ISSN 2074-1863 Ufa mathematical journal. Volume 3. 3 (2011). Pp. 146-157.

UDC 517.547

STABILITY OF SEQUENCES OF ZEROS FOR CLASSES OF HOLOMORPHIC FUNCTIONS OF MODERATE GROWTH IN THE UNIT DISK

F.B. KHABIBULLIN

Abstract. Let $\Lambda = (\lambda_k)$ and $\Gamma = (\gamma_k)$ be two sequences of points in the unit disk $\mathbb{D} := \{z \in \mathbb{C} \colon |z| < 1\}$ of the complex plane $\mathbb{C}$, and H be a weight space of holomorphic functions on $\mathbb{D}$. Suppose that Λ is the zero subsequence of some nonzero function from H. We give conditions of closeness of the sequence Γ to the sequence Λ, under which the sequence Γ is the zero sequence for some holomorphic function from space $\hat{H} \supset H$. The space $\hat{H}$ can be a little larger than H.

Keywords: holomorphic function, unit disk, weight space, zero sequence, zero subsequence, shift of zeros, stability of zero sequence

1. Introduction. Main <<radial>> results

As usually, $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ are sets of natural, real and complex numbers or their geometric interpretations; $\mathbb{D} := \{z \in \mathbb{C} \colon |z| < 1\}$ is a unit circle.

Let us assume that $\Lambda = (\lambda_k)_{k \in \mathbb{N}}$ is a sequence of complex points, that can be repeated a finite number of times, in a unit circle $\mathbb{D}$ and Λ does not have limit points in $\mathbb{D}$; H is a class of functions holomorphic in $\mathbb{D}$.

A set, or sequence, of all zeroes of the nonzero function f holomorphic in $\mathbb{D}$ (we write $f \not\equiv 0$), enumerated in view of the multiplicity (every point in $\mathbb{D}$ is counted the number of times corresponding to the zero multiplicity of the function f at this point) will be denoted by Zero_f. Λ *is a sequence of zeroes or a null sequence, or a null set, for the class* H if there exists a nonzero function $f \in H$ such that $\mathrm{Zero}_f = \Lambda$.

The sequence Λ *is a subsequence of zeroes, or a zero subset, for the class* H if there is a nonzero function $f \in H$, vanishing on Λ in the sense that zero multiplicity of the function f at every point from Ω is no less than the number of iterations of this point in the sequence Λ. If H is a linear space then, a subsequence of zeroes for H is termed as a *sequence or set of nonuniqueness for* H.

The positivity of the number, function, measure etc. everywhere is considered as $\geqslant 0$, and > 0 is a strict positivity; similar agreement is assumed for negativity as well. For $a \in \mathbb{R}$, as usually, $a^+ := \max\{a, 0\}$, $[a]$ is an integer part of the number a, and for $a > 0$ we assume $\log^+ a := \max\{\log a, 0\}$, $\log^\alpha a := \big(\log a\big)^\alpha$. If the function f is not equal identically to the value $a \in [-\infty, +\infty]$, we write $f \not\equiv a$.

For the subset $D \subset \mathbb{C}$, let us denote by $\overline{D}$, ∂D and $\mathrm{dist}(S, D)$ the closure D, the boundary of the set D and the Euclidean distance from the subset $S \subset \mathbb{C}$ to D, respectively.

The space of all holomorphic in D functions is denoted by $\mathrm{Hol}(D)$. The following weight classes of functions holomorphic in a unit circle is considered.

The work is supported by RFBR, grant 09-01-00046a.
Submitted on 15 July 2011.

Let $M\colon \mathbb{D} \to [-\infty, +\infty)$. The class of all functions $f \in \operatorname{Hol}(\mathbb{D})$, satisfying the estimate $|f(z)| \leqslant C_f \exp M(z)$, $z \in \mathbb{D}$, where $C_f \geqslant 0$ is a constant is denoted by $\operatorname{Hol}(\mathbb{D}; M)$.

It is natural to expect (see, e.g., stability theorems from [1, §§ 18, 19] and D. Luecking's result [2, Theorem 6]), that if $\Lambda \subset \mathbb{D}$ is a (sub)sequence of zeroes for a certain class of holomorphic functions $H \subset \operatorname{Hol}(\mathbb{D})$ then, provided that there are ¡¡small shifts¿¿ of points λ_k to points γ_k of another sequence of points $\Gamma := (\gamma_k)$, the latter generates a (sub)sequence of zeroes for a certain, maybe wider, class $\hat{H} \supset H$ of holomorphic in $\mathbb{D}$ functions. The main results of the work represent an explicit quantitative form of this observation: for weight spaces H of holomorphic functions, generated by classes of the form $\operatorname{Hol}(\mathbb{D}; M)$, the question of transition of the subsequence of zeroes Λ for H is investigated in case of a small shift to the sequence of zeroes Γ for some space $\hat{H} \supset H$, that differs insignificantly from H or even coinciding with H. A symbiosis of the results from [1], [3]–[5] is used for our investigation. A part of the results was discussed in [6], [7] without proofs. Here and in what follows, we do not dwell upon the history of the issue, since it is described with sufficient details in the works [1] and [3]. We would only like to mention a simple result following from the Nevanlinna theorem (see [8], [9]) of the 1920ies for the classical algebra $H^\infty := \operatorname{Hol}(\mathbb{D}; 0)$ of bounded holomorphic functions in $\mathbb{D}$. According to the known Nevanlinna theorem about description of (sub)sequences of zeroes for the algebra H^∞, the *sequence Λ is a sequence of zeroes for H^∞ if and only if* $\sum_{k\in\mathbb{N}}\bigl(1 - |\lambda_k|\bigr) < +\infty$. Whence, the following follows immediately.

Nevanlinna theorem (of stability for H^∞)**.** *Let us assume that $\Lambda := (\lambda_k)_{k\in\mathbb{N}}$ and $\Gamma := (\gamma_k)_{k\in\mathbb{N}}$ are sequences of points in $\mathbb{D}$, and Λ is a subsequence of zeroes for H^∞. If*

$$\limsup_{N\to\infty} \sum_{k=1}^{N} \bigl(|\lambda_k| - |\gamma_k|\bigr) < +\infty,$$

then Γ is a sequence of zeroes for H^∞.

