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APPROXIMATE PROPERTIES OF THE ROOT FUNCTIONS GENERATED BY THE CORRECTLY SOLVABLE BOUNDARY VALUE PROBLEMS FOR HIGHER ORDER ORDINARY DIFFERENTIAL EQUATIONS

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Abstract. In this work properties of systems of root functions generated by the correctly solvable boundary value problems for higher order ordinary differential equations are studied. The biorthogonal systems of functions corresponding to the system of root functions are constructed. The resulting system of root functions is a minimal system. The completeness of the system of root functions in $L_2(0,1)$ is proved. The algorithm for the inverse problem is given by reconstruction of the boundary functions. Moreover, some identities are found for the eigenvalues of the considered operator.

Keywords: ordinary differential equations, the system of root functions, the biorthogonal system, the eigenvalues, the completeness of the system of functions, correctly solvable, boundary value problems, inner boundary conditions, nonlocal boundary conditions

1. Introduction

Let L be a differential operator in a functional space $L_2(0,1)$ such that the inverse operator L^{-1} is completely continuous. Then, according to the statement from [1](p. 10), spectrum of the operator L consists of a finite or countable set of isolated eigenvalues of a finite algebraic multiplicity without finite accumulation points. With every eigenvalue λ_s of the geometric multiplicity m_s, one associates a chain of eigen and adjoint functions of the operator L

$$E_s = \{y_{s,0}(x,), y_{s,1}(x), ..., y_{s,m_s-1}(x)\}.$$

The union of various such chains

$$\{E_s : \lambda_s - \text{eigenvalue of the operator L}\}$$

is called the system of root functions of the operator L. Thus, the differential operator L is a source of a system of root functions. Systems of root functions are minimal families. The corresponding system of biorthogonal functions is constructed.

On the basis of asymptotic properties of entire functions one can deduce certain statements on behavior of the Fourier coefficients by the system of root functions of elements from $L_2(0,1)$. The behaviour of such sequence of the Fourier coefficients can differ significantly from the behaviour of the sequence of the Fourier coefficients according to the classical trigonometric system.

Let n be a natural number larger than two. Let us consider an operator L generated by an ordinary differential expression

$$l(y) \equiv y^{(n)}(x) = f(x) \tag{1}$$

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in the functional space $L_2(0,1)$ and having the inner boundary-value conditions

$$y^{(\nu-1)}(0)-\delta_{\nu k}\int_0^1 (l(y))\overline{\sigma(x)}dx=0, \nu=\overline{1;n}, \tag{2}$$

where k is a fixed natural number not larger than n, $\delta_{k\nu}$ is the Kronecker symbol, the function $\sigma(x)$ from the space $L_2(0,1)$, $\overline{\sigma(x)}$ indicates the complex conjugation.

It follows from the works [3], [4], that inner boundary-value conditions (2) with any kind of $\sigma(x)$ describe correctly solvable problems, corresponding to the expression $l(\cdot)$. The work [5] also gives a description of a class of correct problems for the Laplace operator.

The main result of the present work is the following theorem.

Theorem 1. *Is there exists a nonzero limit*

$$\lim_{\varepsilon\to 1-0}\frac{1}{1-\varepsilon}\int_\varepsilon^1 \sigma(x)dx=\alpha,$$

then the system of root functions of the operator L is complete in $L_2(0,1)$.

Theorem 1 is proved in section 5.

2. Resolvent of the operator L

Theorem 2. *Resolvent of the operator L is determined by the formula*

$$y(x,\lambda)=(L-\lambda I)^{-1}f(x)=R(\lambda)f(x)=(K-\lambda I)^{-1}f(x)+\psi(x,\lambda)<f;K^*(K^*-\bar{\lambda}I)^{-1}\sigma>, \tag{3}$$

where K is the operator corresponding to the zero Cauchy conditions at zero, K^* is the operator conjugate to the operator K, $\psi(x,\lambda)$ is the solution of a homogeneous equation, satisfying all (but for one) boundary-value conditions of the operator L:

$$l(\psi(x,\lambda))=\lambda\psi(x,\lambda),$$

$$U_\nu(\psi)\equiv\psi^{(\nu-1)}(0,\lambda)-\delta_{k,\nu}\int_0^1 l(\psi(y,\lambda))\overline{\sigma(y)}dy=\delta_{k,\nu}, \nu=\overline{1;n},$$

where $\delta_{k\nu}$ is the Kronecker symbol.

To prove Theorem 2 we will need the following lemma.

Lemma 1. *If $\varphi(x)$ is a solution of a homogeneous equation $l(\varphi)=0$ and satisfies the relations $\varphi^{(j-1)}(0)=\alpha_j, j=1,...,n$, then the function*

$$\psi(x,\lambda)=L(L-\lambda I)^{-1}\varphi(x)$$

represents a solution of the equation

$$l(\psi(x,\lambda))=\lambda\psi(x,\lambda)$$

and satisfies the conditions

$$U(\psi^{(j-1)}(x,\lambda))=\alpha_j, j=1,...,n.$$

Here α_j is chosen as follows: $\alpha_j=\delta_{jk}$, where k is a fixed natural number.

The lemma is verified directly. Indeed, one can readily see that

$$\psi(x,\lambda)=\varphi(x)+\lambda(L-\lambda I)^{-1}\varphi(x).$$

It remains only to apply the expressions $l(\cdot)$ and $U_j(\cdot)$ to both sides of the latter equality.

