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PERTURBATION OF THE SHRÖDINGER OPERATOR BY A NARROW POTENTIAL

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Abstract. A discrete spectrum of the Schrödinger operator perturbed by a potential on the real line is studied. The potential depends on two small parameters. One of the parameters describes the length of the support of the potential and the inverse of the other parameter corresponds to the magnitude of the potential.

Keywords: Schrödinger operator, perturbation, matching of asymptotic expansions

1. Introduction

The paper is considers perturbation of a discrete spectrum of a lower semibounded selfadjoint Schrödinger operator in $L_2(\mathbb{R})$:

$$\mathcal{H}_0 := -\frac{d^2}{dx^2} + \mathcal{W},$$

where $\mathcal{W}$ is the operator of multiplication by a real function $W(x)$ locally integrable in $\mathbb{R}$ such that

$$\int\limits_{-\infty}^{\infty} W(x)|y(x)|^2 dx \geqslant c\|y\|^2_{L_2(\mathbb{R})}, \qquad c > -\infty \tag{1}$$

for any functions y from $L_2(\mathbb{R})$, for which the integral exists.

The perturbed selfadjoint operator (considered also in $L_2(\mathbb{R})$) has the form:

$$\mathcal{H}_{\mu,\varepsilon} := -\frac{d^2}{dx^2} + \mathcal{W} + \mu^{-1}\mathcal{V}_\varepsilon,$$

where $\mathcal{V}_\varepsilon$ is the operator of multiplication by the function $V\left(\frac{x}{\varepsilon}\right)$, $V(\xi)$ is the real finite function from $L_\infty(\mathbb{R})$,

$$\mu > 0, \qquad 0 < \varepsilon \ll 1.$$

Operators $\mathcal{H}_0$ and $\mathcal{H}_{\mu,\varepsilon}$ are understood as the Friedrichs extensions (see, e.g., [1, Chapter VI, § 2]) of the corresponding symmetric differential expressions

$$H_0 = -\frac{d^2}{dx^2} + W(x), \qquad H_{\mu,\varepsilon} = -\frac{d^2}{dx^2} + W(x) + \mu^{-1}V\left(\frac{x}{\varepsilon}\right)$$

with $C_0^\infty(\mathbb{R})$.

Namely, let us denote by $(\cdot,\cdot)_{L_2(\mathbb{R})}$ a scalar product in $L_2(\mathbb{R})$, and by $\mathfrak{h}_0$ and $\mathfrak{h}_{\mu,\varepsilon}$ quadratic form on $C_0^\infty(\mathbb{R})$, generated by the operators H_0 and $H_{\mu,\varepsilon}$:

$$\mathfrak{h}_0[y] := (H_0 y, y)_{L_2(\mathbb{R})}, \qquad \mathfrak{h}_{\mu,\varepsilon}[y] = (H_{\mu,\varepsilon} y, y)_{L_2(\mathbb{R})}.$$

Since the differential expressions H_0 and $H_{\mu,\varepsilon}$ are symmetric, defined densely in $L_2(\mathbb{R})$ and lower bounded, then the quadratic forms $\mathfrak{h}_0$ and $\mathfrak{h}_{\mu,\varepsilon}$ are symmetric as well, defined densely in $L_2(\mathbb{R})$ and lower semibounded, these forms being closable (see, e.g., [1, Chapter VI, Theorem 1.27]). Let us

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define operators $\mathcal{H}_0$ and $\mathcal{H}_{\mu,\varepsilon}$ as selfadjoint lower semibounded operators in $L_2(\mathbb{R})$, associated with the closure of these forms (see, e.g., [1, Chapter VI, Theorem 2.6]).

The paper investigates the behaviour of eigenvalues of the operator $\mathcal{H}_{\mu,\varepsilon}$ when $\mu, \varepsilon \to 0$.

Similar problems were investigated in [2] for the case when $\mu = \varepsilon^{-2}$, and $\int\limits_{-\infty}^{\infty} V(t)\,dt = 0$.

2. Formulation of results

In the following section the below two statements will be proved.

Lemma 1. *Eigenvalues of operators $\mathcal{H}_0$ (if they exist) are simple.*

Lemma 2. *Let*

$$\varepsilon\mu^{-1} = o\,(1)\,. \tag{2}$$

Then $\mathcal{H}_{\mu,\varepsilon} \to \mathcal{H}_0$ if $\varepsilon \to 0$ in the generalized sense.

These two lemmas and [1, Chapter IV, Theorem 3.16] entail

Theorem 1. *Let λ_0 be an eigenvalue of the operator $\mathcal{H}_0$ and the condition (2) hold. Then, the unique and simple eigenvalue $\lambda^{\mu,\varepsilon}$ of the operator $\mathcal{H}_{\mu,\varepsilon}$ converges to it, and for the corresponding projector $\mathcal{P}_{\mu,\varepsilon}$ there is a convergence in norm to the projector $\mathcal{P}_0$, corresponding to the eigenvalue λ_0.*

The work contains mainly the construction of complete asymptotics of the eigenvalue $\lambda^{\mu,\varepsilon}$ when $\mu, \varepsilon \to 0$. To this end, let us make an additional assumption that $V \in C_0^\infty(\mathbb{R})$, the function W is infinitely differentiable in some vicinity of zero (i.e. there is $\delta > 0$ such that $W \in C^\infty[-\delta, \delta]$). To justify the asymptotics strictly we will need a more rigid condition than (2). Namely, let us consider that there is $\gamma > 0$ such that

$$\varepsilon\mu^{-1} = O\,(\varepsilon^\gamma)\,. \tag{3}$$

In what follows we denote by ψ_0 the eigenfunction of the operator $\mathcal{H}_0$ normed in $L_2(\mathbb{R})$, corresponding to the eigenvalue λ_0, and use the following notation:

$$\langle g(t) \rangle := \int\limits_{-\infty}^{\infty} g(t)\,dt.$$

The following theorem is proved in the paper.

