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THE FATOU SET OF AN ENTIRE FUNCTION WITH THE FEJÉR GAPS

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Abstract. The paper considers the Fatou set of an entire transcendental function, i.e. the largest open set of the complex plane where the family of iterations of the given function forms a normal family. We assume that the entire function, in general, is of an infinite order. We give the sufficient condition on the indexes of the series (it is stronger than the Fejér gap condition), under which every component of the Fatou set is bounded. The same result under stronger restrictions was earlier obtained by Yu. Wang.

Keywords: entire functions, Fejér gaps, iterations of functions, Fatou set

1. Introduction

Let f be a non-linear entire function of the complex variable z. Let us define natural iterations of the function f as follows:

$$f^0(z) = z, \;\; f^1(z) = f(z), \; \dots, \; f^{k+1}(z) = f(f^k(z)) \quad (k = 1, 2, \dots). \tag{1}$$

Following Montel [1], let us term a class N of functions analytical in the domain D of the complex plane $\mathbb{C}$ as normal in D if a subsequence $\{f_{k_p}\}$ can be singled out of any sequence $\{f_k\}$ of functions from N and if it has the following property: either $\{f_{k_p}(z)\}$, or $\{f_{k_p}^{-1}(z)\}$ converge everywhere in D and uniformly on every compact subset M of the domain D. In this case the sequence $\{f_{k_p}\}$ is said to be converging locally uniformly in D [2].

The maximal open set of the complex plane, where the family of iterations $\{f^k\}$ determined by (1) is normal (in Montel's sense), is called the Fatou set $\mathcal{F}(f)$ of the function $f(z)$. Complement of the Fatou set is called the Julia set $\mathcal{J}(f) = \mathbb{C} \setminus \mathcal{F}(f)$.

If f is a polynomial of the power 2 or higher, the set $\mathcal{F}(f)$ contains a component $K = \{z\colon f^k(z) \to \infty\}$, which is unbounded. For instance, the Fatou set of the function $f(z) = z^2$ contains an unbounded component $\{z\colon |z| > 1\}$. If f is a transcendental entire function, the set $\mathcal{J}(f)$ is always unbounded, and the set $\mathcal{F}(f)$ can have either infinitely many unbounded components, or exactly one, or none at all [3].

Investigation of iterations of entire functions was started in 1926 by P. Fatou [4] and then, after almost 40 years, I. Baker (see the review in [3]) obtained results that influenced the topic remarkably. The following theorem was proved by Baker.

Theorem 1 ([5]). *If for transcendental entire f there is an unbounded invariant component of $\mathcal{F}(f)$, then the growth of f must exceed order $1/2$, minimal type.*

It is demonstrated in [5] that when the parameter a takes sufficiently large positive values, the Fatou set $\mathcal{F}(f)$ of the function

$$f(z) = \frac{\sin\sqrt{z}}{\sqrt{z}} + z + a$$

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contains an unbounded component D, containing a ray $[x_0, \infty)$, $x_0 > 0$, and $f(z)$ has obviously the order $\rho = 1/2$ and a normal type.

In 1981 Baker posed the question [5]: will every component of the set $\mathcal{F}(f)$ be bounded if the entire transcendental function f has a sufficiently small growth order? By virtue of Theorem 1 and the above example it is natural to consider the Baker problem in a class of entire transcendental functions of the order $\rho < 1/2$. Baker himself [5], and later Stallard [6], Anderson and Hinkkanen [7] obtained various sufficient conditions providing that the set $\mathcal{F}(f)$ in the indicated class of functions f does not contain unbounded components.

Investigation of a class of entire transcendental functions of the form

$$f(z) = \sum_{n=1}^{\infty} a_n z^{p_n} \qquad \big(p_n \in \mathbb{N},\ \ 0 < p_n \uparrow \infty\big) \tag{2}$$

is of special interest. Due to having gaps, entire functions of the form (2) possess a series of additional properties, giving us a possibility to argue on components of the set $\mathcal{F}(f)$ in case of any finite and even infinite growth order.