Proof of Theorem 2. Let s introduce the following notation

$$y_0(x,\lambda)=(K-\lambda I)^{-1}f(x),$$
$$C=<f,K^*(K^*-\bar{\lambda}I)^{-1}\sigma>,$$
$$\psi(x,\lambda)=L(L-\lambda I)^{-1}\varphi.$$

Manifestly, the function $y_0(x,\lambda)$ belongs to the domain of definition of the operator K and is a solution of the equation $l(y_0)-\lambda y_0=f(x)$. In other words, the boundary-value conditions $y_0^{(\nu-1)}(0,\lambda)=0, \nu=\overline{1;n}$ are satisfied for the function $y_0(x,\lambda)$. In accordance with the right-hand side of the relation (3), consider the function

$$y(x,\lambda)=y_0(x,\lambda)+C\psi(x,\lambda). \tag{4}$$

Let us apply the differential expression $l(\cdot)-\lambda$ to both parts of the latter equality. By virtue of Lemma 1, we have

$$l(y)-\lambda y=l(y_0)-\lambda y_0=f(x).$$

Thus, the right-hand side (4) satisfies the required nonhomogeneous equation. In order to verify the boundary-value conditions we apply the form $U_k(\cdot)$ to both parts of the equality (4). By virtue of Lemma 1, we have

$$U_\nu(y)=U_\nu(y_0)+C\cdot U_\nu(\psi)=y_0^{(\nu-1)}(0)-\delta_{k\nu}\int_0^1 l(y_0)\overline{\sigma(x)}dx+\delta_{k\nu}C=\delta_{k\nu}(C$$
$$-\int_0^1(f(x)+\lambda y_0(x))\overline{\sigma(x)}dx)=\delta_{k\nu}(< f,K^*(K^*-\overline{\lambda}I)^{-1}\sigma >$$
$$-\int_0^1(f(x)+\lambda(K-\lambda I)^{-1}f(x))\overline{\sigma(x)}dx)=0$$

when $\nu=1,...,n$. Theorem 2 is proved completely. Note that the works [5] and [6] represent resolvents of well-posed problems for the Laplace operator and the biharmonic operator.

Note that $\psi(x,\lambda)$ is expressed via $\kappa(x,\lambda)$ by the formula

$$\psi(x,\lambda)=\frac{\kappa(x,\lambda)}{\Delta(\lambda)},$$

where $\kappa(x,\lambda)$ is a solution of the equation

$$l(\kappa(x,\lambda))=\lambda\kappa(x,\lambda) \tag{5}$$

and satisfies the conditions

$$\kappa^{(\nu-1)}(0,\lambda)=\delta_{k,\nu}, \nu=1,...,n, \tag{6}$$

and

$$\Delta(\lambda)=1-\lambda\int_0^1\kappa(x,\lambda)\overline{\sigma(x)}dx. \tag{7}$$

Then, Theorem 2 entails the validity of the following Corollary:

Corollary 1. *Eigenvalues of the boundary-value problem* (1)–(2) *coincide with zeroes of the entire function* (7).

In particular, it follows that eigenvalues of the operator L are isolated and are of a finite multiplicity without finite accumulation points.

3. The system of root functions of the operator L and the corresponding biorthogonal system

Let λ_s be an eigenvalue of the multiplicity m_s. It means that

$$\Delta(\lambda_s)=0, \Delta^{'}(\lambda_s)=0,...,\Delta^{(m_s-1)}(\lambda_s)=0, \Delta^{(m_s)}(\lambda_s)\neq 0. \tag{8}$$

On page 445 of the monograph [7] there is an expansion theorem that entails that the projector $P_s: L_2[0,1]\to Ker(L-\lambda_s I)^{m_s}$ represents the residue of the resolvent at a singular point λ_s

$$(P_sf)(x)=-\frac{1}{2\pi i}\oint_{|\lambda-\lambda_s|=\delta}(L-\lambda I)^{-1}f(x)d\lambda$$

with a certain $\delta > 0$. In view of the fact that the resolvent $(K-\lambda I)^{-1}$ of the Cauchy operator represents an entire function of λ, the projector P_s from expression of the resolvent (3) takes the following form

$$(P_s f(x)) = -\frac{1}{2\pi i}\oint_{|\lambda-\lambda_s|=\delta}(L-\lambda I)^{-1}f(x)d\lambda =$$

$$= -\frac{1}{2\pi i}\oint_{|\lambda-\lambda_s|=\delta}\frac{\kappa(x,\lambda)}{\Delta(\lambda)} < f(\xi), K^*(K^*-\bar{\lambda}I)^{-1}\sigma(\xi) > d\lambda$$

$$= -res_{\lambda_s}\frac{\kappa(x,\lambda)}{\Delta(\lambda)} < f(\xi), K^*(K^*-\bar{\lambda}I)^{-1}\sigma(\xi) > . \tag{9}$$

Whence, P_s is an integral operator and its kernel has the form

$$P_s(x,\xi) = -\sum_{\gamma=1}^{m_s-1}\lim_{\lambda\to\lambda_s}\frac{1}{\gamma!}\frac{\partial^\gamma\kappa(x,\lambda)}{\partial\lambda^\gamma}\lim_{\lambda\to\lambda_s}\frac{1}{(m_s-\gamma-1)!}\frac{\partial^{m_s-\gamma-1}}{\partial\lambda^{m_s-\gamma-1}}\left(\frac{(\lambda-\lambda_s)^{m_s}M(\xi,\lambda)}{\Delta(\lambda)}\right),$$

where

$$M(\xi,\lambda) = \overline{K^*(K^*-\bar{\lambda}I)^{-1}\sigma(\xi)}.$$

The numbers

$$H_{s,m_s-\gamma-1}(\xi) = \lim_{\lambda\to\lambda_s}\frac{1}{(m_s-\gamma-1)!}\frac{\partial^{m_s-\gamma-1}}{\partial\lambda^{m_s-\gamma-1}}\left(\frac{(\lambda-\lambda_s)^{m_s}M(\xi,\lambda)}{\Delta(\lambda)}\right)$$

represent the Taylor coefficients of the functions $\frac{(\lambda-\lambda_s)^{m_s}M(\xi,\lambda)}{\Delta(\lambda)}$ at the point λ_s.