Theorem 2. *The eigenvalue $\lambda^{\mu,\varepsilon}$ of the operator $\mathcal{H}_{\mu,\varepsilon}$, converging to λ_0, has the asymptotics*

$$\lambda^{\mu,\varepsilon} = \lambda_0 + \sum_{i=1}^{\infty} \sum_{j=1}^{i} \varepsilon^i \mu^{-j} \lambda_{i,j}, \tag{4}$$

where

$$\lambda_{1,1} = \psi_0^2(0)\,\langle V(t) \rangle\,. \tag{5}$$

If

$$\psi_0(0)\,\langle V(t) \rangle = 0 \tag{6}$$

then,

$$\lambda_{i,i} = 0. \tag{7}$$

If

$$\langle V(t) \rangle = 0 \tag{8}$$

then,

$$\lambda_{2,1} = 2\psi_0(0)\psi_0'(0)\,\langle tV(t) \rangle\,. \tag{9}$$

If

$$\psi_0(0) = 0, \tag{10}$$

then

$$\lambda_{i+1,i} = 0, \tag{11}$$

$$\lambda_{3,1} = (\psi_0'(0))^2 \left\langle t^2 V(t) \right\rangle. \tag{12}$$

The theorem entails.

Corollary 1. *If the equality (8) holds, then*

$$\lambda^{\mu,\varepsilon} = \lambda_0 + \varepsilon^2\mu^{-1}\lambda_{2,1} + O\left(\varepsilon^3\mu^{-2}\right),$$

where $\lambda_{2,1}$ *is defined by the equality (9).*

If the equality (10) holds then

$$\lambda^{\mu,\varepsilon} = \lambda_0 + \varepsilon^3\mu^{-1}\lambda_{3,1} + O\left(\varepsilon^4\mu^{-2}\right),$$

where $\lambda_{3,1}$ *is defined by the equality (12).*

3. Proof of Lemmas 1 and 2

Proof of Lemma 1. Let us assume that the operator $\mathcal{H}_0$ has two linearly independent eigenfunctions $y_1,\ y_2$, corresponding to the eigenvalue λ_0. Hence,

$$(y_i', v')_{L_2(\mathbb{R})} = (\lambda y_i - W y_i, v)_{L_2(\mathbb{R})} \tag{13}$$

for any function $v \in W_2^1(\mathbb{R})$. Let us denote by w the Wronskian determinant of the functions y_1 and y_2:

$$w := y_1 y_2' - y_2 y_1'.$$

By definition of the derivative of the generalized function (see, e.g., [3]), for any function $\varphi \in C_0^\infty(\mathbb{R})$ one has

$$\left(w', \varphi\right) = -\left(y_1 y_2' - y_2 y_1', \varphi'\right).$$

Since $y_1, y_2 \in W_2^1(\mathbb{R})$, then $y_1 y_2',\ y_2 y_1' \in L_1(\mathbb{R})$. Therefore,

$$\left(w', \varphi\right) = -\left(y_1 y_2', \varphi'\right)_{L_2(\mathbb{R})} + \left(y_2 y_1', \varphi'\right)_{L_2(\mathbb{R})}. \tag{14}$$

Since $y_i \in W_2^1(\mathbb{R})$ then, for any function $C_0^\infty(\mathbb{R})$ one has:

$$\begin{aligned}\left(y_1 y_2', \varphi'\right)_{L_2(\mathbb{R})} &= \int\limits_{-\infty}^{\infty} y_1 y_2' \varphi' dx = \int\limits_{-\infty}^{\infty} y_2' \left((y_1\varphi)' - y_1'\varphi\right) dx \\ &= \left(y_2', (y_1\varphi)'\right)_{L_2(\mathbb{R})} - \int\limits_{-\infty}^{\infty} y_1' y_2' \varphi dx.\end{aligned}$$

Likewise,

$$\left(y_1' y_2, \varphi'\right)_{L_2(\mathbb{R})} = \left(y_1', (y_2\varphi)'\right)_{L_2(\mathbb{R})} - \int\limits_{-\infty}^{\infty} y_1' y_2' \varphi dx.$$

The latter two equalities and (14) entail

$$\left(w', \varphi\right) = -\left(y_2', (y_1\varphi)'\right)_{L_2(\mathbb{R})} + \left(y_1', (y_2\varphi)'\right)_{L_2(\mathbb{R})}.$$

This equality together with (13) provide sequentially that

$$\left(w', \varphi\right) = -\left(\lambda y_2 - W y_2, y_1\varphi\right)_{L_2(\mathbb{R})} + \left(\lambda y_1 - W y_1, y_2\varphi\right)_{L_2(\mathbb{R})} = 0.$$

Whence, it follows that $w \equiv C$, where C is a constant. However, since $w \in L_1(\mathbb{R})$, it is obvious that $C = 0$. It means that y_1, y_2 are linearly dependent. It follows from the resulting contradiction that Lemma 1 holds true. □