Denote by $D(t)$ an open circle with the centre at zero with the radius t. Let us assume that $\nu^{\mathrm{rad}}(t) := \nu\bigl(D(t)\bigr)$ for the measure ν, determined in the circle. $D(t)$.

Our main interest is focused on three types of functions classes (not necessarily algebras!), defined by the weights M growing sufficiently slowly in the vicinity of the unit circle $\partial\mathbb{D}$ (broadly speaking, slower than the functions $z \mapsto 1/(1 - |z|)$ when $z \to \partial\mathbb{D}$). Let us determine them here first for arbitrary weight functions $M\colon \mathbb{D} \to [-\infty, +\infty)$.

(A) Denote by A_M^∞ a class of functions $f \in \operatorname{Hol}(\mathbb{D})$, satisfying the estimate

$$\bigl|f(z)\bigr| \leqslant C_f \exp\bigl(c_f M(z)\bigr), \quad z \in \mathbb{C} \tag{1}$$

for some positive constants c_f, C_f. If M is a positive function, then A_M^∞ is an algebra. In particular, if $\limsup_{z\to\partial\mathbb{D}} M(z) = +\infty$, this algebra can be otherwise defined as

$$A_M^\infty := \Bigl\{ f \in \operatorname{Hol}(\mathbb{D})\colon \limsup_{z\to\partial\mathbb{D}} \frac{\log|f(z)|}{M(z)} < +\infty \Bigr\}. \tag{2}$$

(H^1) Let M be a positive function. The space

$$H_M^{1^-} := \bigcup_{0\leqslant c<1} \operatorname{Hol}(\mathbb{D}; cM) \tag{3}$$

consists of the functions $f \in \operatorname{Hol}(\mathbb{D})$, satisfying the restriction

$$|f(z)| \leqslant C_f \exp\bigl(c_f M(z)\bigr), \quad z \in \mathbb{D}, \tag{4}$$

with some positive constants $c_f < 1$ and $C_f > 0$. In particular, if $\limsup_{z\to\partial\mathbb{D}} M(z) = +\infty$, the space can be otherwise defined as

$$H_M^{1^-} := \Bigl\{ f \in \operatorname{Hol}(\mathbb{D})\colon \limsup_{z\to\partial\mathbb{D}} \frac{\log|f(z)|}{M(z)} < 1 \Bigr\}. \tag{5}$$

($\mathrm{H}_{\log}$) The space $H_{M+\log}$ was defined in [1] as a set of all holomorphic in $\mathbb{D}$ functions f, satisfying the restriction

$$|f(z)| \leqslant C_f\Big(\frac{1}{1-|z|}\Big)^{c_f} \exp M(z), \quad z \in \mathbb{D},$$

with some constants $C_f, c_f \geqslant 0$. Otherwise, this rather ¡¡rigid¿¿ space can be defined as

$$H_{M+\log} := \left\{ f \in \mathrm{Hol}(\mathbb{D})\colon \limsup_{z\to\partial\mathbb{D}} \frac{\log|f(z)| - M(z)}{\log\frac{1}{1-|z|}} < +\infty \right\}. \tag{6}$$

In what follows, the *moderate growth condition*

$$\int_0^1 \int_0^{2\pi} p(te^{i\theta})\,\mathrm{d}\theta\,\mathrm{d}t < +\infty \tag{7}$$

will be always imposed on the weight function $p \not\equiv -\infty$, with respect to which the weights M will be selected depending on the results. If p a subharmonic function here with the Riesz measure $\nu_p := \frac{1}{2\pi}\Delta p \geqslant 0$, where the Laplace operator acts in the meaning of the theory of generalized functions, then the condition (7) is equivalently bounded (see the beginning of the proof of [3, Theorem 2])

$$\int_0^1 (1-t)^2\,\mathrm{d}\nu_p^{\mathrm{rad}}(t) < +\infty, \quad \nu_p(t) := \nu_p\big(D(t)\big). \tag{8}$$

Thus, the moderate growth condition (7) for a radial function p, for which by definition $p(z) = p\big(|z|\big)$ for all $z \in \mathbb{D}$, has an absolutely simple form:

$$\int_0^1 p(t)\,\mathrm{d}t < +\infty. \tag{9}$$

Moreover, definite regularity conditions of the form

$$\frac{1}{2\pi}\int_0^{2\pi} p\left(z+\varepsilon\big(1-|z|\big)e^{i\theta}\right)\,d\theta + a\log\frac{1}{1-|z|} \leqslant bp(z) + C, \quad z \in \mathbb{D} \tag{10}$$

will be imposed. Here a, b, C are some constants, with the choice limit due to specific spaces under consideration and to the method of the work [1]. This condition will have the following form for a radial increasing function p :

$$p\left(t+\varepsilon(1-t)\right) + a\log\frac{1}{1-t} \leqslant bp(t) + C, \quad 0 \leqslant t < 1. \tag{11}$$

Simplest examples of radial increasing unbounded weights p, satisfying conditions of the form (9) and (11) simultaneously, can be as follows:

[L]: *logarithmic weight* $p\colon z \longmapsto \log^{\alpha}\dfrac{1}{1-|z|}$, $\alpha > 0$, $z \in \mathbb{D}$,

[P]: *power weight* $p\colon z \longmapsto \dfrac{1}{(1-|z|)^{\beta}}$, $0 < \beta < 1$, $z \in \mathbb{D}$.

It should be mentioned that complete descriptions of zero sets for the algebras A_p^∞ with weight functions of the form [P], when $\beta > 1$ were obtained already by F. A. Shamoyan in [10]. While, a complete description of zero sequences for ¡¡rigid¿¿ spaces $\mathrm{Hol}(\mathbb{D}; M)$, where M is a logarithmic weight from [L] with $0 < \alpha < 1$, was given by K. Seip in [11]. At the same time, there are a lot of open problems on description of zero sets and their stability even for specified spaces and algebras defined by weight functions of the form [L] and [P] in accordance with $\alpha \geqslant 1$ and $0 < \beta \leqslant 1$.