Let us introduce the notation:

$$y_{s,j}(x) = \frac{1}{j!}\frac{\partial^j\kappa(x,\lambda_s)}{\partial\lambda^j}, j=\overline{0,m_s-1},$$

$$E_s = \{y_{s,j}(x), j=\overline{0,m_s-1}\}.$$

The following statement follows from [8] (p. 29): $dimE_s = m_s$.

Let us investigate properties of the system of functions E, defined as follows:

$$E = \{E_s : \lambda_s - \text{eigenvalue of the operator L}\}.$$

Lemma 2. *Elements of the chain E_s satisfy the following differential equations:*

$$l(y_{s,j}) = \lambda_s y_{s,j}(x) + y_{s,j-1}(x), j=1,...,m_s-1 \tag{10}$$

$$l(y_{s,0}) = \lambda_s y_{s,0}(x) \tag{11}$$

and inner boundary conditions (2). *Thus the system of functions $y_{s,0}(x), y_{s,1}(x), ..., y_{s,m_s-1}(x)$ is a chain of root functions, generated by the eigenfunction $y_{s,0}(x)$, which is not identically zero.*

Proof of Lemma 2. Let us consider functions $y_{s,0} = \kappa(x,\lambda_s)$. By definition, the function $\psi(x,\lambda) = \frac{\kappa(x,\lambda)}{\Delta(\lambda)}$ satisfies the equation

$$l(\psi(x,\lambda)) = \lambda\psi(x,\lambda)$$

and inner boundary conditions

$$\frac{\partial^{i-1}\psi(x,\lambda)}{\partial x^{i-1}}\mid_{x=0} = \delta_{ki}\lambda\int_0^1\psi(x,\lambda)\overline{\sigma(x)}dx + \delta_{ki}, i=\overline{1,n}.$$

In other words, the equation

$$l(\kappa(x,\lambda)) = \lambda\kappa(x,\lambda) \tag{12}$$

with inner boundary conditions

$$\frac{\partial^{i-1}\kappa(x,\lambda)}{\partial x^{i-1}}\mid_{x=0}=\delta_{ki}(\lambda\int_0^1\kappa(x,\lambda)\overline{\sigma(x)}dx+\Delta(\lambda)), i=\overline{1,n} \tag{13}$$

holds. Since we have $\Delta(\lambda_s)=0$ for eigenvalues λ_s then,

$$l(\kappa(x,\lambda_s))=\lambda_s\kappa(x,\lambda_s)$$

and

$$\frac{\partial^{i-1}\kappa(x,\lambda_s)}{\partial x^{i-1}}\mid_{x=0}=\delta_{ki}\lambda_s\int_0^1\kappa(x,\lambda_s)\overline{\sigma(x)}dx, i=\overline{1,n}.$$

Thus, the validity of the formula (11) with inner boundary conditions (2) is proved for $y_{s,0}(x)$. Since $\kappa^{(k-1)}(0,\lambda)=1$ then, $y_{s,0}(x)$ is not identically zero. Hence, $y_{s,0}(x)$ is the eigenfunction of the operator L, corresponding to the eigenvalue λ_s. Now let $m_s-1\geq j\geq 1$. Let us prove (10) and validity of inner boundary conditions. To this end we have to differentiate (12) and (13) with respect to λ the corresponding number of times, and then substitute the value λ_s instead of λ and take into account the relation (8). Thus, Lemma 2 is proved.

Lemma 3. *The identity*

$$<\kappa(\xi,\lambda),\overline{M(\xi,\mu)}>=\frac{\Delta(\lambda)-\Delta(\mu)}{\mu-\lambda}$$

holds for arbitrary complex numbers λ,μ. *Note that* $M(\xi,\lambda)=\overline{\sigma(\xi)}+\lambda\overline{z(\xi,\bar\lambda)}$. *Here* $z(\xi,\bar\lambda)$ *is the solution of the formally conjugate nonhomogeneous equation* $l^*(z)=\bar\lambda z(\xi,\bar\lambda)+\sigma(\xi)$ *with zero conditions when* $\xi=1$

$$z(1,\bar\lambda)=...=z^{(n-1)}(1,\bar\lambda)=0.$$

Proof of Lemma 3. Let us calculate the following scalar product

$$<l(\kappa(x,\lambda)),\overline{M(x,\mu)}>=<l(\kappa(x,\lambda)),\sigma(x)+\bar\mu z(x,\bar\mu)>$$
$$=\lambda<\kappa(x,\lambda),\sigma(x)>+<\kappa(x,\lambda),\bar\mu\overline{l^*(z(x,\bar\mu))}>$$
$$+\mu\sum_{p=0}^{n-1}(-1)^{n-p}\kappa^{(p)}(0,\lambda)\cdot z^{(n-p-1)}(0,\bar\mu)$$
$$=\lambda<\kappa(x,\lambda),\sigma(x)>+\mu<\kappa(x,\lambda),\overline{M(x,\mu)}>$$
$$+\mu\sum_{p=0}^{n-1}(-1)^{n-p}\kappa^{(p)}(0,\lambda)\cdot z^{(n-p-1)}(0,\bar\mu)$$

for arbitrary λ,μ. Therefore, the equality

$$(\lambda-\mu)<\kappa(x,\lambda),\overline{M(x,\mu)}>=\lambda<\kappa(x,\lambda),\sigma(x)>$$
$$+\mu\sum_{p=0}^{n-1}(-1)^{n-p}\kappa^{(p)}(0,\lambda)\cdot z^{(n-p-1)}(0,\bar\mu)$$

holds.