An entire function of the form (2) is said to have the Fabry gaps if $n = o(p_n)$ when $n \to \infty$, and the Fejér gaps if

$$\sum_{n=1}^{\infty} \frac{1}{p_n} < \infty.$$

Investigation of the Fatou sets $\mathcal{F}(f)$ for functions of the form (2) is closely connected to a series of classical problems (Picard's value, Borel and asymptotic values, Julia's directions, maximum and minimum modulus, the Pólya problem, as well as distribution of values of entire functions with different gap conditions). A review of such investigations is given in the work [8].

The present paper is devoted to investigation of the Fatou sets $\mathcal{F}(f)$ of functions f of the form (2) in the general case, namely for entire functions of an arbitrary growth (and of an infinite order as well).

Let us use the following standard notation for the maximum modulus $M(r)$ and the minimum modulus $m(r)$ of the function f:

$$M(r) = \max_{|z|=r} |f(z)|, \qquad m(r) = \min_{|z|=r} |f(z)|.$$

The starting point of investigation is the following Wang result.

Theorem 2 ([9]). *Let an entire function f of the form (2) satisfy the condition: there exists $T_0 > 1$ such that*

$$\varliminf_{r\to\infty} \frac{\ln M(r^{T_0})}{\ln M(r)} > T_0. \tag{3}$$

If for some $\eta > 0$

$$p_n > n \ln n (\ln\ln n)^{2+\eta} \qquad (n \geqslant n_0), \tag{4}$$

then every component of the set $\mathcal{F}(f)$ is bounded.

Note that for any entire function f and for any $T > 1$ (this follows from the Hadamard three circle theorem, see, e.g., in [10])

$$\varliminf_{r\to\infty} \frac{\ln M(r^{T})}{\ln M(r)} \geqslant T. \tag{5}$$

In Theorem 2 entire functions of an arbitrary growth are discussed therefore, one has to postulate the realization of an estimate stronger than (5).

As for the condition (4), on this condition Hayman [11] demonstrated that for any entire function f of the form (2) when $r \to \infty$ outside a certain set of a logarithmic density, we have

$$\ln M(r) = (1 + o(1)) \ln m(r). \tag{6}$$

In proving Theorem 2 the given estimate is used essentially. Therefore, the Hayman condition (4) in Theorem 2 is imposed by the very estimate (6). In fact, the condition (4) can be substituted by a weaker one [12]: $n = o(p_n)$ when $n \to \infty$, and

$$\sum_{n=1}^{\infty} \frac{1}{p_n} \ln \frac{p_n}{n} < \infty.$$

The aim of the article is to demonstrate that under this condition Theorem 2 remains valid.

2. Auxiliary facts. Main result

In order to prove the main result the following lemma is necessary. It is proved by Baker [5] with the use of the Schottky theorem.

Lemma 1 ([5]). *Lemma 1 ([5]). In a domain D the analytic functions g of the family G omit the values 0, 1. D_0 is a compact connected subset of D on which the functions all satisfy $|g(z)| \geqslant 1$. Then there exist constants U, V, dependent only on D_0 and D such that for any z, z' in D_0 and any g in G we have*

$$|g(z')| < U|g(z)|^V.$$

Recall definitions of the measure, logarithmic measure, and upper logarithmic density of the set $E \subset [0, \infty)$:

$$\operatorname{mes} E = \int_E dt, \qquad \ln\text{-}\operatorname{mes} E = \int_E \frac{dt}{t}, \qquad \ln\text{-}\overline{\operatorname{dens}}\, E = \overline{\lim_{r\to\infty}} \frac{1}{\ln r} \int_{E\cap(1,r)} \frac{dt}{t}.$$

If there is a usual limit in the latter expression, the set E is said to have the logarithmic density $\ln\text{-}\operatorname{dens} E$.