First, for the sake of better visibility, let us summarize the simplified results of the work for the radial function p. On condition of a moderate growth (9), we introduce an *auxiliary function*

$$b_p(r) := \frac{1}{1-r}\int_r^1 (1-t)\,dp(t) = \frac{1}{1-r}\int_r^1 p(t)\,\mathrm{d}t - p(r), \quad 0 \leqslant r < 1, \tag{12}$$

where the convergence of the integrals is provided by the moderate growth condition (9) and the first equality in (25) from Lemma 1, proved in the following Section 2.

Theorem 1 (of stability for the radial weight). *Let us assume that $\Lambda = (\lambda_k)_{k\in\mathbb{N}}$ and $\Gamma = (\gamma_k)_{k\in\mathbb{N}}$ are two sequences of points in $\mathbb{D}$ and*

- *the function $p\colon [0,1) \to [0,+\infty)$ is increasing right-continuous at zero;*
- *the composition $p\circ\exp$ is convex on $(-\infty,0)$, i. e. the function p is convex with respect to a logarithmic function;*
- *the moderate growth condition (9) is satisfied;*
- *the function p is extended on $\mathbb{D}$ as a radial one, namely: $p(z) \equiv p\big(|z|\big)$, $z \in \mathbb{D}$.*

Then,

(S_A) *if for $a = 1$ and some constants $\varepsilon \in (0,1)$, $b, C \geqslant 0$ the radial regularity condition (11) of the weight p is satisfied, and*

$$\limsup_{k\to\infty} \frac{|\lambda_k - \gamma_k|}{1 - \max\{|\lambda_k|, |\gamma_k|\}} < +\infty, \tag{13}$$

and Λ is a subsequence of zeroes for the algebra A_p^∞, then Γ is the zero sequence for the algebra A_M^∞, defined by the weight $M = p + b_p$;

(S_1) *if for any $b > 1$ there are constants $\varepsilon \in (0,1)$ and $C \geqslant 0$ such that the radial regularity condition (11) of the weight p is satisfied when $a = 1$, and*

$$\lim_{k\to\infty} \frac{|\lambda_k - \gamma_k|}{1 - \max\{|\lambda_k|, |\gamma_k|\}} = 0, \tag{14}$$

while Λ is a subsequence of zeroes for the space $H_p^{1^-}$, then there is a constant $c < 1$, for which Γ is a sequence of zeroes for the space $\mathrm{Hol}(\mathbb{D}; M)$ with the function $M := cp + B_c b_p$, where $B_c \geqslant 0$ of some constant.

($S_{\log}$) *When $b = 1$, if for some constants $\varepsilon \in (0,1)$ of a strictly negative $a < 0$ and $C \geqslant 0$, the radial regularity condition (11) of the weight p is satisfied, and*

$$\sum_{k=1}^{\infty} \frac{|\lambda_k - \gamma_k|}{1 - \max\{|\lambda_k|, |\gamma_k|\}} < +\infty, \tag{15}$$

and Λ is a subsequence of zeroes for $\mathrm{Hol}(\mathbb{D}; p)$ then Γ is a sequence of zeroes for the space $H_{M+\log}$ with $M = p + Bb_p$, where $B \geqslant 0$ is a certain constant.

The statements (S_A), (S_1), ($S_{\log}$) of this stability theorem are corollaries of Theorems 2, 3 and 4, respectively (see the justifications after their proofs).

Remark 1. The growth conditions of the right-continuity at zero and convexity with respect to a logarithmic function on the function p in Theorem 1 provide subharmonicity of the extended function $p\colon z \mapsto p(z)$, $z \in \mathbb{D}$ in a unit circle.

Remark 2. For a positive function p increasing to $+\infty$ on $(0,1)$, the condition of its convexity with respect to a logarithmic function on the whole interval $(0,1)$ and its right-continuity at zero can be substituted in the stability Theorem 1 by a weaker condition of convexity of the function p with respect to a logarithmic function on an interval $(t_0, 1)$, where $0 < t_0 < 1$. Indeed, in this case we can extend the function p to the interval $[0, t_0]$ by the rule

$p(t) := \liminf_t \to t_0 + 0p(t)$. The function p thus extended satisfies all the conditions of the stability Theorem 1, and the spaces of holomorphic functions, determined via (2), (5), (6) by the function $p = M$, are the same as determined by the extended function p of the space from (A), (H_p^1), $(H_{\log})$.

2. Nonradial stability theorem of (sub)sequences of zeroes for wight algebras

Let p be a subharmonic in $\mathbb{D}$ function with the Riesz measure ν_p, $p \not\equiv -\infty$.

For $z = re^{i\theta}$, $0 \leqslant r < 1$, $\theta \in \mathbb{R}$ and the numbers $a > 0$, let us introduce a *polar rectangular*

$$\square(z; a) := \Big\{ \zeta = te^{i\psi} \colon \big(r - a(1-r)\big)^+ \leqslant t < 1,\ |\sin(\psi - \theta)| < a(1-r) \Big\} \tag{16}$$

with respect to the size a, its s*-slice* when $s < 1$

$$\square_s(z; a) := \Big\{ \zeta = te^{i\psi} \colon \big(r - a(1-r)\big)^+ \leqslant t < s,\ |\sin(\psi - \theta)| < a(1-r) \Big\},$$

the function of the measure distribution ν_p in (16) by the rule $\nu_p(s, z; a) := \nu_p\big(\square_s(z; a)\big)$, as well as an *a-extended auxiliary function*

$$b^{[a]}_{\nu_p}(z) := \frac{1}{(1-|z|)^2} \int_{\square(z;a)} (1-|\zeta|)^2 \,\mathrm{d}\nu_p(\zeta) = \frac{1}{(1-r)^2} \int_{\big(r-a(1-r)\big)^+}^{1} (1-s)^2 \,\mathrm{d}\nu_p(s, z; a), \tag{17}$$

which is finite for all $z \in \mathbb{D}$ on condition (8). Moreover, we will use a special notation for averaging

$$\mathrm{Av}^{[\varepsilon]}_M(z) := \frac{1}{2\pi} \int_0^{2\pi} M\big(z + \varepsilon(1-|z|)e^{i\theta}\big) \,\mathrm{d}\theta, \quad 0 < \varepsilon < 1 \tag{18}$$

with the integrability condition for the function $M \colon \mathbb{D} \to [-\infty, \infty]$ with respect to circles.

