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THE SINGULAR STURM-LIOUVILLE OPERATORS WITH NONSMOOTH POTENTIALS IN A SPACE OF VECTOR FUNCTIONS

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Abstract. This paper deals with the Sturm-Liouville operators generated on the semi-axis by the differential expression $l[y] = -(y' - Py)' - P(y' - Py) - P^2 y$, where $'$ is a derivative in terms of the theory of distributions and P is a real-valued symmetrical matrix with elements $p_{ij} \in L^2_{loc}(R_+)$ $(i, j = 1, 2, \dots, n)$. The minimal closed symmetric operator L_0 generated by this expression in the Hilbert space $\mathcal{L}^2_n(R_+)$ is constructed. Sufficient conditions of minimality and maximality of deficiency numbers of the operator L_0 in terms of elements of a matrix P are presented. Moreover, it is established, that the condition of maximality of deficiency numbers of the operator L_0 (in the case when elements of the matrix P are step functions with an infinite number of jumps) is equivalent to the condition of maximality of deficiency numbers of the operator generated by a generalized Jacobi matrix in the space l^2_n.

Keywords: Quasi-derivative, Sturm-Liouville operator, singular potential, distributions, generalized Jacobi matrices, deficiency numbers, deficiency index.

1. Introduction

Our goal is to construct a spectral theory of operators generated by an expression of the form

$$l[y] = -(y' - Py)' - P(y' - Py) - P^2 y, \tag{1}$$

in the space $\mathcal{L}^2_n(R_+)$, where $n \in \mathcal{N}$, $R_+ := [0, +\infty)$, $P := (p_{ij})^n_{i,j=1}$ is a real-valued symmetric matrix function with elements measurable on the R_+ function and satisfying the condition $p^2_{ij} \in L^1_{loc}(R_+)$, $\mathcal{L}^2_n(R_+)$ is the Hilbert space of all complex-valued, measurable n-component vector functions, whose sum of squares of components moduli is Lebesgue integrable on R_+. The expression (1) defines the minimal closed symmetric operator L_0 with the domain of definition D_0 in the space $\mathcal{L}^2_n(R_+)$. We provide a correct definition of the operator in section 2.

On the other hand, let us assume now that $'$ denotes a derivative in terms of the distribution theory namely, the generalized function $p'\psi$, determined by the equality

$$(p'\psi)(\phi) = -\int_0^{+\infty} p(\psi\phi)'$$

for any infinitely differentiable finite function ϕ on $(0, +\infty)$ is considered as the product of the derivative p' of the scalar function $p \in L^2_{loc}(R_+)$ by locally absolutely continuous scalar function ψ as usually. Then, let us define the product of the matrix P', whose elements are generalized functions p'_{ij}, by the vector-function $y \in D_0$ as an n-component vector-function $P'y$, whose coordinate of the number i equals $p'_{i1}y_1 + p'_{i2}y_2 + \dots + p'_{in}y_n$ $(i = 1, 2, \dots, n)$. Then, in terms of

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the distribution theory, the following natural equality becomes obvious: $(Py)' = P'y + Py'$. Due to this equality the operator L_0 generated by the expression (1) in the Hilbert space $\mathcal{L}_n^2(R_+)$ can be understood as an operator generated by the expression

$$l[y] = -y'' + P'y \tag{2}$$

in the same space.

The above definition of the operator L_0 generated by the expression (2) with the matrix potential-distribution gives us a possibility to include it into the class of operators generated by quasi-differential expressions with locally summable coefficients in the space $\mathcal{L}_n^2(R_+)$ and thus, it allows us to construct the spectral theory of this operator.

Note that problems connected with investigation of the scalar Sturm-Liouville operator with a short-range potential (δ-function type) appeared in physical literature. Mathematical investigation of such physical models was started in the 60ies of the last century in the works [1], [2]. The modern state and new trends in developing spectral theory of such operators is described in the monographs [3], [4]. Meanwhile, the correct definition of the Sturm-Liouville operator with a scalar potential-distribution of the first order was given for the first time in the works [5], [6] apparently by several equivalent ways. Spectral properties of such operators were investigated rather thoroughly in the same works especially for the case of a finite interval. We used one of approaches suggested in these works while determining the operator L_0 generated by the expression (2). It should also be mentioned that the recent works [7], [8] contain a detailed spectral analysis of operators generated by an expression of the form (2) for the case when $n = 1$ and P is a step function with an infinite number of jumps on a semi-axis.

The present paper is devoted to construction of the spectral theory of the operator L_0 in particular, to determining the deficiency numbers of the operator in terms of the elements p_{ij} of the matrix P. Theorems 1 and 2 give sufficient conditions for realization of maximality and, accordingly, minimality of deficiency numbers of the operator L_0. Theorem 3 claims that the maximality condition for deficiency numbers of the operator L_0 (in case when elements of the matrix P are step functions with an infinite number of jumps) is equivalent to the maximality condition of deficiency numbers of the operator generated by a generalized Jacobian matrix in the space l_n^2. Some corollaries of these theorems are given and the corresponding examples are constructed. Note that a part of the obtained results is new for the scalar case as well.

2. Quasi-derivatives and quasi-differential operators. Deficiency indices.

Let us assume that real-valued functions p_{ij} $(i, j = 1, 2, \dots, n)$, which are elements of the matrix-function P, are defined on the semi-axis R_+ and satisfy the following conditions:

a) $p_{ij} = p_{ji}$;
b) $p_{ij}^2 \in L^1(\alpha, \beta)$ for any $\alpha, \beta \in R_+$, i.e. p_{ij}^2 are locally absolutely integrable on R_+ ($p_{ij}^2 \in L_{loc}^1(R_+)$).