In view of the relation (6), we obtain

$$(\lambda-\mu)<\kappa(x,\lambda),\overline{M(x,\mu)}>=\lambda<\kappa(x,\lambda),\sigma(x)>+\mu(-1)^{n-k+1}z^{(n-k)}(0,\bar\mu)\kappa^{(k-1)}(0,\lambda).$$

Whence, we obtain for $\lambda=\mu$

$$-\mu<\kappa(x,\mu),\sigma(x)>=\mu(-1)^{n-k+1}z^{(n-k)}(0,\bar\mu)\kappa^{(k-1)}(0,\lambda).$$

It follows from (6) that $\kappa^{(k-1)}(0,\mu)\equiv 1$. Hence, the equality

$$(-1)^{n-k+1}z^{(n-k-1)}(0,\bar{\mu})=\frac{-1}{\kappa^{(k-1)}(0,\mu)}\cdot<\kappa(x,\mu),\sigma(x)>$$

holds. Finally,

$$(\lambda-\mu)<\kappa(x,\lambda),\overline{M(x,\mu)}>=\lambda<\kappa(x,\lambda),\sigma(x)>-\mu<\kappa(x,\mu),\sigma(x)>.$$

Recall that

$$\Delta(\lambda)=1-\lambda<\kappa(x,\lambda),\sigma(x)>.$$

As a result, we obtain

$$(\lambda-\mu)<\kappa(x,\lambda),\overline{M(x,\mu)}>=(\Delta(\mu)-\Delta(\lambda)).$$

Whence follows what we require. Lemma 3 is proved.

The analysis of the formula (9) leads to the following notation:

$$E_l^{'}=\{h_{l,m_l-1}(x),h_{l,m_l-2}(x),...,h_{l,0}(x)\},$$

where

$$h_{l,m_l-1-j}(x)=-\frac{1}{(m_l-1-j)!}\lim_{\mu\to\lambda_l}\frac{\partial^{m_l-1-j}}{\partial\mu^{m_l-1-j}}\left(\frac{(\mu-\lambda_l)^{m_l}M(x,\mu)}{\Delta(\mu)}\right),j=0,1,...,m_l-1.$$

Let us introduce the following family of functions

$$E^{'}=\{E_l^{'}:\lambda_l\quad\text{is an arbitrary eigenvalue of the operator L}\}.$$

The following theorem holds.

Theorem 3. *The system of functions $E^{'}$ is biorthogonal to the system of functions E, i.e.*

$$<y_{s,i}(x),h_{l,m_l-1-j}(x)>=\begin{cases}1, & if(s,i)=(l,j);\\ 0, & if(s,i)\neq(l,j).\end{cases}$$

Proof of Theorem 3. Let us consider two eigenvalues λ_s and λ_l. The pairs (s,i) and (l,j) correspond to them, where $i=0,1,\dots,m_s-1$ and $j=0,1,\dots,m_l-1$. Note that the scalar product

$$<y_{s,i}(x),h_{l,m_l-1-j}(x)>=$$
$$=-\lim_{\lambda\to\lambda_s}\lim_{\mu\to\lambda_l}\frac{1}{i!}\frac{d^i}{d\lambda^i}\frac{1}{(m_l-1-j)!}\frac{d^{m_l-1-j}}{d\mu^{m_l-1-j}}\left(<\kappa(x,\lambda),\overline{M(x,\mu)}>\frac{(\mu-\lambda_l)^{m_l}}{\Delta(\mu)}\right).$$

Invoking Lemma 3, we obtain the equality

$$<y_{s,i}(x),h_{l,m_l-1-j}(x)>=$$
$$=-\lim_{\lambda\to\lambda_s}\lim_{\mu\to\lambda_l}\frac{1}{i!}\frac{d^i}{d\lambda^i}\frac{1}{(m_l-1-j)!}\frac{d^{m_l-1-j}}{d\mu^{m_l-1-j}}\left(\frac{\Delta(\lambda)-\Delta(\mu)}{\mu-\lambda}\frac{(\mu-\lambda_l)^{m_l}}{\Delta(\mu)}\right).\qquad(14)$$

Let us introduce the notation

$$H_{l,k}(\lambda)=\lim_{\mu\to\lambda_l}\frac{1}{k!}\frac{d^k}{d\mu^k}\left(\frac{\Delta(\lambda)-\Delta(\mu)}{\lambda-\mu}\frac{(\mu-\lambda_l)^{m_l}}{\Delta(\mu)}\right).\qquad(15)$$

