau}{2q}} \|P^*_{l_k}\|^{n\tau}_{L_{2q\theta}(\mathbb{R}_n;\rho_n)}\right\}^{\frac{1}{\tau}} \geq$$

$$\geq C'_{q\tau\theta n} \left\{\sum_{k=0}^{M} l_k^{\frac{\tau n}{4q}} l_k^{-\frac{\tau n}{4q}}\right\}^{\frac{1}{\tau}} = C'_{q\tau\theta n}(M+1)^{\frac{1}{\tau}} \to +\infty,$$

when $M \to +\infty$. It means that $f_0 \notin L_{q\tau}(\mathbb{R}_n;\rho_n), 1 < p < q < +\infty$. Let $\varepsilon > 0$ be an arbitrary positive number such that $p < (q-\varepsilon) < q < +\infty, 1 \leqslant \theta \leqslant +\infty, 1 \leqslant \tau < +\infty$. Then, according to Lemma 4:

$$\sum_{k=0}^{M} l_k^{\tau(\frac{n}{2p}-\frac{n}{2(q-\varepsilon)})} \|\Delta_{l_k,\dots,l_k}(f_0)\|_{L_{p\theta}(\mathbb{R}_n;\rho_n)}^{\tau} \leqslant$$

$$\leqslant (C''_{p\theta})^{\tau n} \sum_{k=0}^{M} l_k^{\tau(\frac{n}{2p}-\frac{n}{2(q-\varepsilon)})} \cdot l_k^{\frac{\tau n}{2q}} \left\|P^*_{l_k}\right\|_{L_{p\theta}(\mathbb{R}_n;\rho_n)}^{\tau n} \leqslant$$

$$(C'''_{p\theta})^{\tau n} \sum_{k=0}^{+\infty} l_k^{-\tau(\frac{1}{2(q-\varepsilon)}-\frac{1}{2q})} < +\infty, \forall m \in \mathbb{N}.$$

Hence, according to Theorem 1: $f_0 \in L_{q-\varepsilon,\tau}(\mathbb{R}_n;\rho_n), 1 \leqslant \tau < +\infty$, which proves the theorem.

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