Theorem B from [12] (the formulation is given in connection with power series of the form (2)) will also be necessary.

Theorem 3 ([12]). *Let $n = o(p_n)$ when $n \to \infty$ and*

$$\sum_{n=1}^{\infty} \frac{1}{p_n} \ln \frac{p_n}{n} < \infty. \tag{7}$$

Then, there is a set $E \subset [0, \infty)$ of a zero logarithmic density such that for any entire function f of the form (2) when $r \to \infty$ outside E

$$\ln M(r) = (1 + o(1)) \ln m(r). \tag{8}$$

Finally, let us enumerate the main properties of the Fatou and Julia sets. They are formulated in a separate lemma.

Lemma 2. *The following statements are true for the Fatou $\mathcal{F}(f)$ and Julia $\mathcal{J}(f)$ sets of the entire function f [3], [13]:*

1. *The set $\mathcal{F}(f)$ is open, and $\mathcal{J}(f)$ is closed;*
2. *The sets $\mathcal{F}(f)$ and $\mathcal{J}(f)$ are completely invariant under f (i.e. every set coincides with its image as well as with the complete preimage):*

$$1^\circ.\quad f^{-1}(\mathcal{F}(f)) = f(\mathcal{F}(f)) = \mathcal{F}(f);$$
$$2^\circ.\quad f^{-1}(\mathcal{J}(f)) = f(\mathcal{J}(f)) = \mathcal{J}(f).$$

3. *For any $k > 0$, the Fatou (Julia) set of a k-multiple iteration of the function f coincides with the Fatou (Julia) set of the function f itself:*

$$3^\circ.\quad \mathcal{F}(f^k) = \mathcal{F}(f);$$
$$4^\circ.\quad \mathcal{J}(f^k) = \mathcal{J}(f).$$

4. *Any unbounded component of the set $\mathcal{F}(f)$ of an entire transcendental function f is simply connected.*
5. *The set $\mathcal{J}(f)$ of an entire transcendental function f is unbounded.*

The main result of the article is

Theorem 4. *Let f be an entire transcendental function, given by a gap power series (2), and the estimate (3) hold for it when $T_0 > 1$. If the condition (7) holds, then every component of the set $\mathcal{F}(f)$ is bounded.*

3. Proof of Theorem 4

According to the condition (3) with some $T_1 > T_0 > 1$ the following estimate holds:

$$\frac{\ln M(r^{T_0})}{\ln M(r)} \geqslant T_1, \qquad r \geqslant x_0. \tag{9}$$

Let $T_0 < T < T_1$, and $q > 1$ such that $qT < T_1$. Then $(1-\varepsilon)T_1 \geqslant qT$ for some $\varepsilon > 0$.

According to Theorem 3, by such choice of $\varepsilon > 0$ there exists a set $E \subset [0,\infty)$ of a zero logarithmic density such that

$$m(r) > M(r)^{1-\varepsilon} \tag{10}$$

when $r \in [0,\infty) \setminus E$.

Furthermore, the function f is transcendental therefore, $M(r)$ grows quicker than any power of r^N. Let $R_1 > 0$ such that

$$M(r) > 2r^{qT} \quad \text{when} \quad r \geqslant R_1.$$

Taking this into account, let us consider the sequence $\{R_n\}$, where $R_{n+1} = M(R_n)$ $(n \geqslant 1)$. It is clear that $R_n \uparrow \infty$ when $n \to \infty$, and $J_n \subset I_n$, where

$$J_n = [R_n^T, R_n^{qT}], \qquad I_n = [R_n, R_{n+1}] \qquad (q > 1,\ T > 1).$$

Let us demonstrate that when $n \geqslant n_1$, the inequality (10) holds for all points of the interval J_n. Indeed, if there is a subsequence of intervals J_{n_k} where (10) does not hold, one arrives at the contradiction:

$$\ln\text{-dens}\, E = \ln\text{-}\overline{\text{dens}}\, E \geqslant \varlimsup_{k\to\infty} \frac{1}{\ln R_{n_k}^{qT}} \int\limits_{E\cap(1,R_{n_k}^{qT})} \frac{dt}{t} \geqslant \varlimsup_{k\to\infty} \frac{1}{qT\ln R_{n_k}} \int\limits_{R_{n_k}^T}^{R_{n_k}^{qT}} \frac{dt}{t} = 1 - \frac{1}{q}.$$