Let us consider the function

$$F(\mu)=\frac{\Delta(\lambda)-\Delta(\mu)}{\lambda-\mu}\frac{(\mu-\lambda_l)^{m_l}}{\Delta(\mu)}$$

and expand it in the vicinity of the point $\mu=\lambda_l$ into the Taylor series. Then,

$$F(\mu)=H_{l,0}(\lambda)+H_{l,1}(\lambda)(\mu-\lambda_l)+H_{l,2}(\lambda)(\mu-\lambda_l)^2+\cdots+H_{l,m_l-1}(\lambda)(\mu-\lambda_l)^{m_l-1}+\dots,$$

i.e. $H_{l,k}(\lambda)$ is the k-th Taylor coefficient of the corresponding expansion in the neighborhood $\mu=\lambda_l$. Straightforward calculation of the coefficient of the Taylor series of the function $F(\mu)$ leads to the following series for $k=0,1,\dots,m_l-1$:

$$H_{l,k}(\lambda)=\Delta(\lambda)\left(A_{l,m_l-1}\frac{1}{(\lambda-\lambda_l)^{k+1}}+A_{l,m_l-2}\frac{1}{(\lambda-\lambda_l)^{k}}+\dots+A_{l,m_l-k-1}\frac{1}{\lambda-\lambda_l}\right), \qquad (16)$$

where the numbers $A_{l,m_l-1},\dots,A_{l,0}$ are determined from the identity

$$\frac{1}{\Delta(\mu)}\equiv\frac{A_{l,m_l-1}}{(\mu-\lambda_l)^{m_l}}+\frac{A_{l,m_l-2}}{(\mu-\lambda_l)^{m_l-1}}+\dots+\frac{A_{l,0}}{\mu-\lambda_l}+\sum_{q=0}^{\infty}B_{l,q}(\mu-\lambda_l)^q.$$

If $\lambda_s\neq\lambda_l$, the correlations (14), (15) and (16) when $i=0,1,\dots,m_l-1$ provide

$$<y_{s,i}(x),h_{l,m_l-1-j}(x)>=\lim_{\lambda\to\lambda_s}\frac{1}{i!}\frac{d^i}{d\lambda^i}H_{l,m_l-1-j}(\lambda)=$$

$$=\lim_{\lambda\to\lambda_s}\frac{1}{i!}\frac{d^i}{d\lambda^i}(\Delta(\lambda)\sum_{p=1}^{m_l-j}\frac{A_{l,j+p-1}}{(\lambda-\lambda_l)^p})=\frac{1}{i!}\sum_{t=0}^{i}C_i^t\Delta^{(t)}(\lambda_s)\sum_{p=1}^{k+1}A_{i-t,p}\frac{A_{l,m_l-k+p-2}}{(\lambda_s-\lambda_l)^{p+i-t}}=0,$$

because $\Delta^{(t)}(\lambda_s)=0$ for any $t<m_s$, and where $C_i^t, A_{i-t,p}$ are element of combinatorics.

Let us consider the case $\lambda_s=\lambda_l$. Transform the right-hand side of (16).

$$H_{l,k}(\lambda)=\Delta(\lambda)\sum_{p=1}^{k+1}\frac{A_{l,m_l-k+p-2}}{(\lambda_s-\lambda_l)^p}=$$

$$=\Delta(\lambda)\left(A_{l,m_l-1}\frac{1}{(\lambda-\lambda_l)^{k+1}}+A_{l,m_l-2}\frac{1}{(\lambda-\lambda_l)^{k}}+\dots+A_{l,m_l-k-1}\frac{1}{\lambda-\lambda_l}\right)=$$

$$=\Delta(\lambda)(\lambda-\lambda_l)^{m_l-k-1}\left(A_{l,m_l-1}\frac{1}{(\lambda-\lambda_l)^{m_l}}+A_{l,m_l-2}\frac{1}{(\lambda-\lambda_l)^{m_l-1}}+\dots+A_{l,m_l-k-1}\frac{1}{(\lambda-\lambda_l)^{m_l-k}}\right)$$

$$=\Delta(\lambda)(\lambda-\lambda_l)^{m_l-k-1}\left(\frac{1}{\Delta(\lambda)}-A_{l,m_l-2}\frac{1}{(\lambda-\lambda_l)^{m_l-k-1}}-\dots-A_{l,0}\frac{1}{\lambda-\lambda_l}-\sum_{q=m_l}^{\infty}B_{lq}(\lambda-\lambda_l)^q\right)=$$

$$=(\lambda-\lambda_l)^{m_l-k-1}+\sum_{q=m_l}^{\infty}c_{lq}^k(\lambda-\lambda_l)^q, s=0,1.\dots,m_l-1. \qquad (17)$$

The relations (14), (15) and (17) provide

$$<y_{s,i}(x),h_{l,m_l-1-j}(x)>=\frac{1}{i!}\lim_{\lambda\to\lambda_l}\frac{d^i}{d\lambda^i}H_{l,m_l-1-j}=$$

$$=\frac{1}{i!}\lim_{\lambda\to\lambda_l}\frac{d^i}{d\lambda^i}\left((\lambda-\lambda_l)^j+\sum_{q=m_l}^{\infty}c_{lq}^{m_l-1-j}(\lambda-\lambda_l)^q\right).$$

Whence, the required statement follows for $i=j$.

Theorem 3 is proved.

Thus, it follows from Lemma 2 and Theorem 3 that the system of functions E represents a system of root functions of the operator L, and the system of functions E' is biorthogonal to the system E. Hence, the system of functions E ia a minimal system of functions [2] (p. 171).

Only the existence of a biorthogonal system of functions is proved in [9] for the operator of the first order without giving the explicit formulae. In the initial terms of the boundary-value problem of Theorem 3 the biorthogonal system of functions is written out explicitly.

4. Functional series, generated by a system of root functions

Since the spectrum of the operator L is discrete, there is an increasing without limit sequence $\{R_N\}$ of radii such that the corresponding circles $|\lambda| = R_N$ do not possess points of the spectrum of the operator. Let $A_N = \{\lambda \in C : |\lambda| = R_N\}$ and $\sigma(L) = \{\lambda_1, \lambda_2, \dots\}$. In what follows we consider that R_N are chosen so that the inequality $dist(A_N, \sigma(L)) > \delta > 0$ holds for all N. Let us consider a subsequence of partial sums in accordance with the chosen circles

$$(S_N f)(x) = -\frac{1}{2\pi i} \oint_{|\lambda|=R_N} (L - \lambda I)^{-1} f(x) d\lambda \tag{18}$$

for an arbitrary function $f(\cdot)$ from the space $L_2(0, 1)$.