Let us determine the first quasi-derivative of the given locally absolutely continuous vector-function $y(x) = (y_1(x), y_2(x), \dots, y_n(x))^t$ ($y \in AC_{loc}(R_+)$; t is the transportation symbol, assuming that $y_P^{[1]} = y' - Py$. Then, regarding that the vector function $y_P^{[1]}$ is already defined and is locally absolutely continuous, let us determine the second quasi-derivative of the vector-function y, assuming that $y_P^{[2]} := (y_P^{[1]})' + Py_P^{[1]} + P^2 y$, and the quasi-differential expression:

$$l[y](x) := -y_P^{[2]}(x), \qquad x \in R_+. \tag{3}$$

Note that , the condition b) provides the validity of the existence and uniqueness theorem for solutions of the system of first-order differential equations

$$Y' = FY,$$

corresponding to equation $l[y] = 0$, and the condition a) entails that the following matrix identity holds:

$$F = -J^{-1}F^*J, \tag{4}$$

where the matrices F and J have the form $F := \begin{pmatrix} P & I \\ -P^2 & -P \end{pmatrix}$, $J := \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$ in the block representation, and $F^* = \begin{pmatrix} P & -P^2 \\ I & -P \end{pmatrix}$ is a matrix conjugate to F, and I is a unit matrix of the order n here and in what follows.

Thus, the definition range Δ of the expression $l[y]$ is a set of all locally absolutely continuous vector-functions y on R_+ such that the vector function $y_P^{[1]}$ ia also locally absolutely continuous on R_+. Now let us prove that the following lemma holds.

Lemma 1. (The Green formula) *Let P be a quadratic matrix of the order n ($n \geq 1$), satisfying the conditions a) and b). Then, for any two vector functions $u, v \in \Delta$ and for any two numbers α and β such that $0 \leq \alpha \leq \beta < \infty$, the formula*

$$\int\limits_{\alpha}^{\beta} \{(l[u](x), v(x)) - (u(x), l[v](x))\}dx = [u(x), v(x)](\beta) - [u(x), v(x)](\alpha), \tag{5}$$

holds. Here $(g, h) = \sum\limits_{s=1}^{n} g_s \overline{h_s}$ is a scalar product of the vectors g and h, and the bilinear form $[u, v]$ is defined by the equality: $[u, v](x) := (u^{[1]}(x), v(x)) - (u(x), v^{[1]}(x))$.

Proof. Let $u, v \in \Delta$. Then, there is a pair of vector-functions h, g with locally summable on R_+ components such that

$$l[u] = g \qquad \text{and} \qquad l[v] = h. \tag{6}$$

Conditions (6) can be written in the matrix form:

$$U' = FU + G \qquad \text{and} \qquad V' = FV + H, \tag{7}$$

where the matrix F is defined above and $2n$-dimensional vector columns U, V, G, H have the form: $U := (u, u^{[1]})^t$, $V =: (v, v^{[1]})^t$, $G =: (0, l[u])^t$, $H =: (0, l[v])^t$ (Recall that quasi-derivatives are determined by means of the matrix P).

Multiplying both equalities (7) in the left-hand side by the constant matrix J (see above) and invoking the symmetry condition (4), we obtain the following matrix equalities

$$(JU)' = -F^*JU + JG, \qquad (JV)' = -F^*JV + JH.$$

Then, let us differentiate the scalar product (JU, V):

$$(JU, V)' = ((JU)', V) + (JU, V') =$$
$$(-F^*JU + JG, V) + (JU, FV + H) = (JG, V) + (JU, H),$$

where

$$(JU, V) = \sum_{j=0}^{n} \{u_j \overline{v_j^{[1]}} - u_j^{[1]} \overline{v_j}\} = (u, v^{[1]}) - (u^{[1]}, v) = -[u, v],$$

$$(JG, V) = -\sum_{j=0}^{n} l_j[u] v_j = -(l[u], v)$$

and

$$(JU, H) = \sum_{j=0}^{n} u_j \overline{l_j[v]} = (u, l[v]),$$

where l_j is the jth component of the vector l. Thus, we have proved that

$$(l[u], v) - (u, l[v]) = [u, v]'.$$

It remains only to integrate the resulting equality. Lemma 1 is proved.
Due to the formula (5), the expression $l[y]$ is said to be a symmetric (formally conjugate) vector quasi-differential expression of the second order.

Let us denote by D_0' the set of all complex valued vector functions from Δ that are finite on $(0, +\infty)$. Repeating the same reasoning as in the scalar case (see [9], p. 133), and applying the Green formula it is established that the set D_0' is dense everywhere in $\mathcal{L}_n^2(R_+)$, and the expression l determines on the set D_0' a symmetric (open) operator in $\mathcal{L}_n^2(R_+)$ with the definition range D_0' by the formula $L_0' = l[y]$. Let us denote by the symbols L_0, and D_0 the closure of this operator, and the range of its definition, respectively. Then, we denote by n_+ (n_-) the maximal number of linearly independent solutions to the equation

$$l[y] = \lambda y, \tag{8}$$

that belong to the space $\mathcal{L}_n^2(R_+)$ when $\Im\lambda > 0$ $(\Im\lambda < 0)$. The numbers n_+ and n_- coincide with the deficiency numbers of the minimal closed symmetric operator L_0 (see [10]), preserve their values in semi-planes, are equal to each other and are enclosed between n and $2n$.

Indeed, the fact that the numbers n_+ and n_- cannot be smaller than n is proved similarly to Theorem 2 in [11] (this fact can also be established on the basis of S.A. Orlov's results [12]); and the fact that these numbers cannot be larger than $2n$ is evident.