Proof of Lemma 2. It follows from the definition of the forms $\mathfrak{h}_0$ and $\mathfrak{h}_{\mu,\varepsilon}$ and the function V that

$$|(\mathfrak{h}_{\mu,\varepsilon} - \mathfrak{h}_0)[y]| = \mu^{-1}\left|\int\limits_{-\infty}^{\infty} V\left(\frac{x}{\varepsilon}\right)|y(x)|^2 dx\right| \leqslant C\mu^{-1}\int\limits_{-a\varepsilon}^{a\varepsilon} |y(x)|^2 dx,$$

where $C > 0$ are some fixed numbers, $a > 0$ is any number such that $\operatorname{supp}V(x) \subset [-a, a]$. Similarly to the proof of the Friedrichs inequality, one can easily prove the following its analogue (see, e.g., [4]):

$$\int_{-\varepsilon a}^{\varepsilon a} |y|^2 dx \leqslant C_1 \varepsilon \int_{-\infty}^{\infty} (|y'|^2 + |y|^2) dx,$$

where C_1 is a constant independent of $y \in C_0^\infty(\mathbb{R})$. Hence,

$$|(\mathfrak{h}_{\mu,\varepsilon} - \mathfrak{h}_0)[y]| \leqslant C_2 \mu^{-1} \varepsilon \int_{-\infty}^{\infty} (|y'|^2 + |y|^2) dx.$$

Since

$$\mathfrak{h}_0[y] = \int_{-\infty}^{\infty} |y'|^2 dx + \int_{-\infty}^{\infty} W(x) |y(x)|^2 dx,$$

by virtue of (1) one obtains the following estimate:

$$|(\mathfrak{h}_{\mu,\varepsilon} - \mathfrak{h}_0)[y]| \leqslant - C_2 \mu^{-1} \varepsilon \int_{-\infty}^{\infty} (W(x) - 1) |y(x)|^2 dx + C_2 \mu^{-1} \varepsilon \mathfrak{h}_0[y]$$

$$\leqslant C_2 \mu^{-1} \varepsilon \left(|c - 1| \int_{-\infty}^{\infty} |y(x)|^2 dx + \mathfrak{h}_0[y] \right).$$

Since by virtue of (2) the quadratic forms $\mathfrak{h}_0$ and $\mathfrak{h}_{\mu,\varepsilon}$ are defined densely in $L_2(\mathbb{R})$, lower bounded and closable, and $\mu^{-1}\varepsilon \to 0$ when $\varepsilon \to 0$ then, it follows from the latter estimate and [1, Chapter VI, Theorem 3.6] that the statement of the lemma holds true. $\square$

4. Construction of asymptotics

The matched asymptotic expansions approach [5] will be used in constructing asymptotics of the eigenvalue $\lambda^{\mu,\varepsilon}$ and the corresponding eigenfunction $\psi^{\mu,\varepsilon}$.

Theorem 1 entails that for the eigenfunction $\psi^{\mu,\varepsilon}$ normed in $L_2(\mathbb{R})$ and corresponding to the eigenvalue $\lambda^{\mu,\varepsilon}$ converging to λ_0, there is a convergence $\psi^{\mu,\varepsilon} \to \psi_0$ in $L_2(\mathbb{R})$. Therefore, outside the vicinity of the origin of coordinates (where the perturbation of the operator $\mathcal{H}_{\mu,\varepsilon}$ is concentrated), we will look for the asymptotics (*external decompositions*) of the function $\psi^{\mu,\varepsilon}$ similarly to (4) in the form

$$\psi^{ex,\pm}(x, \mu, \varepsilon) = \psi_0(x) + \sum_{i=1}^{\infty} \sum_{j=1}^{i} \varepsilon^i \mu^{-j} \psi_{i,j}^{\pm}(x), \qquad x \in \mathbb{R}_\pm. \tag{15}$$

Outside the vicinity of zero $V\left(\frac{x}{\varepsilon}\right) \equiv 0$. Since external decompositions will be used outside the vicinity of zero as well, by virtue of the definition of the operators $\mathcal{H}_{\mu,\varepsilon}$, $\mathcal{H}_0$ we obtain the following equations (in a generalized sense) for external decompositions:

$$H_0 \psi^{ex,\pm} = \lambda^{\mu,\varepsilon} \psi^{ex,\pm}, \quad x \in \mathbb{R}_\pm.$$

Here and in what follows, $\mathbb{R}_\pm = \{x : \pm x > 0\}$. Substituting the series (4) and (15) into these equations, collecting the coefficients of ε, μ in the same powers, we obtain the equality, which obviously holds

$$H_0 \psi_0 = \lambda_0 \psi_0 \tag{16}$$

and the following equations for the remaining coefficients of external decompositions:

$$\varepsilon^i \mu^{-j} : H_0 \psi_{i,j}^{\pm} = \lambda_{i,j} \psi_0 + \lambda_0 \psi_{i,j}^{\pm} + \sum_{p=1}^{i-1} \sum_{q=1}^{j-1} \lambda_{p,q} \psi_{i-p,j-q}^{\pm}, \quad x \in \mathbb{R}_\pm, \tag{17}$$

where $i \geq 1, \quad 1 \leqslant j \leqslant i$. In what follows we impose the normalizing conditions:

$$\int\limits_{-\infty}^{0} \psi_{i,j}^{-}\psi_0\, dx + \int\limits_{0}^{\infty} \psi_{i,j}^{+}\psi_0\, dx = 0 \tag{18}$$

on the coefficients $\psi_{i,j}^{\pm}$.

Solutions of equations (17) are considered in $W^2_{2,loc}(\mathbb{R}_\pm) \cap L_2(\mathbb{R}_\pm)$. Since the function W is infinitely differentiable in the vicinity of zero, then for any constants $\lambda_{i,j}$ the solutions $\psi_{i,j}^{\pm}$ of the system of recurrent equations (17) from $W^2_{2,loc}(\mathbb{R}_\pm)$ belong to $C^\infty[0,\delta]$, $C^\infty[-\delta,0]$, respectively.