Let us substitute the relation (3) into (18) according to the theorem on residues and in view of the relations (18) and write the partial sum in the form

$$(S_N f)(x) = \sum_{|\lambda_s|<R_N} \sum_{j=0}^{m_s-1} < f, h_{m_s-1-j} > y_{s,j}(x). \tag{19}$$

According to the right-hand side (19), the sequence of Fourier coefficients $f \in L_2(0, 1)$ with respect to the system E is defined by the formula

$$c(f) = \{c_{s,i}(f) =< f, h_{s,i} >, i = 0, 1, \dots, m_s - 1,$$

$$\lambda_s \text{ is an arbitrary eigenvalue of the operator L}\}.$$

Naturally, the questions on convergence and summability of the subsequence $\{S_N f\}$ with respect to the norm $L_2(0, 1)$, on the behaviour of the Fourier coefficients $c_{s,i}$, etc arise. These questions are suggested by the theory of trigonometric Fourier series.

5. Completeness of the system of root functions

As it has been demonstrated above, the function $\kappa(x, \lambda)$ is a solution to the problem (5)–(6).

Let us partition the whole complex ρ-plane into $2n$ sectors $S_\nu, \nu = 0, 1, 2, ..., 2n - 1$, determined by the inequality

$$\frac{\nu\pi}{n} \leqslant arg(\rho) \leqslant \frac{(\nu + 1)\pi}{n}.$$

Denote by $\omega_1, \omega_2, ..., \omega_n$ all different roots in the n-th power -1. While ω^* will indicate one of the roots such that

$$Re(\omega^* \rho) = \max\{Re(\omega_1 \rho), Re(\omega_2 \rho), ..., Re(\omega_n \rho)\}$$

holds. Note that $Re(\omega^* \rho) > 0$ holds for all ρ.

It follows from [8] (p. 55) that the general solution (5) can be written as follows

$$\kappa(x, \lambda) = c_1 \exp(\omega_1 \rho x) + ... + c_n \exp(\omega_n \rho x), \tag{20}$$

where $\lambda = -\rho^n; \omega_i^n = -1, i = \overline{1;n}$ and $\{c_i, i = \overline{1;n}\}$ are some constants. Substituting (20) into (6), we obtain that

$$\kappa(x, \lambda) = \frac{1}{n} \sum_{p=1}^{n} \frac{\exp(\omega_p \rho x)}{(\omega_p \rho)^{k-1}}.$$

The estimate

$$C_1 \frac{\exp(Re(\omega^* \rho x))}{| \rho |^{k-1}} \leqslant | \kappa(x, \lambda) | \leqslant C_2 \frac{\exp(Re(\omega^* \rho x))}{| \rho |^{k-1}}, \tag{21}$$

holds when $| \rho | \to \infty$ from the resulting formula for some constants C_1 and C_2.

Here we investigate the problem of completeness of the system E in the functional space $L_2(0, 1)$. We will need the following lemma:

Lemma 4. *The set of functions*

$$D = \{\kappa(x,\mu), \mu - \textit{arbitrary complex number}\ \}$$

is dense in the space $L_2(0,1)$.

Proof of Lemma 4. To this end, it is sufficient to demonstrate that for any $h(x) \in L_2(0,1)$ from $< \kappa(x,\mu), h(x) >= 0, \forall \mu$ it follows that $h(x) = 0$ almost everywhere in $L_2(0,1)$. The formulae (5)–(6) provide that $\kappa(x,\mu)$ is a function analytic in μ. Then, $\kappa(x,\mu)$ can be expanded into the following series:

$$\kappa(x,\mu) = \sum_{i=0}^{\infty} A_i(x)\mu^i,$$

where

$$l(A_0(x)) = 0, A_0^{(\nu-1)}(0) = \delta_{k\nu}, \nu = \overline{1,n}$$

and for $i = 0, 1, ...$

$$l(A_{i+1}(x)) = A_i(x), A_{i+1}^{(\nu-1)}(0) = 0, \nu = \overline{1,n}.$$

One can readily verify that

$$A_i(x) = \frac{1}{(ni+k-1)!} x^{ni+k-1}, i = 0, 1, \dots.$$

Then, it follows that $< A_i(x), h(x) >= 0, i = 0, 1, ...$ for any $h(x) \in L_2(0,1)$ from $< \kappa(x,\mu), h(x) >= 0, \forall \mu$. Since the series $\sum_{i=1}^{\infty} \frac{1}{ni+k-1}$ diverges, the system of functions $\{A_i(x)\}_{i=0}^{\infty}$ is complete in $L_2(0,1)$ according to the Muntz theorem. Thus, $h(x) = 0$ almost everywhere in $L_2(0,1)$, i.e. the set of functions D is dense in $L_2(0,1)$. Which was to be proved.