Theorem 2. *Let p be a positive subharmonic function with the Riesz measure ν_p, and the condition* (7) *or the restriction* (8) *equivalent to it be satisfied. Moreover, we assume that p for $a = 1$ and some constants $\varepsilon \in (0,1)$, $b, C \geqslant 0$ satisfies the nonradial regularity condition* (10), *i. e.*

$$\mathrm{Av}^{[\varepsilon]}_p(z) + \log \frac{1}{1-|z|} \leqslant bp(z) + C, \quad z \in \mathbb{D}. \tag{19}$$

If for two sequences of the points $\Lambda = (\lambda_k)_{k\in\mathbb{N}}$ and $\Gamma = (\gamma_k)_{k\in\mathbb{N}}$ in $\mathbb{D}$ the condition of their closeness (13) *is satisfied and Λ is a subsequence of zeroes for the algebra A^∞_p, then Γ is a sequence of zeroes for the algebra A^∞_M for $M = p + b^{[6]}_{\nu_p}$.*

Proof. It follows from conditions of Theorem 2 that all conditions from [1, Theorem 0.2(S_1)] for the *convex* domain $\Omega = \mathbb{D}$ are satisfied. In view of the remark after the formulation of [1, Theorem 0.2] on strengthening this result for convex domains we conclude that, the sequence Γ is a subsequence of zeroes for the same algebra A^∞_p (even without the conditions (7)–(8)). In other words, Γ is a subsequence of zeroes for the class $\mathrm{Hol}(\mathbb{D}; M)$, where $M := cp$, c is some constant; ν_M is the Riesz measure of the subharmonic function M.

Now under the conditions of Theorem 2, according to [3, Theorem 2(U)] the subsequence of zeroes Γ for our space $\mathrm{Hol}(\mathbb{D}; M)$ becomes a sequence of zeroes for the space $\mathrm{Hol}\big(\mathbb{D}; A^{[\varepsilon]}_M + C_\varepsilon b^{[6]}_{\nu_M}\big)$ with any constant $\varepsilon \in (0,1)$ and some constant C_ε. However, it follows immediately from the regularity condition (10) and the form of the function $M = cp$ that

$$A^{[\varepsilon]}_M + C_\varepsilon b^{[6]}_{\nu_M} \leqslant bp + Cb^{[6]}_{\nu_M} \leqslant \max\{b, Cc\}\big(p + b^{[6]}_{\nu_p}\big)$$

everywhere on $\mathbb{D}$ for some constants $b, C, c \geqslant 0$. Thus, Γ is a sequence of zeroes for the algebra A^∞_M with a larger weight $M = p + b^{[6]}_{\nu_p}$. □

Let us deduce the part (S_A) of the radial Theorem 1 from Introduction from Theorem 2. By virtue of increment of the positive radial function p, the condition of its regularity (11) is even stronger than the nonradial regularity condition (10) of Theorem 2. The condition (7) for the radial function is (9). According to Remark 1, the extension of p to $\mathbb{D}$ is a subharmonic function. Thus, all conditions of Theorem 2 are satisfied.

When $a > 0$, we introduce an *a-extended auxiliary radial function*

$$b_p^{[a]}(r) := \frac{1}{1-r}\int_{\left(r-a(1-r)\right)^+}^{1^-}(1-t)\,\mathrm{d}p(t), \quad 0 \leqslant r < 1. \tag{20}$$

It remains only to demonstrate that the following holds.

Proposition 1. *Let the function $p\colon [0,1) \to [0,+\infty)$ satisfy conditions of Theorem 1, and its extension to $\mathbb{D}$ be denoted by p as well. Then, for an a-extended auxiliary function from (17) and the auxiliary function b_p from (12), the estimates*

$$b_{\nu_p}^{[a]}(z) \leqslant a b_p^{[a]}(r) + B_a \leqslant (8a+1)b_p(r) + C_a, \quad 0 \leqslant r = |z| < 1 \tag{21}$$

hold for some constants $B_a, C_a \geqslant 0$ in the notation $r := |z|$.

Proof. Let us denote the Riesz measure of the extended subharmonic function p by ν_p. The density of the Rieasz measure for such function is readily written out in polar coordinates via the initial function $p\colon [0,1) \to [0,+\infty)$ by means of the Laplace operator, namely:

$$d\nu_p(z) = \frac{1}{2\pi}\,\mathrm{d}\theta \otimes \mathrm{d}\big(tp'_-(t)\big), \quad z = te^{i\theta},\ r \geqslant 0,$$

where p'_- is the left derivative, $\otimes$ is the product of measures. Then, by Definitions (17) and (16) when $z = re^{i\theta}$ and with the condition that

$$\frac{a}{a+1} \leqslant r \leqslant 1 \iff \big(r - a(1-r)\big) \geqslant 0, \tag{22}$$

taking into account that

$$\textit{thefunction } t \mapsto tp'_-(t) \textit{ is increasing with respect to } t \in (0,1] \textit{ and is positive,} \tag{23}$$

for an increasing convex with respect to the logarithm of the function p, we have

$$\begin{aligned}
b_{\nu_p}^{[a]}(z) &= \frac{1}{2\pi(1-r)^2}\int_{\theta-\arcsin a(1-r)}^{\theta+\arcsin a(1-r)}\int_{r-a(1-r)}^{1}(1-t)^2\,\mathrm{d}\big(tp'_-(t)\big)\,\mathrm{d}\theta \\
&= \frac{\arcsin a(1-r)}{\pi(1-r)^2}\int_{r-a(1-r)}^{1}(1-t)^2\,\mathrm{d}\big(tp'_-(t)\big) \\
&\leqslant \frac{a(1-r)}{2(1-r)^2}\Big(\lim_{t\to 1-0}(1-t)^2 rp'_-(t) - \big(1-r+a(1-r)\big)^2\big((r-a(1-r))p'_-(r-a(1-r))\big) \\
&+2\int_{r-a(1-r)}^{1}(1-t)tp'_-(t)\,\mathrm{d}t\Big) \leqslant \frac{a}{2(1-r)}\left(\lim_{t\to 1-0}(1-t)^2 p'_-(t) + 2\int_{r-a(1-r)}^{1}(1-t)\,\mathrm{d}p(t)\right).
\end{aligned} \tag{24}$$