Now let us demonstrate that $n_+ = n_-$. Let an n-component vector function y be a solution to Equation (8), belonging to the space $\mathcal{L}_n^2(R_+)$ (for the sake of definiteness we assume that $\Im\lambda > 0$). In the equality (8), let us turn to the conjugate equation

$$l[\overline{y}] = \lambda_1 \overline{y}, \tag{9}$$

where $\lambda_1 = \overline{\lambda}$ and $\Im\lambda_1 < 0$. Since $\int\limits_0^{+\infty} ||y(x)||^2 dx = \int\limits_0^{+\infty} ||\overline{y}(x)||^2 dx$, it means that as soon as y is a solution to Equation (8) (with $\Im\lambda > 0$), belonging to the space $\mathcal{L}_n^2(R_+)$, $\overline{y}$ becomes a solution of the same equation (with $\Im\lambda < 0$), belonging to $\mathcal{L}_n^2(R_+)$.

It follows from the above reasoning that the pair of numbers (n_+, n_-), called the deficiency index of the operator L_0, can take one of their the values: (n, n), $(n+1, n+1)$, ..., $(2n, 2n)$. By analogy to the spectral theory of scalar differential Sturm-Liouville operators on a semi-axis, it is said that for the operator L_0 in the first case, and in the latter case the cases of the limit point, and of the limit circle are realized, respectively (see, e.g., [13]). Meanwhile, matrix circles on the set of real symmetric matrices of the order n appear to be analogues of the Weil circles on a complex plane (see [11]).

3. Asymptotic integration of systems of quasi-differential equations

Let the matrix $P^{(1)} := (p_{ij}^{(1)})$ possess the same properties as the matrix P: $p_{ij} = p_{ji}$ and $(p_{ij}^{(1)})^2 \in L_{loc}^1(R_+)$ $(i, j = 1, 2, \ldots, n)$, and n-component vector functions y and $y_{P^{(1)}}^{[1]} := y' - P^{(1)}y$ be determined and be locally absolutely continuous on the semi-axis. The mentioned conditions allow us to define the symmetric quasi-differential expression

$$s[y] := -(y_{P^{(1)}}^{[1]})' - P^{(1)} y_{P^{(1)}}^{[1]} - (P^{(1)})^2 y \tag{10}$$

as well as in the case of the expression l. The expression (10) defines the minimal closed symmetric operator S_0 in the Hilbert space $\mathcal{L}_n^2(R_+)$. Let us denote by $\mathcal{D}_0$ the definition range of the operator S_0.

Further, let us consider the symmetric quasi-differential vector equations

$$l[y] = -(y_P^{[1]})' - Py_P^{[1]} - P^2 y = 0 \tag{11}$$

and

$$s[y] = -(y_{P^{(1)}}^{[1]})' - P^{(1)}y_{P^{(1)}}^{[1]} - (P^{(1)})^2 y = 0. \tag{12}$$

Manifestly, every one of them is equivalent to the system of differential equations of the first order

$$\begin{pmatrix} y \\ y_Q^{[1]} \end{pmatrix}' = \begin{pmatrix} Q & E \\ -Q^2 & -Q \end{pmatrix} \begin{pmatrix} y \\ y_Q^{[1]} \end{pmatrix}, \tag{13}$$

where $Q = P$ in case of Equation (11), and in case of Equation (12) $Q = P^{(1)}$, respectively.

The equivalence (11) (or (12)) and (13) is understood in the sense that if an n-component vector-function y is a solution (11) (or (12)), then a $2n$-component vector function $Y = (y, y_Q^{[1]})^t$ is a solution (13) and vice versa, if $Y := (Y_1, Y_2, \dots, Y_{2n})^t$ is a solution to the system (13), then $y := y_0 = (y_1^0, y_2^0, \dots, y_n^0)$ is a solution to Equation (11) (or (12)) and

$$Y_k = \begin{cases} y_k, & k = 1, 2, \dots, n \\ (y_{k-n})_Q^{[1]}, & k = n+1, n+2, \dots, 2n \end{cases}$$

(for more details see , e.g., [14, Ch. V]).

Denote by T the fundamental matrix of the linear homogeneous system (13) with $Q = P^{(1)}$. Obviously, the columns of the matrix T are $2n$-dimensional columns of the form $(u_j, u_j^{[1]})^t$ $(j = 1, 2, \dots, 2n)$, where u_j are linearly independent vector solutions of Equation (12) (recall that quasi-derivatives are determined by means of the matrix $P^{(1)}$). The following theorem holds.

Theorem 1. . *Let the matrices P, $P^{(1)}$ and T be such that*

$$\int_0^{+\infty} \left\| T^{-1} \begin{pmatrix} P - P^{(1)} & 0 \\ -P^2 + (P^{(1)})^2 & -P + P^{(1)} \end{pmatrix} T \right\|^{1} < +\infty. \tag{14}$$

Then, for any complex numbers $\alpha_1, \alpha_2, \dots, \alpha_{2n}$, Equation (11) has the solution $\phi(x)$, satisfying the conditions:

$$\begin{aligned} \phi(x) &= \sum_{j=1}^{2n} [\alpha_j + a_j(x)] u_j, \\ \phi_P^{[1]}(x) &= \sum_{j=1}^{2n} [\alpha_j + a_j(x)] (u_j)_{P^{(1)}}^{[1]}(x), \end{aligned} \tag{15}$$

where $a_i(x) \to o$ when $x \to +\infty$ $(i = 1, 2, \dots, 2n)$.