It is natural to look for the asymptotics (*external decomposition*) of the function $\psi^{\mu,\varepsilon}$ in the vicinity of the origin of coordinates in the form of the expansion in functions depending on the variable $\xi = x\varepsilon^{-1}$, corresponding to the length of the support of the perturbed potential $V\left(\frac{x}{\varepsilon}\right)$.

The structure of the inner decomposition $\psi^{in}(\xi,\mu,\varepsilon)$ is determined on the basis of the following considerations. The Taylor series have the following form at zero of coefficients of external decompositions:

$$\begin{aligned} &\psi_0(x) = \sum_{k=0}^{\infty} P_k^{(0)}(x), \qquad P_k^{(0)}(x) = \frac{\psi_0^{(k)}(0)}{k!}x^k, \quad x \to 0, \\ &\psi_{i,j}^{\pm}(x) = \sum_{k=0}^{\infty} P_k^{(i,j,\pm)}(x), \qquad P_k^{(i,j,\pm)}(x) = \frac{\left(\psi_{i,j}^{\pm}\right)^{(k)}(0)}{k!}x^k, \quad x \to \pm 0. \end{aligned} \tag{19}$$

In (15) substituting instead of the functions $\psi_0(x)$ and $\psi_{i,j}^{\pm}(x)$ their asymptotics at zero (19) and substituting the variable $x = \xi\varepsilon$, we obtain that

$$\psi^{ex,\pm}(x,\mu,\varepsilon) = \sum_{i=1}^{\infty} \varepsilon^i V_{i,0}(\xi) + \sum_{i=1}^{\infty}\sum_{j=1}^{i} \varepsilon^i \mu^{-j} V_{i,j}^{\pm}(\xi), \qquad \xi\varepsilon \to \pm 0, \tag{20}$$

where

$$V_{i,0}(\xi) = P_i^{(0)}(\xi), \qquad V_{i,j}^{\pm}(\xi) = \sum_{q=0}^{i-j} P_q^{(i-q,j,\pm)}(\xi), \quad 1 \leqslant j \leqslant i. \tag{21}$$

In accordance with the matched asymptotic expansions approach, we look for the inner decomposition in the form

$$\psi^{in}(\xi,\mu,\varepsilon) = \sum_{i=1}^{\infty} \varepsilon^i v_{i,0}(\xi) + \sum_{i=1}^{\infty}\sum_{j=1}^{i} \varepsilon^i \mu^{-j} v_{i,j}(\xi), \tag{22}$$

where

$$v_{i,0}(\xi) = V_{i,0}(\xi), \qquad v_{i,j}(\xi) = V_{i,j}^{\pm}(\xi), \qquad \xi \to \pm\infty. \tag{23}$$

Substituting the series (4) and (22) into the equation

$$H_{\mu,\varepsilon}\psi^{in} = \lambda^{\mu,\varepsilon}\psi^{in},$$

substituting the function $W(x)$ there by its expansion into the Taylor series at zero, turning to the variable $\xi = x\varepsilon^{-1}$ and collecting the coefficients of ε, μ in the same powers, we obtain the following equations for coefficients of the inner decomposition:

$$\varepsilon^i: \ \frac{d^2 v_{i,0}}{d\xi^2} = \sum_{t=0}^{i-2} \frac{W^{(i-t-2)}(0)}{(i-t-2)!}\xi^{i-t-2} v_{t,0} - \lambda_0 v_{i-2,0}, \qquad i \geqslant 0, \tag{24}$$

$$\begin{aligned} \varepsilon^i\mu^{-j}: \ \frac{d^2 v_{i,j}}{d\xi^2} =& \sum_{t=0}^{i-2} \frac{W^{(i-t-2)}(0)}{(i-t-2)!}\xi^{i-t-2} v_{t,j} - \lambda_0 v_{i-2,j} + \\ &+ V(\xi)v_{i-2,j-1} + \sum_{p=1}^{i-1}\sum_{q=1}^{j-1} \lambda_{p,q} v_{i-p-2,j-q}, \qquad 1 \leqslant j \leqslant i. \end{aligned} \tag{25}$$

For the sake of brevity, here and in what follows the coefficients $\lambda_{p,s}$, $v_{p,s}$, whose indices do not correspond to the indices from (4) and (20) are taken as vanishing.

Thus, matching of asymptotic decompositions is reduced to the proof of existence of constants $\lambda_{i,j}$ such that the equalities (23) hold for solutions of equations (17), (18) and equations (24), (25).

By virtue of the definition (21), (19) of the polynomials $V_{i,0}$ and Equation (16), the functions

$$v_{0,0}(\xi) \equiv \psi_0(0), \quad v_{1,0}(\xi) = \psi_0'(0)\xi, \quad v_{k,0}(\xi) = \frac{\psi_0^{(k)}(0)}{k!}\xi^k, \quad k \geqslant 2, \tag{26}$$

satisfy (24), (23).

Equations (17) for $\psi_{i,i}^{\pm}$ have the form:

$$H_0\psi_{i,i}^{\pm} = \lambda_0\psi_{i,i}^{\pm} + \lambda_{i,i}\psi_0 + \sum_{p=1}^{i-1}\lambda_{p,p}\psi_{i-p,i-p}^{\pm}, \quad x \in \mathbb{R}_{\pm}, \qquad i \geqslant 1. \tag{27}$$

Let us introduce the following conjugation condition for $\psi_{i,i}^{\pm}$ at zero from the matched conditions (23). Equations (25) and the equalities (21), (19) for $v_{i,i}$ and $V_{i,i}^{\pm}$ have the form:

$$\frac{d^2v_{i,i}}{d\xi^2} = 0, \qquad V_{i,i}^{\pm}(\xi) \equiv \psi_{i,i}^{\pm}(0), \qquad i \geqslant 1. \tag{28}$$