In what follows we need

Lemma 1. *For the function p from Proposition 1, the auxiliary function (12) is finite on $(0,1]$, and the equalities*

$$\lim_{r\to 1-0}(1-r)p(r) = 0, \qquad \lim_{r\to 1-0}(1-t)^2 p'_-(r) = 0 \tag{25}$$

hold.

Proof of Lemma 1. The moderate growth condition (9) and the growth of p provide

$$0=\lim_{r\to 1-0}\int_r^1 p(t)\,\mathrm{d}t \geqslant \lim_{r\to 1-0} p(r)\int_r^1 \mathrm{d}t=\lim_{r\to 1-0} p(r)(1-r)\geqslant 0, \tag{26}$$

and the first equality (25) is proved. Using it, one obtains

$$\int_r^1 (1-t)\,\mathrm{d}p(t)=-(1-r)p(r)+\int_r^1 p(t)\,\mathrm{d}t \tag{27}$$

integrating by parts. This yields that the auxiliary function b_p is finite according to the moderate growth condition (9) for p. Moreover, the left-hand side is tending to zero when $r\to 1-0$. It follows from (23) that

$$\int_r^1 (1-t)\,\mathrm{d}p(t)=\int_r^1 \frac{1-t}{t}\,tp'_-(t)\,\mathrm{d}t\geqslant rp'_-(r)\int_r^1\frac{1-t}{t}\,\mathrm{d}t\geqslant rp'_-(r)\frac{1}{2}(1-r)^2. \tag{28}$$

Whence, the second equality from (25) follows. The lemma is proved. □

According to the second equality from (25), one can exclude the limit in the right-hand side of (24) and obtain the estimate

$$b_{\nu_p}^{[a]}(z)\leqslant \frac{a}{(1-r)}\int_{r-a(1-r)}^1 (1-t)\,\mathrm{d}p(t),$$

which proves the first inequality in (21) with the restriction (22). The function $b_{\nu_p}^{[a]}$ is upper bounded by some constant independent of r. We have $r<\frac{a}{1+a}$ for the remaining values.

Let us turn to the proof of the second inequality from (21). Let us make the upper estimate of the integral

$$\begin{aligned}\int_{r-a(1-r)}^r (1-t)\,\mathrm{d}p(t)&=(1-r)p(r)-\big(1-r+a(1-r)\big)p\big(r-a(1-r)\big)\\ &+\int_{r-a(1-r)}^r p(t)\,\mathrm{d}t\leqslant(1-r)p(r)-(1-r)(1+a)p\big(r-a(1-r)\big)+p\big(r-a(1-r)\big)a(1-r)\\ &=(1-r)\big(p(r)-p\big(r-a(1-r)\big)\big)\end{aligned}$$

via the auxiliary function b_p with the condition (22). Whence,

$$\frac{1}{1-r}\int_{r-a(1-r)}^r (1-t)\,\mathrm{d}p(t)\leqslant\big(p(r)-p\big(r-a(1-r)\big)\big). \tag{29}$$

Upon substituting $r=e^x$, $r_a=r-a(1-r)=e^{x_a}$ for a convex function $P(x):=p(e^x)$, $-\infty<x<0$, one obtains the estimate [12, Corrolary 1.1.6]

$$\begin{aligned}p(r)-p(r_a)=P(x)-P(x_a)&\leqslant P'_-(x)(x-x_a)=p'_-(r)r(\log r-\log r_a)\\ &=p'_-(r)r\log\Big(1+\frac{a(1-r)}{r-a(1-r)}\Big)\leqslant p'_-(r)\,\frac{a(1-r)}{r-a(1-r)}.\end{aligned}$$

Whence, if

$$r-a(1-r)\geqslant\frac{1}{2}\iff r\geqslant\frac{a+1/2}{a+1}, \tag{30}$$

one obtains

$$p(r)-p(r_a)\leqslant 2a(1-r)p'_-(r)$$

and according to (29)

$$\frac{1}{1-r}\int_{r-a(1-r)}^r (1-t)\,\mathrm{d}p(t)\leqslant 2a(1-r)p'_-(r). \tag{31}$$

On the other hand, due to (28) and (30)

$$\frac{1}{1-r}\int_r^1 (1-t)\,\mathrm{d}p(t) \geqslant rp'_-(r)\,\frac{1}{2}\,(1-r) \geqslant \frac{1}{4}(1-r)p'_-(r).$$

Whence, according to (31)

$$\frac{1}{1-r}\int_{r-a(1-r)}^{r} (1-t)\,\mathrm{d}p(t) \leqslant 8a\,\frac{1}{1-r}\int_r^1 (1-t)\,\mathrm{d}p(t). \tag{32}$$

The latter yields the second inequality from (21) by definition of an a-extended auxiliary radial function $b_p^{[a]}$ from (20). The proposition is proved. □

This demonstrates that the part (S_A) of the radial Theorem 1 is a direct corollary of Theorem 2.

Let us consider now the radial theorem applied to specific logarithmic and power weights p from Propositions [L] and [P].