Thus, when $n \geqslant n_1$ every interval J_n contains a point ρ_n that does not belong to E.

Thus, taking into account the estimates (9), (10), one obtains

$$m(\rho_n) > M(\rho_n)^{1-\varepsilon} \geqslant [M(R_n^T)]^{1-\varepsilon} > M(R_n)^{(1-\varepsilon)T_1}, \qquad n \geqslant n_1.$$

Since $(1-\varepsilon)T_1 \geqslant qT$ then

$$m(\rho_n) > M(R_n)^{qT} = R_{n+1}^{qT}, \tag{11}$$

where $q > 1,\ T > 1$, when $n \geqslant n_1$.

Our aim is to demonstrate that every component of the set $\mathcal{F}(f)$ is bounded. Let s assume the converse. Let $\mathcal{F}(f)$ have an unbounded component D. Then, according to Lemma 2 it is simply connected.

Then, let us use some Baker ideas. Since D is a component of $\mathcal{F}(f)$ and it is unbounded, there is a number $n_2 \geqslant n_1$ such that $D \cap A_n \neq \varnothing$ for all $n \geqslant n_2$, where $A_n = \{z \colon |z| = R_n\}$.

Let us also introduce the circles

$$C_n = \{z \colon |z| = \rho_n\}, \qquad B_n = \{z \colon |z| = R_n^{qT}\} \qquad (q > 1,\ T > 1).$$

Recall that $R_n^T \leqslant \rho_n \leqslant R_n^{qT}$, $\quad R_n < R_n^T < R_n^{qT} < R_{n+1}$.

Let $n \geqslant n_2$. Since the set D is connected and $D \cap A_n \neq \varnothing$, then there is a curve γ in D, that connects a certain point $a_n \in A_n$ with a certain point $b_{n+1} \in B_{n+1}$. Let c_{n+1} be a point of the curve γ, via which C_{n+1} passes, and $|c_{n+1}| = \rho_{n+1}$. Since $m(\rho_{n+1}) \to \infty$ when $n \to \infty$ (it is clear from the estimate (11)), then $f(D)$ is an unlimited connected subset of $\mathcal{F}(f)$, containing a continuum $f(\gamma)$.

Since a_n is a point of γ, $|a_n| = R_n$ then $|f(a_n)| \leqslant M(R_n) = R_{n+1}$. On the other hand, it follows from (11) that

$$|f(c_{n+1})| \geqslant m(\rho_{n+1}) > R_{n+2}^{qT}.$$

Hence, the curve $f(\gamma)$ contains the arc $\gamma^{(1)}$, connecting a certain point $a_{n+1}^{(1)} \in A_{n+1}$ with a certain point $b_{n+2}^{(1)}$ of the circle B_{n+2}. Meanwhile, $\gamma^{(1)}$ contains a certain point $c_{n+2}^{(1)}$ of the circle C_{n+2}. Continuing the induction argument, one obtains that $f^k(D)$ is an unlimited connected subset of $\mathcal{F}(f)$, containing the arc $\gamma^{(k)}$ of the curve $f^k(\gamma)$, connecting the points $a_{n+k}^{(k)} \in A_{n+k}$ and $b_{n+k+1}^{(k)} \in B_{n+k+1}$ and containing the point $c_{n+k+1}^{(k)} \in C_{n+k+1}$, where n ($n \geqslant n_2$) is fixed, $k \geqslant 1$. Moreover,

$$\min_{z \in \gamma^{(k)}} |f^k(z)| = |f(z_k)| \geqslant R_{n+k},$$

where z_k is a certain point γ.