Proof. In the system (13) with $Q = P$ let us carry out the linear substitution

$$Y = Tz, \tag{16}$$

where the vector column z has the form $z = (z_1, z_2, \dots, z_{2n})^t$, and differentiate

$$Y' = T'z + Tz'.$$

Then, take into account that the matrix T is a fundamental matrix of solutions of the system (13) with $Q = P^{(1)}$, namely:

$$T' = \begin{pmatrix} P^{(1)} & I \\ -(P^{(1)})^2 & -P^{(1)} \end{pmatrix} T$$

[1]——.—— indicates the sum of absolute values of all elements of the matrix

As a result of the mentioned transformations the system takes the form:

$$z' = T^{-1}\begin{pmatrix} P - P^{(1)} & 0 \\ -P^2 + (P^{(1)})^2 & -P + P^{(1)} \end{pmatrix} Tz. \tag{17}$$

By virtue of the assumption (14), one can apply the result of the problem 1.4 (c) from [15, Ch. X, §1, p. 331] to the system (17) namely, for any complex numbers α_i (i=1,2,...,2n) the system (17) has a unique solution, for which the following asymptotic formulae hold:

$$\begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_{2n} \end{pmatrix} = \begin{pmatrix} \alpha_1 + a_1(x) \\ \alpha_2 + a_2(x) \\ \vdots \\ \alpha_{2n} + a_{2n}(x) \end{pmatrix},$$

where $a_i(x) \to o$ when $x \to +\infty$ $(i = 1, 2, \dots, 2n)$.
It remains only to take into account the relation (16) between the vector column z and the solution of the initial system ϕ. Theorem 1 is proved.

Remark 1. *If we make the supplementary hypothesis, that the matrix T is determined by means of the initial data $T(0) = I_{2n}$ then the inverse matrix to T is determined by the correlation*

$$T^{-1}(x) = -(JT^*J)(x),$$

where I_{2n} is a unit matrix of the order $2n$, and the constant matrix J is determined in (4).

Proof. Firs of all note that the symmetry of the matrix $P^{(1)}$ (as well as in case of the matrix P) entails symmetry of the quasi-differential expression $s[y]$, namely:

$$F_1(x) = -J^{-1}F_1^*(x)J, \tag{18}$$

where the matrix F_1 and the matrix F_1^* conjugate to it have the following form in the block representation:

$$F_1 := \begin{pmatrix} P^{(1)} & I \\ -(P^{(1)})^2 & -P^{(1)} \end{pmatrix}, \qquad F_1^* = \begin{pmatrix} P^{(1)} & -(P^{(1)})^2 \\ I & -P^{(1)} \end{pmatrix}.$$

The definition provides that

$$T^{-1}(x)T(x) = I_{2n}.$$

Differentiating the equality and invoking the fact that T is a fundamental matrix of solutions to the system (13) with $Q = P^{(1)}$, we express $(T^{-1}(x))'$

$$(T^{-1}(x))' = -T^{-1}(x)F_1(x).$$

Let us turn to conjugate matrices

$$((T^*)^{-1}(x))' = -F_1^*(x)(T^*)^{-1}(x), \tag{19}$$

take into account the condition (18) and assume that $(T^*)^{-1}(x) = J\Psi(x)$. This yields a matrix differential equation

$$\Psi'(x) = F_1(x)\Psi(x).$$

Thus, the matrix $\Psi(x)$ defined above is a fundamental matrix of the linear homogeneous system (13) with $Q = P^{(1)}$ and is connected to the matrix $T(x)$ and some constant matrix C by the following equality

$$\Psi(x) = T(x)C$$

(see, e.g., [16],p. 82, Theorem 2.3).
On the other hand, by virtue of definition of the matrix $\Psi(x)$ we have:

$$(T^*)^{-1}(x) = JT(x)C.$$

Meanwhile, the auxiliary initial condition $T(0) = I_{2n}$ entails the matrix equality $C = J^{-1}$. Thus, we obtain that

$$(T^*)^{-1}(x) = JT(x)J^{-1}.$$

Further, the definition of the matrix J entails $J^{-1} = J^* = -J$. Hence,

$$(T^*)^{-1}(x) = -JT(x)J.$$

It remains to turn to conjugate matrices. Remark 1 is proved.

Corollary 1. *Let us assume that conditions of Theorem 1 hold. Then, the limit circle case is realized for the operator L_0 if and only if this case is realized for the operator S_0 as well.*

Proof. Let $k, j \in \{1, 2, \ldots, 2n\}$. Let us take $\alpha_k = \delta_{kj}$ (δ is the Kronecker symbol) from Theorem 1, as constants $\alpha_1, \alpha_2, \ldots, \alpha_{2n}$ and determine the vector-functions v_j assuming that $v_j = \phi(x)$. Then, by virtue of the formulae (15):

$$v_j(x) = (1 + a_j(x))u_j(x) + \sum_{k=1,\ k\neq j}^{2n} a_k(x)u_k(x),$$

where $u_j(x)$ are linearly independent vector solutions of Equation (12) and $a_j(x) = o(1)$ when $x \to +\infty$.
Denote by $T^{(1)}$ the matrix with $2n$-dimensional columns being the columns $(v_j, (v_j)_P^{[1]})^t$ ($j = 1, 2, \ldots, 2n$). Straightforward calculations demonstrate that

$$detT^{(1)} = (1 + \sum_{k=1}^{2n} a_k(x))detT,$$

where $detT \neq 0$. Thus, the system of vectors v_j ($j = 1, 2, \ldots, 2n$) is linearly independent. On the other hand, simple calculations show that

$$\sum_{j=1}^{2n} ||v_j||^2 = \sum_{j=1}^{2n} ||u_j||^2 + o(\sum_{j=1}^{2n} ||u_j||^2)$$

and hence,

$$\lim_{x\to+\infty} \frac{\sum_{j=1}^{2n} ||v_j||^2}{\sum_{j=1}^{2n} ||u_j||^2} = 1.$$

Thus, the improper integrals

$$\int_0^{+\infty} \sum_{j=1}^{2n} ||v_j||^2 \quad \text{and} \quad \int_0^{+\infty} \sum_{j=1}^{2n} ||u_j||^2$$

converge or diverge simultaneously. Corollary (1) is proved.