- *For the logarithmic weight from* [L] *with* $\alpha \geqslant 1$, *the regularity condition of the weight function of Theorem* 1 *from* $[\mathrm{S}_A]$ *is satisfied and*

$$b_p(r) \leqslant C_\alpha \log^{\alpha-1}\frac{1}{1-r}\,, \quad C_\alpha \textit{ is a constant}, \tag{33}$$

  when $p(r) := \log^{\alpha}\dfrac{1}{1-r}$, $0 \leqslant r < 1$ (see [5, Lemma 2])*. Thus, in this case, the algebra* A_M^∞ *with the weight* $M = p + b_p$ *coincides with the initial algebra* A_p^∞, *i. e. none extension of the algebra* A_p^∞ *takes place.*
- *For the power weight from* [P] *with* $0 \leqslant \beta < 1$, *the regularity condition of the weight function from Theorem* 1 *of Section* $[\mathrm{S}_A]$ *is satisfied. One can readily calculate that*

$$b_p(r) \leqslant C_\beta \frac{1}{(1-r)^\beta}\,, \quad C_\beta \textit{ is a constant},$$

  when $p(r) := \dfrac{1}{(1-r)^\beta}$, $0 \leqslant r < 1$. *Thus, the algebra* A_M^∞ *with the weight* $M = p + b_p$ *coincides with the initial algebra* A_p^∞ *as well, i. e. none extension of the algebra* A_p^∞ *takes place again.*

Remark 3. In all results of the works [3]–[5], where a 6-extended auxiliary function $b_p^{[6]}$ or correspondingly $b_M^{[6]}$ is involved in formulations for the radial function p or M in $\mathbb{D}$, Proposition 1 allows us to substitute it by a simpler auxiliary function b_p or b_M, respectively.

Examples. Let us draw examples of nonradial weight functions to which the nonradial stability Theorem 2 can be applied.

Let $E \subset \partial\mathbb{D}$ be a subset on a unit circle. Assume that

$$d_{\mathbb{D}}(z,E) := \inf\{|w-z| \colon w \in \partial\mathbb{D}\} = \operatorname{dist}(z,E), \quad z \in \mathbb{D},$$

is the distance from the point $z \in \mathbb{D}$ to the unit circle.

(P_E) the functions

$$p \colon z \mapsto \frac{1}{\bigl(d_D(z,E)\bigr)^\beta}\,, \quad z \in \mathbb{D}$$

are continuous positive subharmonic for constants $\beta \geqslant 0$ (see [13]–[16] together with applications), and when $0 \leqslant \beta < 1$, they satisfy the moderate growth condition (7), because this function is majorized by a power function $z \mapsto \frac{1}{(1-|z|)^\beta}$, $z \in \mathbb{D}$.

Let us consider the functions $l_1\colon z\mapsto \log|z|$ and $L_1\colon z\mapsto \log(1+|z|)$, $z\in\mathbb{C}$ in order to construct examples of nonradial weights of the logarithmic growth similar to [L]. The functions l_1 and L_1 are subharmonic in $\mathbb{C}$: l_1 as a logarithm of the module of the holomorphic function $z\mapsto z$, $z\in\mathbb{C}$, and L_1 as a positive continuous function with the values of the Laplace operator $\Delta L_1(z)=\frac{1}{r(1+r)^2}\geqslant 0$ positive everywhere when $r:=|z|>0$ and the value $L_1(0)=0$.

Let us consider a convex growing function

$$\psi_\alpha(x):=\begin{cases}0, & x\leqslant 0\\ x^\alpha, & x>0\end{cases}$$

for the constant $\alpha\geqslant 1$, assuming that $\psi_\alpha(-\infty):=\lim_{x\to-\infty}\psi_\alpha(x)=0$. According to [12, Theorem 3.2.18], the compositions

$$(\psi_\alpha\circ l_1)(z):=(\log^+|z|)^\alpha:=(\log^+)^\alpha|z|,\quad (\psi_\alpha\circ L_1)(z)=\log^\alpha\bigl(1+|z|\bigr),\quad z\in\mathbb{D},$$

are subharmonic as well for every $\alpha\geqslant 1$. Whence, it follows that compositions of these functions with any function $f\in\operatorname{Hol}(\mathbb{D})$, i. e. the functions $(\log^+)^\alpha|f|$ and $\log^\alpha\bigl(1+|f|\bigr)$ are subharmonic, positive and continuous in $\mathbb{D}$ [17, Coprollary 2.5.7]. In particular, for every point $w\in\partial\mathbb{D}$, the functions

$$z\mapsto(\log^+)^\alpha\frac{1}{|z-w|},\quad z\mapsto\log^\alpha\Bigl(1+\frac{1}{|z-w|}\Bigr)$$

are the same when $\alpha\geqslant 1$ and $f(z)\equiv 1/(z-w)$, $z\in\mathbb{D}$. Hence, the exact upper boundaries of these functions with respect to $w\in E$, equal to

$$(\log^+)^\alpha\frac{1}{d_{\mathbb{D}}(\cdot,E)},\quad \log^\alpha\Bigl(1+\frac{1}{d_{\mathbb{D}}(\cdot,E)}\Bigr),\tag{34}$$

respectively, being continuous are also subharmonic positive functions, but not radial if E is a subset of the circle $\partial\mathbb{D}$ which is not dense everywhere.

(L_E) The functions (34), nonradial when $\overline{E}\neq\partial\mathbb{D}$, are continuous positive and subharmonic when $\alpha\geqslant 1$ and satisfy the moderate growth condition (7), because these functions are majorized by the logarithmic function $z\mapsto\log^\alpha\bigl(1+1/(1-|z|)\bigr)$, $z\in\mathbb{D}$.

3. Nonradial stability theorems of (sub)sequences of zeroes for weighted spaces

Main results of this section deal with weight spaces of holomorphic functions, that are not algebras, i. e. a product of two functions from the space can belong already not to this space.

The space $H_p^{1^-}$. Similarly to [1], we impose the following additional regularity condition[1] on a subharmonic positive weight function p :

(LD_0^1) *when $a=1$ for any number $b>1$ there are numbers ε, $0<\varepsilon<1$, and C_b, such that* (10) *is satisfied, i. e. in the notation* (18)*, the restriction*

$$\operatorname{Av}_p^{[\varepsilon]}(z)+\log\frac{1}{1-|z|}\leqslant bp(z)+C_b,\quad z\in\mathbb{D}\tag{35}$$

takes place for averaging $\operatorname{Av}_p^{[\varepsilon]}$.