Proof of Theorem 1. It is sufficient to approximate an arbitrary element from D with a prescribed accuracy by linear combinations from E. To this end we consider

$$S_N\kappa(x,\mu) = -\frac{1}{2\pi i} \oint_{|\lambda|=R_N} \frac{< \kappa(x,\mu), \overline{M(x,\lambda)} >}{\Delta(\lambda)} \kappa(x,\lambda) d\lambda.$$

Invoking Lemma 3, let us rewrite the latter relation

$$S_N\kappa(x,\mu) = \frac{1}{2\pi i} \oint_{|\lambda|=R_N} \frac{\Delta(\lambda) - \Delta(\mu)}{\lambda - \mu} \frac{\kappa(x,\lambda)}{\Delta(\lambda)} d\lambda$$

$$= \frac{1}{2\pi i} \oint_{|\lambda|=R_N} \frac{\kappa(x,\lambda)}{\lambda - \mu} d\lambda - \frac{1}{2\pi i} \oint_{|\lambda|=R_N} \frac{\Delta(\mu)}{\Delta(\lambda)} \frac{\kappa(x,\lambda)}{\lambda - \mu} d\lambda.$$

Using the Cauchy integral form, we obtain

$$S_N\kappa(x,\mu) = \kappa(x,\mu) - \frac{1}{2\pi i} \oint_{|\lambda|=R_N} \frac{\Delta(\mu)}{\Delta(\lambda)} \frac{\kappa(x,\lambda)}{\lambda - \mu} d\lambda.$$

Let us denote the inaccuracy $\kappa(x,\mu) - S_N\kappa(x,\mu)$ by $Q_N(x,\mu)$. Then, we have the integral representation

$$Q_N(x,\mu) = \frac{1}{2\pi i} \oint_{|\lambda|=R_N} \frac{\Delta(\mu)}{\Delta(\lambda)} \frac{\kappa(x,\lambda)}{\lambda - \mu} d\lambda$$

for the inaccuracy. Let us obtain the sufficient condition on $\sigma(x) \in L_2(0,1)$ in order to

$$\lim_{R_N \to \infty} \|Q_N(x,\mu)\| = 0,$$

where $\|\cdot\|$ is the norm of the space $L_2(0,1)$. Let us consider the norm of the inaccuracy $Q_N(x,\mu)$

$$\|Q_N(x,\mu)\| = \left(\int_0^1 \mid \frac{1}{2\pi i} \oint_{|\lambda|=R_N} \frac{\Delta(\mu)}{\Delta(\lambda)} \frac{\kappa(x,\lambda)}{\lambda - \mu} d\lambda \mid^2 dx \right)^{\frac{1}{2}}.$$

To begin with, we estimate $|\Delta(\lambda)|$ on the circles $|\lambda| = R_N$ when $R_N \to \infty$. Let us assume that there is a limit $\lim_{\varepsilon\to 1-0} \frac{1}{\varepsilon}\int_\varepsilon^1 \sigma(x)dx = \alpha_1 \neq 0$. Then the lower estimate

$$|\Delta(\lambda)| \geq c_1 R_N^{\frac{n-k+1}{n}} \exp(Re(\omega^*\rho)) > 0 \tag{22}$$

holds for the function $\Delta(\lambda)$.

Writing λ in the form $\lambda = R_N \exp(i\theta)$ and due to (21) and (22), we obtain

$$\|Q_N(x,\mu)\| \leqslant \frac{|\Delta(\mu)|}{2\pi}\left(\int_0^1 |\oint_{|\lambda|=R_N} \frac{1}{\Delta(\lambda)}\frac{\kappa(x,\lambda)}{\lambda-\mu}d\lambda|^2\, dx\right)^{\frac{1}{2}}$$

$$\leqslant \frac{|\Delta(\mu)|}{2\pi}\left(\int_0^1 |\int_0^{2\pi} \frac{|\kappa(x,\lambda)|}{||\lambda|-|\mu||\cdot|\Delta(\lambda)|}R_N d\theta|^2\, dx\right)^{\frac{1}{2}}$$

$$\leqslant C\,|\Delta(\mu)|\left(\int_0^1 |\int_0^{2\pi} \frac{\frac{\exp(Re(\omega^*\rho x))}{R_N^{\frac{k-1}{n}}}}{R_N(1-|\frac{\mu}{R_N}|)\cdot R_N^{\frac{n-k+1}{n}}\exp(Re(\omega^*\rho))}R_N d\theta|^2\, dx\right)^{\frac{1}{2}}$$

$$= C\frac{|\Delta(\mu)|}{R_N}\left(\int_0^1 |\int_0^{2\pi} \frac{\exp(Re(\omega^*\rho(x-1)))}{(1-|\frac{\mu}{R_N}|)}d\theta|^2\, dx\right)^{\frac{1}{2}}$$

for the norm $Q_N(x,\mu)$. Thus, we arrive to the limiting relation

$$\lim_{R_N\to\infty} \|Q_N(\cdot,\mu)\| = 0.$$

The theorem is proved.

Lemma 5. *Let us assume that there is a nonzero limit*

$$\lim_{\varepsilon\to 1-0} \frac{1}{1-\varepsilon}\int_\varepsilon^1 \sigma(x)dx = \alpha.$$

Then, the limiting relation

$$\lim_{R_N\to\infty} \|\kappa(x,\mu) - \sum_{|\lambda_s|<R_N}\sum_{j=0}^{m_s-1} < \kappa(t,\mu), h_{s,m_s-1-j}(t) > y_{s,j}(x)\| = 0$$

holds for any complex number μ.

Lemma 5 and the fact that $\overline{D} = L_2(0,1)$ provide the statement of Theorem 1.

6. The inverse problem

The problem on eigenvalues

$$l(y) = \lambda y(x), x \in (0,1), \tag{23}$$

$$y^{(\nu-1)}(0) - \delta_{\nu k}\int_0^1 (l(y))\overline{\sigma(x)}dx = 0, \nu = \overline{1;n} \tag{24}$$

is studied. Given the complete set of eigenvalues $\{\lambda_s, s = 1,2,...\}$ of the boundary problem (23)-(24). Restore the boundary function $\sigma(x)$ from $L_2(0,1)$.