4. Examples of realization of the limit circle case for the operator L_0

Let $n = 2$ and $\alpha > 2, 0 < \beta < \alpha$. We define the matrix $P^{(1)}$, assuming that

$$P^{(1)} = \begin{pmatrix} -\frac{x^{\alpha+1}}{\alpha+1} & \frac{x^{\beta+1}}{\beta+1} \\ \frac{x^{\beta+1}}{\beta+1} & -\frac{x^{\alpha+1}}{\alpha+1} \end{pmatrix}. \tag{20}$$

Then, the differential operator S_0, generated by the expression s in the Hilbert space $\mathcal{L}_2^2(R_+)$, can be treated as an operator generated by the expression

$$-y'' + (P^{(1)})'y,$$

in the same space $\mathcal{L}_2^2(R_+)$, where $(P^{(1)})'(x) = (p_{ij}^1)'(x)$ $(i, j = 1, 2)$ is a derivative matrix $P^{(1)}(x)$, and the homogeneous quasi-differential equation (12) coincides with the equation

$$-y'' - \begin{pmatrix} x^\alpha & -x^\beta \\ -x^\beta & x^\alpha \end{pmatrix} y = 0. \tag{21}$$

The following lemma holds.

Lemma 2. *Equation (21) has four linearly independent solutions $y^j(x)$ $(j = 1, 2, 3, 4)$ such that the following asymptotic formulae hold when $x \to +\infty$:*

$$\begin{aligned} y^1(x), y^2(x) &\sim \psi_1(x) exp \int\limits_{x_0}^{x} \pm i(s^\alpha + s^\beta)^{1/2} ds, \\ y^3(x), y^4(x) &\sim \psi_2(x) exp \int\limits_{x_0}^{x} \pm i(s^\alpha - s^\beta)^{1/2} ds, \end{aligned} \tag{22}$$

where $\psi_1(x) = \frac{1}{2(x^\alpha + x^\beta)^{1/4}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, $\psi_2(x) = \frac{1}{2(x^\alpha - x^\beta)^{1/4}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Proof. Manifestly, the vector equation (21) is equivalent to the system of two scalar differential equations of the second order

$$\begin{cases} z'' = (x^\beta - x^\alpha) z \\ t'' = -(x^\beta + x^\alpha) t, \end{cases} \tag{23}$$

where $z = y_1 + y_2$ and $t = y_1 - y_2$. For Equations (23) with $x \to +\infty$ asymptotic formulae of the Liouville-Green type (see, e.g., [17], p. 68) are well known, namely

$$z_1, z_2 \sim (x^\alpha - x^\beta)^{-1/4} \exp(\pm i \int\limits_{x_o}^{x} (s^\alpha - s^\beta)^{1/2} ds), \tag{24}$$

and accordingly

$$t_1, t_2 \sim (x^\alpha + x^\beta)^{-1/4} \exp(\pm i \int\limits_{x_o}^{x} (s^\alpha + s^\beta)^{1/2} ds). \tag{25}$$

It remains to take into account the relations between z, t and y. Lemma (2) is proved.

On the basis of the asymptotic formulae (22) and conditions on the coefficients α and β, one can make the conclusion that all solutions to Equation (21) belong to the space $\mathcal{L}_2^2(R_+)$, i.e. the limit circle case is realized for the operator S_0 .

Let x_n $(n = 0, 1, \ldots)$ be an increasing sequence of positive numbers such that $x_0 = 0$, $\lim\limits_{n \to +\infty} x_n = +\infty$. Let us choose an arbitrary point $\nu_k \in [x_k; x_{k+1})$ and determine elements $p_{ij}(x)$ of the matrix P, assuming that: $p_{11}(x) = p_{22}(x) = -\frac{\nu_k^{\alpha+1}}{\alpha+1}$, $p_{12}(x) = p_{21}(x) = \frac{\nu_k^{\beta+1}}{\beta+1}$ when $x \in [x_k, x_{k+1})$. The following lemma holds.

Lemma 3. *Let us assume that the above conditions are satisfied and*

$$\sum_{k=1}^{+\infty} x_{k+1}^{\alpha}(x_{k+1}-x_k)^2 < +\infty, \tag{26}$$

$$\sum_{k=1}^{+\infty} \frac{x_{k+1}^{2\alpha+1}}{(x_k^{\alpha}+x_k^{\beta})^{1/2}}(x_{k+1}-x_k)^2 < +\infty. \tag{27}$$

Then, matrices P and $P^{(1)}$ satisfy the condition (14) of Theorem 1.

Proof. The asymptotic formulae for the matrix T and hence, for the matrix T^{-1} when $x \to +\infty$ are written out explicitly by virtue of the formulae (22). Meanwhile, easy but cumbersome calculations show that the convergence of the following integrals provides the convergence of the integral (14)

α) $\int\limits_{x_0}^{+\infty} |p_{ij}(x)-p_{ij}^{(1)}(x)|dx, \quad i,j=1,2,$

β) $\int\limits_{x_0}^{+\infty} \frac{|p_{ij}^2(x)-(p_{ij}^{(1)})^2(x)|}{(x^{\alpha}+x^{\beta})^{1/2}}dx, \quad i,j=1,2,$