Theorem 3. *Let us assume that both the moderate growth condition* (7) *and the restriction* (LD_0^1) *hold for a positive subharmonic function p in $\mathbb{D}$. If for two sequences of points $\Lambda=(\lambda_k)_{k\in\mathbb{N}}$ and $\Gamma=(\gamma_k)_{k\in\mathbb{N}}$ in $\mathbb{D}$ the condition of their closeness* (14) *is satisfied and Λ is a sequence of zeroes for the space $H_p^{1^-}$, then there are constants $c<1$ and $B_c\geqslant 0$ such that Γ is a sequence of zeroes for the space* $\operatorname{Hol}(\mathbb{D};M)$ *when*

$$M=cp+B_cb_{\nu_p}^{[6]}.\tag{36}$$

[1]The numbering from [1] is used.

In particular, if p is a logarithmic weight of the form [L] *with $\alpha > 1$, then the second addend in the righthand side* (36) *disappears, and Λ is a sequence of zeroes for the space $H_p^{1^-}$.*

Proof. [1, Theorem 0.2(S_3)] proves that under the conditions of Theorem 3, the sequence of points Γ is a sequence of nonuniqueness, or a subsequence of zeroes, for the space $H_p^1 := \bigcup_{0\leqslant c<1} \mathrm{Hol}(\mathbb{D}, cp)$ (even without equivalent conditions (7)–(8)), i.e. for some $c' < 1$ for the space $\mathrm{Hol}(\mathbb{D}; c'p)$. Moreover, for the function $c'p$ and its Riesz measure $\nu_{c'p}$, the equivalent conditions (7)–(8) are still satisfied with the change of p by $c'p$. Under these conditions it is claimed in [3, Theorem 2, section. (U)] that every subsequence of zeroes for the space $\mathrm{Hol}(\mathbb{D}; c'p)$ for any $\varepsilon \in (0, 1)$ becomes a sequence of zeroes for the space

$$\mathrm{Hol}\big(\mathbb{D}; \mathrm{Av}_{c'p}^{[\varepsilon]} + C_\varepsilon b_{c'p}^{[6]}\big). \tag{37}$$

Obviously, $b_{c'p}^{[6]} = c'b_p^{[6]}$. Moreover, by virtue of the regularity condition (LD_0^1) when the value of the number $b > 1$ is sufficiently small for which the restriction $c = c'b < 1$ is satisfied, the inequality

$$\mathrm{Av}_{c'p}^{[\varepsilon]}(z) \leqslant cp(z) + C, \quad z \in \mathbb{D},$$

where C is a constant, holds. Thus, the space (37) is embedded to the weight space $\mathrm{Hol}(\mathbb{D}; M)$ with the weight M from (36), and Γ is a sequence of zeroes for this space, which was to be proved. In particular,

$$p(z) = \log^\alpha \frac{1}{1 - |z|}, \quad \alpha > 1, \tag{38}$$

the conditions (7) and (LD_0^1) are satisfied, and the estimate (33) provides

$$b_{\nu_p}^{[6]}(z) = O\Big(\log^{\alpha-1} \frac{1}{1 - |z|}\Big), \quad z \to \partial\mathbb{D} \tag{39}$$

for the logarithmic weight (38). Whence, with the same choice of the weight p, one can find a constant $d \in (c, 1)$ such that the inequality $cp(z) + B_c b_{\nu_p}^{[6]}(z) \leqslant dp(z)$ is satisfied for all $z \in \mathbb{D} \setminus D(t)$ for a given $t < 1$. Since the holomorphic functions are bounded in the circles $D(t)$, it is sufficient for the space $\mathrm{Hol}\big(\mathbb{D}; cp + B_c b_{\nu_p}^{[6]}\big)$ to be embedded to $\mathrm{Hol}(\mathbb{D}; dp) \subset H_p^{1^-}$. The theorem is proved. □

Let us deduce the part (S_1) of the stability Theorem 1 in Introduction from Theorem 3. Since the function p is growing, the conditions on it are even stronger than the regularity conditions (LD_0^1) of Theorem 3 such that (11) holds. The condition (7) for the radial function is (9). According to Remark 1, the function p extended to $\mathbb{D}$ is subharmonic. Finally, the estimate (21) of Proposition 1 holds (see also Remark 3). This demonstrates that the part (S_1) of the stability Theorem 1 is a direct Corollary of Theorem 3.

The space $H_{p+\log}$. In [1], an auxiliary regularity condition [1] was imposed on the weight, not necessarily radial or positive, subharmonic function p, determining the space $H_{p+\log}$,

(LD_0^0) there are numbers ε, $0 < \varepsilon < 1$, and $c, C \geqslant 0$ such that the inequality

$$\mathrm{Av}_p^{[\varepsilon]}(z) \leqslant p(z) + c \log \frac{1}{1 - |z|} + C, \quad z \in \mathbb{D}. \tag{40}$$

holds in the notation (18) for averaging.

Theorem 4. *Let us assume that for a function $p \not\equiv -\infty$ subharmonic in $\mathbb{D}$ the moderate growth conditions* (7) *and the condition of the weight* (LD_0^0) *regularity are satisfied. If for two sequences of points $\Lambda = (\lambda_k)_{k\in\mathbb{N}}$ and $\Gamma = (\gamma_k)_{k\in\mathbb{N}}$ in $\mathbb{D}$, the condition of their closeness* (15) *is*

[1] The numbering from [1] is used again.

satisfied, and Λ *is a subsequence of zeroes for the space* $H_{p+\log}$*, there are constants* $C, B \geqslant 0$ *such that* Γ *is a sequence of zeroes for the space* $\mathrm{Hol}(\mathbb{D}; M)$ *when*

$$M = p + C \log \frac{1}{1 - |\cdot|} + B b^{[6]}_{\nu_p}. \qquad (41)$$

In particular, if p *is a logarithmic weight of the form* [L] *with* $\alpha \geqslant 1$*, then* Γ *is a sequence of zeroes for the space* $\mathrm{Hol}(\mathbb{D}; M)$*, where*

$$M(z) := p(z) + C_\alpha \log^{\max\{1, \alpha-1\}} \frac{1}{1 - |z|}, \ z \in \mathbb{D}, \quad C_\alpha \text{ is a constant.} \qquad (42)$$