Let us introduce the following notation for the functions $U^- \in C^\infty[-\delta, 0]$ and $U^+ \in C^\infty[0, \delta]$:

$$[U](0) := U^+(0) - U^-(0), \qquad [U'](0) := (U^+)'(0) - (U^-)'(0).$$

It follows sequentially from (28) and the matching condition (23) for $v_{i,i}$ that

$$[\psi_{i,i}](0) = 0, \qquad i \geqslant 1, \tag{29}$$

$$v_{i,i}(\xi) = V_{i,i}^{\pm}(\xi) \equiv \psi_{i,i}^{+}(0) = \psi_{i,i}^{-}(0) := \psi_{i,i}(0), \quad i \geqslant 1. \tag{30}$$

Equations (25) and the equalities (21), (19) for $v_{i+1,i}$ and $V_{i+1,i}^{\pm}$ have the form:

$$\frac{d^2v_{i+1,i}}{d\xi^2} = V(\xi)v_{i-1,i-1}, \qquad i \geqslant 1 \tag{31}$$

$$V_{i+1,i}^{\pm}(\xi) = \left(\psi_{i,i}^{\pm}\right)'(0)\xi + \psi_{i+1,i}^{\pm}(0), \qquad i \geqslant 1. \tag{32}$$

Invoking (26), we obtain that the functions

$$\begin{aligned} v_{2,1}(\xi) =& \psi_0(0)\int_{-\infty}^{\xi}\int_{-\infty}^{\tau} V(t)dtd\tau + a_{2,1}\xi + b_{2,1}, \\ v_{j+1,j}(\xi) =& \psi_{j-1,j-1}(0)\int_{-\infty}^{\xi}\int_{-\infty}^{\tau} V(t)dtd\tau + a_{j+1,j}\xi + b_{j+1,j}, \qquad j \geqslant 2, \end{aligned} \tag{33}$$

are solutions of Equation (31) with any constants $a_{p+1,p}$, $b_{p+1,p}$. It follows from (33) that

$$\begin{aligned} v_{i+1,i}(\xi) =& a_{i+1,i}\xi + b_{i+1,i}, \quad \xi \to -\infty, \qquad i \geqslant 1, \\ v_{2,1}(\xi) =& \psi_0(0)\left(\langle V(t)\rangle \xi - \langle tV(t)\rangle\right) + a_{2,1}\xi + b_{2,1}, \quad \xi \to +\infty \\ v_{j+1,j}(\xi) =& \psi_{j-1,j-1}(0)\left(\langle V(t)\rangle \xi - \langle tV(t)\rangle\right) + \\ & + a_{j+1,j}\xi + b_{j+1,j}, \quad \xi \to +\infty, \qquad j \geqslant 2. \end{aligned} \tag{34}$$

Comparing (32) and the right-hand sides (34), we conclude that the following statement holds.

Lemma 3. *If the following conjugation conditions are satisfied at zero*

$$[\psi_{1,1}'](0) = \psi_0(0)\langle V(t)\rangle, \qquad [\psi_{j,j}'](0) = \psi_{j-1,j-1}(0)\langle V(t)\rangle, \qquad j \geqslant 2 \tag{35}$$

then, there exist constants $a_{i+1,i}$ *such that*

$$\begin{aligned} v_{2,1}(\xi) - V^+_{2,1}(\xi) =& b_{2,1} - \psi^+_{2,1}(0) - \psi_0(0) \langle tV(t)\rangle, \quad \xi \to +\infty, \\ v_{2,1}(\xi) - V^-_{2,1}(\xi) =& b_{2,1} - \psi^-_{2,1}(0), \quad \xi \to -\infty, \\ v_{j+1,j}(\xi) - V^+_{j+1,j}(\xi) =& b_{j+1,j} - \psi^+_{j+1,j}(0) - \psi^+_{j-1,j-1}(0) \langle tV(t)\rangle, \ \xi \to +\infty, \\ v_{j+1,j}(\xi) - V^-_{j+1,j}(\xi) =& b_{j+1,j} - \psi^-_{j+1,j}(0), \quad \xi \to -\infty, \qquad j \geqslant 2. \end{aligned} \tag{36}$$

Moreover, if

$$\begin{aligned} [\psi_{2,1}]\,(0) =& -\psi_0(0) \langle tV(t)\rangle, \\ [\psi_{j+1,j}](0) =& -\psi_{j-1,j-1}(0) \langle tV(t)\rangle, \qquad j \geqslant 2 \end{aligned} \tag{37}$$

then there exist also constants $b_{i+1,i}$ *such that*

$$v_{i+1,i}(\xi) - V^{\pm}_{i+1,i}(\xi) = 0, \qquad \xi \to \pm\infty,$$

i.e. (23) is satisfied for $j = i + 1$.

Similarly to [4], one can readily demonstrate that the following statement holds.

Lemma 4. *Let* $F^\pm \in L_2(\mathbb{R}_\pm)$, $F^+ \in C^\infty[0, \delta]$, $F^- \in C^\infty[-\delta, 0]$ *and*

$$\int\limits_{-\infty}^{0} F^-\psi_0\,dx + \int\limits_{0}^{\infty} F^+\psi_0\,dx = 0.$$

Then for any numbers α, β *there are functions* $U^\pm \in W^2_{2,loc}(\mathbb{R}_\pm) \cap L_2(\mathbb{R}_\pm)$, $U^+ \in C^\infty[0, \delta]$, $U^- \in C^\infty[-\delta, 0]$, *that are solutions to the boundary-value problem*

$$H_0U^\pm = \lambda_0U^\pm + F^\pm + \Lambda\psi_0, \quad x \gtrless 0, \qquad [U](0) = \beta, \quad [U'](0) = \alpha,$$

$$\int\limits_{-\infty}^{0} U^-\psi_0\,dx + \int\limits_{0}^{\infty} U^+\psi_0\,dx = 0$$

when

$$\Lambda = \alpha\psi_0(0) - \beta\psi_0'(0).$$

The lemma entails

Corollary 2. *If* $\psi_0(0) = \beta = 0$ *then* $\Lambda = 0$.