γ) $\int\limits_{x_0}^{+\infty} \frac{|p_{i2}(p_{11}+p_{22})-p_{12}^{(1)}(p_{11}^{(1)}+p_{22}^{(1)})|}{(x^{\alpha}+x^{\beta})^{1/2}}dx,$

where $x_0 > 1$.
On the other hand, one can readily see that when x, $\nu_k \in [x_k, x_{k+1})$ and with the above choice of the matrices P and $P^{(1)}$,the following inequalities hold

$$\begin{aligned} |p_{ij}(x)-p_{ij}^{(1)}(x)| = |p_{ij}^{(1)}(\nu_k)-p_{ij}^{(1)}(x)| \leqslant |p_{ij}^{(1)}(x_{k+1})-p_{ij}^{(1)}(x_k)| \leqslant \\ \leqslant |(p_{ij}^{(1)})'(x_{k+1})|(x_{k+1}-x_k) = \begin{cases} x_{k+1}^{\alpha}(x_{k+1}-x_k), \ i=j \\ x_{k+1}^{\beta}(x_{k+1}-x_k), \ i \neq j \end{cases}, \end{aligned} \tag{28}$$

$$|p_{ij}^2(x)-(p_{ij}^{(1)})^2(x)| \leqslant |((p_{ij}^{(1)})^2)'(x_{k+1})|(x_{k+1}-x_k) = \begin{cases} \dfrac{2x_{k+1}^{2\alpha+1}}{\alpha+1}(x_{k+1}-x_k), \ i=j \\ \dfrac{2x_{k+1}^{2\beta+1}}{\beta+1}(x_{k+1}-x_k), \ i\neq j \end{cases} \tag{29}$$

and

$$\begin{aligned} |p_{12}(p_{11}+p_{22})(x)-p_{12}^{(1)}(p_{11}^{(1)}+p_{22}^{(1)})(x)| \leqslant \\ \leqslant |(p_{12}^{(1)}(p_{11}^{(1)}+p_{22}^{(1)}))'(x_{k+1})|(x_{k+1}-x_k) = \frac{2(\alpha+\beta+2)}{(\alpha+1)(\beta+1)}x_{k+1}^{\alpha+\beta+1}(x_{k+1}-x_k). \end{aligned} \tag{30}$$

Let us demonstrate now that convergence of the series (26) and (27) is provided by convergence of the integrals α), β), γ). Note that convergence of the integrals α), β), γ) is proved uniformly therefore, we limit ourselves by the **Proof** of convergence of the integral α) when $i=j=1$. Indeed,
$\int\limits_{x_0}^{+\infty} |p_{11}(x)-p_{11}^{(1)}(x)|dx \leqslant \sum\limits_{k=k_0}^{+\infty} \int\limits_{x_k}^{x_{k+1}} x_{k+1}^{\alpha}|x_{k+1}-x_k| \int\limits_{x_k}^{x_{k+1}} 1dx = \sum\limits_{k=k_0}^{+\infty} x_{k+1}^{\alpha}(x_{k+1}-x_k)^2$. Lemma 3 is proved.

Thus, the statement of Corollary 1 holds for the matrices P and $P^{(1)}$, i.e. the deficiency index of the operator L_0 is maximal and equals to $(4,4)$.

Summing up the above, it can be mentioned that we constructed examples of realization of the limit circle case for the operator L_0, generated by the quasi-differential expression (1) with the matrix P such that

$$P'(x) = \sum_{k=0}^{+\infty} \begin{pmatrix} \alpha_k & \beta_k \\ \beta_k & \gamma_k \end{pmatrix} \delta(x - x_k), \tag{31}$$

where $'$ denotes a derivative in the sense of the distribution theory and the constants α_k, β_k, γ_k are determined by the equalities: $\alpha_k = \gamma_k = \frac{\nu_k^{\alpha+1} - \nu_{k+1}^{\alpha+1}}{\alpha+1}$, $\beta_k = \frac{\nu_{k+1}^{\beta+1} - \nu_k^{\beta+1}}{\beta+1}$.

Remark 2. *One can take, e.g., a sequence with the general term $x_k = \ln k$ $(k = 1, 2, \dots)$ as a suitable sequence of the points x_k.*

Proof. Convergence of the series (26) and (27) is proved likewise. Therefore, let us demonstrate that, e.g., the series (26) converges. Indeed,

$$\sum_{k=1}^{+\infty} x_{k+1}^{\alpha} (x_{k+1} - x_k)^2 = \sum_{k=1}^{+\infty} \ln^{\alpha}(k+1) \ln^2 \frac{k+1}{k},$$

and the latter series converges since $\ln^{\alpha}(k+1) \ln^2 \frac{k+1}{k} \sim \frac{ln^{\alpha}(k+1)}{k^2}$ when $k \to +\infty$ and the series $\sum\limits_{k=1}^{+\infty} \frac{ln^{\alpha}(k+1)}{k^2} < +\infty$.

5. The sufficient condition for realization of the limit point case for the operator L_0

Let us denote by O a zero matrix of the order n. As usually, for real symmetric matrices A and B, the inequality $A \geq B$ means that for any $u \in R^n$ the inequality $(Au, u) \geq (Bu, u)$ is satisfied. The following theorem holds.

Theorem 2. *Let us assume that there exists a sequence of pairwise disjoint intervals $(a_k, b_k) \subset R_+$ $(k = 1, 2, \dots)$ such that*
1. elements p_{ij} $(i, j = 1, 2, \dots, n)$ of the matrix P are absolutely continuous on $[a_k, b_k]$;
2. $P'(x) \geq O$ when $x \in [a_k, b_k]$;
3.

$$\sum_{k=1}^{+\infty} (b_k - a_k)^2 = +\infty. \tag{32}$$

Then $n_+ = n_- = n$.

Indeed, let the elements p_{ij} of the matrix P satisfy the conditions mentioned in the beginning of Section 2 and the conditions *1-3* of Theorem 2 on the intervals $[a_k, b_k]$. Then the quasi-differential expression $l[y]$ coincides with the ordinary vector differential expression (2) on the interval $[a_k, b_k]$ when k is fixed. Meanwhile, the **Proof** of Theorem 2 is conducted repeating almost word for word the reasoning from [18], which demonstrates that the conditions *1–3* provide the validity of this theorem independently of the behaviour of the elements p_{ij} of the matrix P outside $\bigcup\limits_{k=1}^{+\infty} [a_k, b_k]$, and is omitted here.