Corollary 1 entails the statement about the direct problem

Theorem 4. *For any boundary function* $\sigma(x)$ *from* $L_2(0,1)$, *the identity*

$$\sum_{s=1}^\infty \frac{1}{\lambda_s} = \int_0^1 \kappa(x,0)\overline{\sigma(x)}dx$$

holds.

The main result of the section is the algorithm for finding the boundary function $\sigma(x)$ by the spectrum $\{\lambda_s, s = 1, 2, ...\}$:

Let us assume that the sequence of nonzero complex numbers $\{\lambda_s, s = 1, 2, ...\}$ is given without finite accumulation points posessing the following properties:

1)the series $\sum_{s=1}^{\infty} \frac{1}{\lambda_s}$ converges

2)the system of functions $\{\kappa(x, \lambda_s), s = 1, 2, ...\}$ is complete and minimal in $L_2(0, 1)$

3) the following series $\sum_{s=1}^{\infty} \frac{1}{\lambda_s} h(x, \lambda_s)$ converges to $\{\kappa(x, \lambda_s), s = 1, 2, ...\}$ in the sense $L_2(0, 1)$ for the biorthogonal system of functions $\{h(x, \lambda_s), s = 1, 2, ...\}$.

Then, the boundary function $\sigma(x)$ is restored by the formula

$$\sigma(x) = \sum_{s=1}^{\infty} \frac{1}{\lambda_s} h(x, \lambda_s)$$

and belongs to $L_2(0, 1)$.

7. On identities for eigenvalues

Let us denote by $\{\lambda_k\}_{k=1}^{\infty}$ eigenvalues of the operator L, numbered in the increasing order with respect to the module in view of their multiplicities, and by $R(\lambda) = (L - \lambda I)^{-1}$ we indicate the resolvent of the operator L.

According to the formula (3)

$$R(\lambda) f(x) = (K - \lambda I)^{-1} f(x) + \psi(x, \lambda) < f; K^*(K^* - \bar{\lambda} I)^{-1} \sigma > . \quad (25)$$

Let us calculate the trace of (25). To begin with, we calculate the trace of the right-hand side (25). The trace of the first addend equals to zero, since the first addend is the Volterra operator. The trace of the second addend, i.e.

$$Sp(K) = \int_0^1 (K^*(K^* - \lambda I)^{-1} \sigma(t))(L(L - \lambda I)^{-1} \varphi(t)) dt$$
$$= < L(L - \lambda I)^{-1} \varphi, K^*(K^* - \lambda I)^{-1} \sigma >,$$

where $Kf = \int_0^1 (K^*(K^* - \lambda I)^{-1} \sigma(t))(L(L - \lambda I)^{-1} \varphi(x)) f(t) dt$.

Since the trace of the right-hand side exists in the identity (25), then the trace of the left-hand side exists as well and the following identity holds

$$Sp(R(\lambda)) = \sum_{k=1}^{\infty} \frac{1}{\lambda_k - \lambda} = < L(L - \lambda I)^{-1} \varphi, K^*(K^* - \lambda I)^{-1} \sigma > . \quad (26)$$

The identity (26) holds for all λ from $U = \{\lambda : |\lambda| < |\lambda_1|\}$, and the following expansion series hold:

$$\frac{1}{\lambda_k - \lambda} = \frac{1}{\lambda_k} \left(\frac{1}{1 - \frac{\lambda}{\lambda_k}}\right) = \frac{1}{\lambda_k} \left(1 + \frac{\lambda}{\lambda_k} + \left(\frac{\lambda}{\lambda_k}\right)^2 + ... + \left(\frac{\lambda}{\lambda_k}\right)^n + ...\right),$$

$$K^*(K^* - \lambda I)^{-1} = (I - \lambda (K^*)^{-1})^{-1} = I + \lambda (K^*)^{-1} + \lambda^2 (K^*)^{-2} + ... + \lambda^j (K^*)^{-j} + ...,$$

$$L(L - \lambda I)^{-1} = (I - \lambda (L)^{-1})^{-1} = I + \lambda (L)^{-1} + \lambda^2 (L)^{-2} + ... + \lambda^j (L)^{-j} + ...$$

Thus, we arrived to the following identity equivalent to (26):

$$\sum_{k=1}^{\infty} \sum_{l=0}^{\infty} \left(\frac{1}{\lambda_k}\right)^{l+1} \lambda^l = < \sum_{j=0}^{\infty} \lambda^j (L)^{-j} \varphi, \sum_{i=0}^{\infty} \lambda^i (K^*)^{-i} \sigma > . \quad (27)$$

Since (27) holds for all λ from $U = \{\lambda : |\lambda| < |\lambda_1|\}$ then, equating coefficients of λ^l, we obtain that

$$\sum_{k=1}^{\infty} \left(\frac{1}{\lambda_k}\right)^{l+1} = \sum_{j+i=l}^{\infty} < (L)^{-j} \varphi, (K^*)^{-i} \sigma > .$$

for any $l \in Z_+$.

Let us formulate the obtained results in the form of the following theorem.

Theorem 5. *Let $\{\lambda_k\}_{k=1}^{\infty}$ be eigenvalues of the problem (23)–(24), numbered in the increasing order in view of their multiplicities then, the following identity holds for any $l \in Z_+$:*

$$\sum_{k=1}^{\infty}(\frac{1}{\lambda_k})^{l+1} = \sum_{j+i=l}^{\infty} < (L)^{-j}\varphi, (K^*)^{-i}\sigma > .$$

In particular, when $l = 0$ we obtain

$$\sum_{k=1}^{\infty}\frac{1}{\lambda_k} =< \varphi, \sigma > .$$

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