The lemma entails that when $\lambda_{1,1}$, defined by the equality (5),

$$\lambda_{p,p} = \psi_0(0) < V(t) > \widetilde{\lambda}_{p,p}, \qquad p \geqslant 2, \tag{38}$$

where $\widetilde{\lambda}_{p,p}$ are some constants, there are functions of the form

$$\psi^\pm_{i,i}(x) = \psi_0(0) < V(t) > \widetilde{\psi}^\pm_{i,i}(x), \qquad i \geqslant 1, \tag{39}$$

satisfying (27), (29), (35), (18). Finding $\psi^\pm_{i,i}(x)$, we finally determine $v_{i,i}$ in accordance with (30), achieve the matching condition (23) for $v_{i,i}$, and by virtue of Lemma 3 find the functions $v_{i+1,i}$ with the accuracy up to arbitrary addends $b_{i+1,i}$, obtaining the equality (36) for $v_{i+1,i}$.

In particular, it follows from (5), (38), (39), (30) that

$$\text{if } \psi_0(0) \langle V(t)\rangle = 0 \text{ then } \psi^\pm_{i,i}(x) = v_{i,i}(\xi) \equiv \lambda_{i,i} = 0, \qquad i \geqslant 1. \tag{40}$$

Let us turn to the following steps in construction of asymptotics. Let us denote

$$\widetilde{V}^\pm_{i,j}(\xi) := V^\pm_{i,j}(\xi) - (\psi^\pm_{i-1,j})'(0)\xi - \psi^\pm_{i,j}(0), \quad 1 \leqslant j \leqslant i - 1. \tag{41}$$

By virtue of Equation (17) and Definitions (21), (19), (41) of the polynomials $V^\pm_{i,j}$, $\widetilde{V}^\pm_{i,j}$, we achieve the validity of the following statement.

Lemma 5. *The polynomials $\widetilde{V}^{\pm}_{i,j}(\xi)$ can depend only on $\lambda_{p,q}$ and $\psi^{\pm}_{p,q}(x)$ when $p \leqslant j-1$, $q \leqslant j$ and satisfy the equalities*

$$\left(\widetilde{V}^{\pm}_{i,j}\right)'(0) = \widetilde{V}^{\pm}_{i,j}(0) = 0,$$

$$\begin{aligned}\frac{d^2\widetilde{V}^{\pm}_{i,j}}{d\xi^2} =& \sum_{t=0}^{i-2} \frac{W^{(i-t-2)}(0)}{(i-t-2)!}\xi^{i-t-2}V^{\pm}_{t,j} - \lambda_0 V^{\pm}_{i-2,j} + \\ &+ \sum_{p=1}^{i-1}\sum_{q=1}^{j-1} \lambda_{p,q} V^{\pm}_{i-p-2,j-q}, \qquad \xi \in \mathbb{R}_{\pm}, \qquad 1 \leqslant j \leqslant i-1.\end{aligned}$$

The validity of the following statement is demonstrated in [4].

Lemma 6. *Let the function $f \in C^{\infty}(\mathbb{R})$ coincide with the polynomials $f^{\pm}(\xi)$ when $\xi \to \pm\infty$, and the following equalities hold for the polynomials $\widetilde{v}^{\pm}(\xi)$:*

$$\frac{d^2\widetilde{v}^{\pm}}{d\xi^2} = f^{\pm}, \qquad (\widetilde{v}^{\pm})'(0) = \widetilde{v}^{\pm}(0) = 0.$$

Then, for the general solution of the equation

$$\frac{d^2 v}{d\xi^2} = f$$

the equalities

$$\begin{aligned} v(\xi) =& \widetilde{v}^{-}(\xi) + a\xi + b, \quad \xi \to -\infty, \\ v(\xi) =& \widetilde{v}^{+}(\xi) + A\xi + B + a\xi + b, \quad \xi \to +\infty, \end{aligned}$$

hold, where A, B are some constants, depending on f, and a, b are arbitrary constants.

At the following step by virtue of Lemmas 5, 6, we find the solutions $v_{q+2,q}$ of Equations (25) such that

$$\begin{aligned} v_{q+2,q}(\xi) - V^{+}_{q+2,q}(\xi) =& \left(A_{q+2,q} + a_{q+2,q} - \left(\psi^{+}_{q+1,q}\right)'(0)\right)\xi + \\ &+ B_{q+2,q} + b_{q+2,q} - \psi^{+}_{q+2,q}(0), \quad \xi \to +\infty, \\ v_{q+2,q}(\xi) - V^{-}_{q+2,q}(\xi) =& \left(a_{q+2,q} - \left(\psi^{-}_{q+1,q}\right)'(0)\right)\xi + \\ &+ b_{q+2,q} - \psi^{-}_{q+2,q}(0), \quad \xi \to -\infty, \qquad q \geqslant 1, \end{aligned} \tag{42}$$

where $A_{q+2,q}$, $B_{q+2,q}$ are completely determined constants independent of $A_{p+2,p}$, $B_{p+2,p}$ when $p > q$, and $a_{q+2,q}$, $b_{q+2,q}$ are arbitrary constants. In addition to the conjugation conditions (37), let us also impose the conjugation conditions