Example 1. *Let $0 =: x_0 < x_1 < x_2 < \dots$ and $\lim\limits_{k \to +\infty} x_k = +\infty$. Suppose that $P(x) = C_k$ when $x \in [x_{k-1}, x_k)$, where C_k is a symmetric real numerical matrix and $\sum\limits_{k=1}^{+\infty} (x_k - x_{k-1})^2 = +\infty$. Then, the deficiency index of the operator L_0 equals (n, n).*

Indeed, if $[x_{k-1}, x_k)$ is divided into three equal parts and the middle third of this interval is taken as $[a_k, b_k]$, then all conditions of Theorem 2 become satisfied.

6. Special case of the operator L_0 and the generalized Jacobi matrix

Let us assume that x_k and C_k are the same as in Example 1, i.e. x_k $(k = 0, 1, \ldots)$ is an increasing sequence of positive numbers such that $x_0 = 0$ and $\lim\limits_{n\to+\infty} x_n = +\infty$, C_k is a symmetric real numerical matrix, and let $\mathcal{A}_k = (\alpha^k_{ij})^n_{i,j=1} := C_{k+1} - C_k$. In this case, the expression (2) takes the following form:

$$l[y] = -y'' + \sum_{k=1}^{+\infty} \mathcal{A}_k \delta(x - x_k) y. \qquad (33)$$

The following theorem holds.

Theorem 3. *Minimal closed symmetric operator L_0, generated by the expression (33) in the space $\mathcal{L}^2_n(R_+)$ has the deficiency index $(2n, 2n)$ if and only if all solutions of the difference vector equation*

$$-\frac{Z_{k+1}}{r_{k+1}r_{k+2}d_{k+1}} + \frac{1}{r^2_{k+1}}[\mathcal{A}_k + (\frac{1}{d_k} + \frac{1}{d_{k+1}})I]Z_k - \frac{Z_{k-1}}{r_k r_{k+1} d_k} = 0, \quad k = 1, 2, \ldots, \qquad (34)$$

where $d_k := x_k - x_{k-1}$, $r_{k+1} := \sqrt{d_{k+1} + d_k}$, belong to the space l^2_n.

Proof. Consider a homogeneous system of linear differential equations of the second order

$$-y'' + \sum_{k=1}^{+\infty} \mathcal{A}_k \delta(x - x_k) y = 0. \qquad (35)$$

Manifestly, the vector-functions $u^1_k(x), u^2_k(x), \ldots, u^{2n}_k(x)$, $x \in R_+$, when $x \in (x_{k-1}, x_k)$ determined by the equalities

$$u^i_k = \frac{1}{\sqrt{d_k}} e_i, \quad u^{i+n}_k = [-\frac{\sqrt{3}}{\sqrt{d_k}} + \frac{2\sqrt{3}}{d_k\sqrt{d_k}}(x - x_{k-1})]e_i, \quad i = 1, 2, \ldots, n,$$

where $x_0 = 0$, e_i is a canonical basis of the space R^n, generate an orthonormal system of solutions to Equation (35). Therefore, an arbitrary solutions $y(x)$ to the system (35) is a locally absolutely continuous function on R_+ and has the form

$$y(x) = A^1_k u^1_k(x) + A^2_k u^2_k(x) + \ldots + A^{2n}_k u^{2n}_k(x)$$

when $x \in (x_{k-1}, x_k)$. Hence,

$$\int_0^{+\infty} ||y(x)||^2 dx = \sum_{k=1}^{+\infty} \int_{x_{k-1}}^{x_k} ||y(x)||^2 dx = \sum_{k=1}^{+\infty} \{(A^1_k)^2 + (A^2_k)^2 + \ldots + (A^{2n}_k)^2\}. \qquad (36)$$

On the other hand, arbitrary solution of the system (35) is a continuous piecewise linear function, i.e. $y = X_k + Y_k(x - x_{k-1})$ when $x \in (x_{k-1}, x_k)$, where coordinates with the number i of vector columns X_k and Y_k are determined by equalities

$$X^i_k = \frac{1}{\sqrt{d_k}}(A^i_k - \sqrt{3}A^{i+n}_k), \qquad Y^i_k = \frac{2\sqrt{3}}{d_k^{3/2}} A^{i+n}_k, \quad i = 1, 2, \ldots, n.$$

Now let us take into account the continuity condition of the vector function y and the condition of absolute continuity of its first quasi-derivative generated by means of the matrix P, $y^{[1]}_P = y' - Py$, i.e. $y(x_k-) = y(x_k+) = y(x_k)$ and $y^{[1]}_P(x_k-) = y^{[1]}_P(x_k+)$. Since $y(x_k-) = X_k + Y_k d_k$, and $y(x_k+) = X_{k+1}$ then, the first of the mentioned conditions is equivalent to the equality:

$$X_k + Y_k d_k = X_{k+1}.$$

Likewise, the second condition is equivalent to the correlation

$$y'(x_k+) - y'(x_k-) = \mathcal{A}_k y(x_k).$$

Hence, unifying the results one can make the conclusion that the vector columns X_k and Y_k satisfy the system of equations

$$\begin{cases} X_k + Y_k d_k = X_{k+1} \\ Y_{k+1} - Y_k = \mathcal{A}_k X_{k+1} \end{cases} \quad k = 1, 2, \dots .$$

Excluding Y_k, we obtain that the vector X_k satisfies the vector difference equation

$$\frac{1}{d_{k+1}} X_{k+2} - [\mathcal{A}_k + (\frac{1}{d_k} + \frac{1}{d_{k+1}})I] X_{k+1} + \frac{1}{d_k} X_k = 0, \quad k = 1, 2, \dots . \tag{37}$$

Now, invoking the connection between X_k, Y_k and A_k, note that

$$A_k^i = \frac{\sqrt{d}_k}{2}(X_{k+1}^i + X_k^i), \;\; A_k^{i+n} = \frac{\sqrt{d}_k}{2\sqrt{3}}(X_{k+1}^i - X_k^i), \;\; i = 1, 2, \dots, n, \;\; k = 1, 2, \dots .$$