The family $\{f^k\}$ is normal in D. Hence, there is a subsequence $\{f^{k_p}\}$, converging locally uniformly in D. Without loss of generality, one can assume that $z_{k_p} \to z_0 \in \gamma$. Since $|f(z_{k_p})| \to \infty$ when $k_p \to \infty$, the sequence $\{f^{k_p}\}$ converges to infinity uniformly on γ. Hence,

$$\min_{z \in \gamma} |f^{k_p}(z)| \geqslant s \tag{12}$$

for any $s > 0$ when $k_p \geqslant N(s) > n_3$.

Consider a family of functions $G = \{g_{k_p}\}_{k_p \geqslant N}$, where

$$g_{k_p}(z) = \frac{f^{k_p}(z) - a}{b - a},$$

a, b are arbitrary, but fixed points from the Julia set $\mathcal{J}(f)$ such that $a \neq b$. The value N will be chosen later.

Let us verify that the family of functions G for some N satisfies the conditions of Lemma 1 if we take the considered unbounded component of the set $\mathcal{F}(f)$ as the domain D and assume that $D_0 = \gamma$.

Since according to Lemma 2 for all $k \geqslant 1$, for all $a, b \in \mathcal{J}(f)$ when $z \in D \subset \mathcal{F}(f)$ iterations $f^k(z)$ omit the values a, b then, functions $g_{k_p}(z)$ omit the values 0 and 1 in D with any $p \geqslant 1$. Moreover, choosing $s_0 = |a| + |b - a|$ in (12), one obtains that when $k_p \geqslant N(s_0) > n_3$

$$|g_{k_p}(z)| = \frac{|f^{k_p}(z) - a|}{|b - a|} \geqslant \frac{\big||f^{k_p}(z)| - |a|\big|}{|b - a|} \geqslant 1, \qquad z \in \gamma.$$

Thus, the family of functions G satisfies the conditions of Lemma 1 when $N = N(s_0)$. Therefore, there are constants U, V, depending only on γ and D, such that

$$|g_{k_p}(z')| < U |g_{k_p}(z)|^V \tag{13}$$

for all $z, z' \in \gamma$.

It is verified that for all $z \in \gamma$

$$A|f^{k_p}(z)| \leqslant |g_{k_p}| \leqslant B|f^{k_p}(z)|,$$

where

$$A = \frac{1}{s_0}, \quad B = \frac{|a| + s_0}{s_0|b - a|}, \quad s_0 = |a| + |b - a|. \tag{14}$$

Hence, for all $z, z' \in \gamma$ when $k_p \geqslant N$

$$|f^{k_p}(z')| < U^*|f^{k_p}(z)|^V, \qquad U^* = \frac{UB^V}{A}.$$

Let $k_p \geqslant N$, z, z' be points of γ such that:

$$1) \quad f^{k_p}(z) = a^{(k_p)}_{n+k_p}, \qquad a^{(k_p)}_{n+k_p} \in A_{n+k_p};$$

$$2) \quad f^{k_p}(z') = c^{(k_p)}_{n+k_p+1}, \qquad c^{(k_p)}_{n+k_p+1} \in C_{n+k_p+1}.$$

Then, for $k_p \geqslant N$

$$M(R_{n+k_p}) = R_{n+k_p+1} < |c^{(k_p)}_{n+k_p+1}| = |f^{k_p}(z')| <$$

$$< U^*|f^{k_p}(z)|^V = U^*|a^{(k_p)}_{n+k_p}|^V = U^* R^V_{n+k_p},$$

which contradicts the fact that f is a transcendental function since $R_{n+k_p} \to \infty$ when $k_p \to \infty$.

The theorem is proved.

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