$$[\psi'_{q+1,q}](0) = A_{q+2,q}, \qquad q \geqslant 1. \tag{43}$$

By virtue of Lemma 4, there are constants $\lambda_{q+1,q}$ and functions $\psi^{\pm}_{q+1,q}(x)$ satisfying (17), (18) when $i = q+1$, $j = q$ and the conjugation conditions (37), (43). Upon determining $\psi^{\pm}_{q+1,q}(x)$, we find sequentially $b_{q+1,q}$, $a_{q+1,q}$, define the functions $v_{q+1,q}(\xi)$ completely, achieving the equality of the functions $v_{q+1,q}(\xi) = V^{\pm}_{q+1,q}(\xi)$ when $\xi \to \pm\infty$, and the functions $v_{q+2,q}(\xi)$ with the accuracy to arbitrary addends $b_{q+2,q}$, obtaining the equalities

$$\begin{aligned} v_{q+2,q}(\xi) - V^{+}_{q+2,q}(\xi) =& B_{q+2,q} + b_{q+2,q} - \psi^{+}_{q+2,q}(0), \quad \xi \to +\infty, \\ v_{q+2,q}(\xi) - V^{-}_{q+2,q}(\xi) =& b_{q+2,q} - \psi^{-}_{q+2,q}(0), \quad \xi \to -\infty, \qquad q \geqslant 1 \end{aligned}$$

(an analogue of the equalities (36) at the previous step), etc.

As a result we conclude that the following lemma holds.

Lemma 7. *There are series (4), (15), (22) such that the equalities (17), (24), (25), (23) hold, where the polynomials $V_{i,0}(\xi)$ and $V^{\pm}_{i,j}(\xi)$ are defined by the equalities (21), (19).*

The equalities (5), (26), (40) hold for coefficients of these series.

Note that the equalities (7) are contained in (40) under the condition (6).

Let us demonstrate that the equality (9) holds under the condition (8). By virtue of Lemma 4 we have:

$$\lambda_{2,1} = [\psi'_{2,1}](0)\psi_0(0) - [\psi_{2,1}](0)\psi'_0(0). \tag{44}$$

The value $[\psi_{2,1}](0)$ is defined in (37), and the equality (43) has the form

$$[\psi'_{2,1}](0) = A_{3,1} \tag{45}$$

when $q = 1$. While,

$$\begin{aligned} v_{3,1}(\xi) &= (A_{3,1} + a_{3,1})\,\xi + B_{3,1} + b_{3,1}, \quad \xi \to +\infty, \\ v_{3,1}(\xi) &= a_{3,1}\xi + b_{3,1}, \quad \xi \to -\infty \end{aligned} \tag{46}$$

according to (42), and $v_{3,1}$ is the solution of Equation (25) when $i = 3$, $j = 1$. Since $v_{1,1}(\xi) \equiv 0$, $v_{1,0}(\xi) = \psi'_0(0)\xi$ by virtue of (40) and (26), respectively then Equation (25) for $v_{3,1}$ has the form

$$\frac{d^2 v_{3,1}}{d\xi^2} = \psi'_0(0)V(\xi)\xi. \tag{47}$$

Hence,

$$v_{3,1}(\xi) = \psi'_0(0) \int\limits_{-\infty}^{\xi} \int\limits_{-\infty}^{\tau} tV(t)dtd\tau + a_{3,1}\xi + b_{3,1}, \tag{48}$$

and $A_{3,1} = \psi'_0(0)\,\langle tV(t)\rangle$. This equality together with (37), (45), (44) entail the equality (9) under the condition (8).

Let us demonstrate that the equalities (11) hold under the condition (10). Since $v_{j,j} \equiv 0$ when $j \geqslant 0$ by virtue of (40) and the equality $v_{0,0}(\xi) \equiv \psi_0(0) = 0$ then, invoking Equations (31) and Corollary 2, one obtains that the validity of the equalities follows from the chain:

$$\begin{aligned} v_{j,j} \equiv 0, \quad j \geqslant 0 \quad &\Rightarrow \quad \frac{d^2}{d\xi^2} v_{i+1,i} \equiv 0, \quad i \geqslant 1 \quad \Rightarrow \quad v_{i+1,i}(\xi) = a_{i+1,i}\xi + b_{i+1,i} \\ &\Rightarrow \quad [\psi_{i+1,i}](0) = 0 \quad \Rightarrow \quad \lambda_{i+1,i} = 0. \end{aligned}$$

It remains only to demonstrate the validity of the equality (12) under the condition (10). By virtue of Lemma 4 we have:

$$\lambda_{3,1} = -[\psi_{3,1}](0)\psi'_0(0). \tag{49}$$

Since $v_{1,1}(\xi) = 0$, and $v_{1,0}(\xi) = \psi'_0(0)\xi$ by virtue of (26) then Equation (25) for $v_{3,1}$ has the from (47) again. Hence, we obtain the equalities (48) and (46), where $B_{3,1} = -\psi'_0(0)\left\langle t^2 V\right\rangle$. Since $[\psi_{3,1}](0) = B_{3,1}$, the equality (12) follows from (49).