One can readily deduce from these formulae that

$$\begin{aligned} (A_k^1)^2 + (A_k^2)^2 + \ldots + (A_k^{2n})^2 = \frac{d_k}{4}(||x_k||^2 + ||x_{k+1}||^2 + 2\sum_{s=1}^{n} x_k^s \cdot x_{k+1}^s) + \\ + \frac{d_k}{12}(||x_k||^2 + ||x_{k+1}||^2 - 2\sum_{s=1}^{n} x_k^s \cdot x_{k+1}^s). \end{aligned} \tag{38}$$

The relation (38) entails immediately that the following inequalities hold

$$\frac{d_k}{6}\left\{||X_k||^2 + ||X_{k+1}||^2\right\} \le (A_k^1)^2 + (A_k^2)^2 + \ldots + (A_k^{2n})^2 \le \frac{d_k}{2}\left\{||X_k||^2 + ||X_{k+1}||^2\right\}.$$

Thus, the row in the right-hand side of the equality (36) converges if and only if the following series converges

$$\sum_{k=1}^{+\infty} d_k(||X_{k+1}||^2 + ||X_k||^2) = d_1||X_1||^2 + \sum_{k=1}^{+\infty}(d_k + d_{k+1})||X_{k+1}||^2,$$

where X_k satisfies the vector equation (37).
Let us make the substitution

$$Z_k = r_{k+1} X_{k+1}$$

in the system of vector difference equations (37). As a result the latter system is reduced to the form:

$$\frac{Z_{k+1}}{r_{k+2} d_{k+1}} - \frac{1}{r_{k+1}}[\mathcal{A}_k + (\frac{1}{d_k} + \frac{1}{d_{k+1}})I] Z_k + \frac{Z_{k-1}}{r_k d_k} = 0, \quad k = 1, 2, \dots . \tag{39}$$

Multiplying every equation of the system (39) by $-\frac{1}{r_{k+1}}$, we obtain the symmetric system (34).

Thus, any solution $y(x)$ of Equation (35) belongs to the space $\mathcal{L}_n^2(R_+)$ if and only if any solution Z_k of the difference equation (34) belongs to the space l_n^2. Theorem 3 is proved.

Theorem 3 claims that for the operator L_0, generated by the expression (33), the limit circle case is realized if and only if the deficiency numbers of the difference operator, generated by the generalized Jacobian matrix of the form

$$J = \begin{pmatrix} A_0 & B_0 & 0 & 0 & \dots \\ B_0^* & A_1 & B_1 & 0 & \dots \\ 0 & B_1^* & A_2 & B_2 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

in the space l_n^2, where A_0, B_0 are arbitrary quadratic real symmetric matrices of the order n, B^{-1} exists and

$$A_k = \frac{1}{r_{k+1}^2}[\mathcal{A}_k + (\frac{1}{d_k} + \frac{1}{d_{k+1}})I], \ B_k = -\frac{1}{r_{k+1}r_{k+2}d_{k+1}}I, \quad k = 1, 2, \dots, \tag{40}$$

are maximal, i.e. are equal to the number n. The generalized Jacobian matrices of the form J appear in connection with the matrix power moment problem suggested and developed by M.G. Krein (see, e.g., [19]) and are well investigated. In particular, in the works [20] – [22] maximality criteria for deficiency numbers and various criteria for realization of maximality and nonmaximality cases of deficiency numbers for the corresponding difference operators in terms of elements of the matrix J are indicated. Applying these criteria and Theorem 3 in the given case, one can obtain the maximality and nonmaximality conditions of deficiency numbers for the operator L_0, generated by the expression (33), in terms of $\mathcal{A}_k$ and d_k. Namely, the following corollaries are valid.

Corollary 2. *Let us assume that one of the following conditions is satisfied:*

$$\sum_{k=1}^{+\infty} r_{k+1}r_{k+2}d_{k+1} = +\infty$$

or

$$\sum_{k=1}^{+\infty} \frac{r_{k+2}^2 r_{k+3} d_{k+1} d_{k+2}}{r_{k+1}} ||\mathcal{A}_k + \left(\frac{1}{d_k} + \frac{1}{d_{k+1}}\right) I|| = +\infty.$$

Then, the limit circle case is not realized for the operator L_0.

Indeed, the enumerated conditions is the result of application of Theorem 3 from [20] to elements of the matrix J, defined in (40), according to which a completely indeterminate case is not realized for the matrix J, i.e. the deficiency index of the corresponding difference operator is not maximal. Then, according to Theorem 3, the deficiency index of the differential operator L_0 is not maximal as well.

Corollary 3. *Let us assume that elements of the matrix J satisfy the following conditions:*
I $\frac{n^4}{r_k r_{k+3} d_k d_{k+2}} \leqslant \frac{1}{r_{k+1} r_{k+2} d_{k+1}^2}$ *or* $\frac{n^4}{r_k r_{k+3} d_k d_{k+2}} \geq \frac{1}{r_{k+1} r_{k+2} d_{k+1}^2}$ *for all* $k = 1, 2, ...,$
II $\sum\limits_{k=1}^{+\infty} r_{k+1}r_{k+2}d_{k+1} < +\infty$, *III* $\sum\limits_{k=1}^{+\infty} \frac{r_{k+2}d_{k+1}}{r_{k+1}} ||\mathcal{A}_k + \left(\frac{1}{d_k} + \frac{1}{d_{k+1}}\right) I|| < +\infty.$
Then, the limit circle case is realized for the operator L_0.

Indeed, as well as in Corollary 2, Conditions 1–3 are a result of the direct application of Corollary 1 from [22] and Theorem 3 to the generalized Jacobian matrix J, with matrix elements defined in (40).

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