Proof. Note that, the inequality (40) coincides with (10) when $0 > a = -c$ and $b = 1$. In [1, Theorem 0.2(S_4)] it is proved that, in the very conditions of Theorem 4, the sequence of points Γ is a sequence of nonuniqueness or the subsequence of zeroes, for the space $H_{p+\log}$ (even without the equivalence conditions (7)–(8)), i. e. D is a constant for a certain $C \geqslant 0$ for the space $\mathrm{Hol}(\mathbb{D}; \hat{M})$, where $\hat{M}(z) := p(z) + D \log \frac{1}{1-|z|}$, $z \in \mathbb{D}$ is a subharmonic function. Moreover, for the function $\hat{M}$ and for its Riesz measure $\nu_{\hat{M}}$, the equivalence moderate growth conditions (7)–(8) are satisfied as before with the change of p by $\hat{M}$. With these conditions in [3, Theorem 2, section . (U)] it is established, that every subsequence of zeroes for the space $\mathrm{Hol}(\mathbb{D}; \hat{M})$ for any $\varepsilon \in (0, 1)$ becomes a sequence of zeroes for the space $\mathrm{Hol}\big(\mathbb{D}; \mathrm{Av}^{[\varepsilon]}_{\hat{M}} + C_\varepsilon b^{[6]}_{\hat{M}}\big)$. The regularity conditions (LD^0_0) readily provide that the latter space is embedded to the space $\mathrm{Hol}(\mathbb{D}; M)$ with the weight M from (41), and Γ is a sequence of zeroes for this space, which was to be proved.

In particular, the function from (41) is majorized by the function (42) according to (33) for the logarithmic weight p, which proves the final part of Theorem 4. □

Let us deduce the part ($H_{\log}$) from the stability Theorem 1 of Introduction from Theorem 4. By virtue of growth of the function p, conditions imposed on it, such that (11) holds, are even stronger than the regularity conditions (LD^0_0) of Theorem 4. The condition (7) for the radial function is (9). The subharmonic property of the function p extended to $\mathbb{D}$ was indicated in Remark 1. Finally, the inequality (21) holds. This demonstrates that the part ($S_{\log}$) of Theorem 1 is a direct corollary of the stability Theorem 4.

REFERENCES

1. Khabibullin B.N., Cherednikova L.Yu. *Subsequences of zeroes for classes of holomorphic functions, their stability and entropy of the linear connection. II* // Algebra and analysis. 2008. V. 20, . 1. P. 190–236.
2. D.H. Luecking *Zero sequences for Bergman spaces* // Complex Variables. 1996. V. 30. P. 345–362.
3. Kudasheva E.G., Khabibullin B.N. *Distribution of zeroes of holomorphic moderate growth functions in a unit circle and representation of meromorphic functions in it* // Matem. sb. 2009. V. 201, 9. P. 995–126.
4. Khabibullin F.B. *Sequences of zeroes of holomorphic functions in weight space in a unit circle* // Izvestija vuzov. Matem. 2010. No. 3. P. 102–105.
5. Khabibullin F.B. *Sequences of zeroes for holomorphic functions in spaces in circles* // International school-conference for students, and young researchers «Fundamental mathematics and its applications in natural sciences», devoted to the 100th anniversary of Bash SU. Mathematics. Vol I. Ufa. RIZ BashGu. 2009. P. 357–377.
6. Khabibullin F.B. *Sequences of zeroes of moderate growth holomorphic in a circle* // Spectral theory of operators and its applications. (Ufa, 13–15 June 2011). Ufa. RiZ BashGU. 2011. P. 85–86.
7. Khabibullin F.B. *Stability of (sub)sequences of zeroes in spaces of holomorphic functions of the moderate growth in a circle* // Materials of the Xth international Kazan summer school-conference (Kazan, 1–7 July 2011). Theory of functions, its applications and conjugate questions. Kazan mathematical society. Works of N.I. Lobachevskii mathematical centre. 2011. V. 43. P. 358–360.
8. Nevanlinna R. *One-to-one analytical functions.* M.–L.: Gostekhizdat, 1941.

9. *Heyman U. Meromorphic functions.* M: Mir, 1966.
10. Shamoyan F.A., *M.M. Dzhrbashyan factorization theorem and characteristics of zeroes of functions analytical in the circle of functions with a majorant of finite growth* // Izv. AN Arm. SSR. Matematika. 1978. V. XIII, 5–6. P. 405–422.
11. K. Seip *An extension of the Blaschke condition* // J. London Math. Soc. 1995. V. 51. P. 545–558.
12. L. Hörmander *Notions of Convexity.* Progress in mathematics. 1994. V.127. Birkhäser (Boston, Mass.).
13. A. Borichev, L. Golinskii, S. Kupin *A Blaschke-type condition and its application to complex Jacobi matrices* // Bull. of the London Math. Soc. 2009. V. 41. P. 117–123.
14. S. Favorov, L. Golinskii *A Blaschke-type condition for analytic and subharmonic functions and application to contraction operators* // Amer. Math. Soc. Transl. 2009. V. 226, . 2. P. 37–47.
15. S. Favorov, L. Golinskii *On critical points of Blaschke products* // Matematychni Studii. 2010. V.34, No. 2. P. 168–173.
16. S. Favorov, L. Golinskii *Blaschke-type conditions for analytic functions in the unit disk: inverse problems and local analogs* // preprint arXiv:1007.3020 [math.CV], 2010.
17. M. Klimek *Pluripotential Theory.* Clarendon Press (Oxford etc.), 1991.

Farkhat Bulatovich Khabibullin,
Bashkir State University,
Z. Validi Str., 32,
450074, Ufa, Russia
E-mail: khabibullinfb@list.ru

Translated from Russian by E.D. Avdonina.