5. Justification of asymptotics

Let us assume that $\chi(x) \in C_0^\infty(\mathbb{R})$ is a patch functions vanishing when $|x| < 1$ and equal to one when $|x| > 2$, $\widehat{\lambda}_N(\varepsilon,\mu)$, $\widehat{\psi}_N^{\pm}(x,\varepsilon,\mu)$ and $\widehat{v}_N(\xi,\varepsilon,\mu)$ are partial sums in ε up to the order N including the series (4), (15) and (22), respectively. Let us introduce the notation

$$\begin{aligned} \Psi_N(x,\varepsilon,\mu) :=& \chi\left(x\varepsilon^{-\frac{1}{2}}\right)\left(\widehat{\psi}_N^{+}(x,\varepsilon,\mu) + \widehat{\psi}_N^{-}(x,\varepsilon,\mu)\right) + \\ &+ \left(1 - \chi\left(x\varepsilon^{-\frac{1}{2}}\right)\right)\widehat{v}_N\left(x\varepsilon^{-1},\varepsilon,\mu\right). \end{aligned}$$

The definition of Ψ_N entails that the function belongs to the definition domain of the operator $\mathcal{H}_{\mu,\varepsilon}$ (coinciding with the definition domain of the operator $\mathcal{H}_0$) and

$$\|\Psi_N\|_{L_2(\mathbb{R})} \to 1, \qquad \varepsilon \to 0. \tag{50}$$

The following lemma is proved on the basis of statements of Lemma 7.

Lemma 8. *The equality*

$$\mathcal{H}_{\mu,\varepsilon}\Psi_N = \widehat{\lambda}_N \Psi_N + F_N \tag{51}$$

holds and

$$\|F_N\|_{L_2(\mathbb{R})} = O\left(\varepsilon^{\frac{N}{2}-1} + \varepsilon^{\gamma N-1}\right) \tag{52}$$

in it.

Proof. It follows from the definitions of Ψ_N and $\mathcal{H}_{\mu,\varepsilon}$ that

$$F_N = F_{1,N} + F_{2,N} + F_{3,N}, \tag{53}$$

where

$$\begin{aligned}
F_{1,N}(x,\varepsilon,\mu) =& \chi\left(x\varepsilon^{-\frac{1}{2}}\right)\left(H_0 - \widehat{\lambda}_N\right)\left(\widehat{\psi}_N^+(x,\varepsilon,\mu) + \widehat{\psi}_N^-(x,\varepsilon,\mu)\right),\\
F_{2,N}(x,\varepsilon,\mu) =& \left(1 - \chi\left(x\varepsilon^{-\frac{1}{2}}\right)\right)\left(H_{\mu,\varepsilon} - \widehat{\lambda}_N\right)\widehat{v}_N\left(x\varepsilon^{-1},\varepsilon,\mu\right),\\
F_{3,N}(x,\varepsilon,\mu) =& -\left(\widehat{\psi}_N^+(x,\varepsilon,\mu) + \widehat{\psi}_N^-(x,\varepsilon,\mu) - \widehat{v}_N\left(x\varepsilon^{-1},\varepsilon,\mu\right)\right)\frac{d^2}{dx^2}\chi\left(x\varepsilon^{-\frac{1}{2}}\right) -\\
&- 2\frac{d}{dx}\left(\widehat{\psi}_N^+(x,\varepsilon,\mu) + \widehat{\psi}_N^-(x,\varepsilon,\mu) - \widehat{v}_N\left(x\varepsilon^{-1},\varepsilon,\mu\right)\right)\frac{d}{dx}\chi\left(x\varepsilon^{-\frac{1}{2}}\right).
\end{aligned}$$

It follows from the definition of $F_{1,N}$ and the equalities (17) that

$$\|F_{1,N}\|_{L_2(\mathbb{R})} = O\left(\varepsilon^{N+1}\mu^{-N-1}\right). \tag{54}$$

Since the support of the function $F_{2,N}$ belongs to the interval $[-2\varepsilon^{\frac{1}{2}}, 2\varepsilon^{\frac{1}{2}}]$, $v_{i,j}(\xi) = O(\xi^{i-j})$ when $\xi \to \pm\infty$ (see (23), (21), (19)) then, by virtue of the equalities (24), (25), we obtain the following estimate:

$$\|F_{2,N}\|_{L_2(\mathbb{R})} = O\left(\varepsilon^{\frac{N}{2}-\frac{3}{4}}\left(\varepsilon^{\frac{1}{2}} + \frac{\varepsilon}{\mu}\right) + \varepsilon^{\frac{1}{4}}\left(\frac{\varepsilon}{\mu}\right)^{N-1}\left(1 + \frac{\varepsilon^{\frac{1}{4}}}{\mu}\right)\right). \tag{55}$$

The definitions (21), (19) of the polynomials $V_{i,j}^{\pm}$ and the equalities (23) also entail that when $x \in [-2\varepsilon^{\frac{1}{2}}, -\varepsilon^{\frac{1}{2}}] \cup [\varepsilon^{\frac{1}{2}}, 2\varepsilon^{\frac{1}{2}}]$, the differentiable equality

$$\begin{aligned}
&\widehat{\psi}_N^+(x,\varepsilon,\mu) + \widehat{\psi}_N^-(x,\varepsilon,\mu) - \widehat{v}_N\left(x\varepsilon^{-1},\varepsilon,\mu\right) =\\
&= O\left(x^{N+1} + \left(\frac{\varepsilon}{\mu}\right)^N x\right)
\end{aligned} \tag{56}$$

holds. Since the support of the function $F_{3,N}$ belongs to $[-2\varepsilon^{\frac{1}{2}}, -\varepsilon^{\frac{1}{2}}] \cup [\varepsilon^{\frac{1}{2}}, 2\varepsilon^{\frac{1}{2}}]$ and

$$\frac{d}{dx}\chi\left(x\varepsilon^{-\frac{1}{2}}\right) = O(\varepsilon^{-\frac{1}{2}}), \qquad \frac{d^2}{dx^2}\chi\left(x\varepsilon^{-\frac{1}{2}}\right) = O(\varepsilon^{-1}),$$

then (56) provides the estimate

$$\|F_{3,N}\|_{L_2(\mathbb{R})} = O\left(\varepsilon^{-\frac{1}{4}}\left(\varepsilon^{\frac{1}{2}} + \frac{\varepsilon}{\mu}\right)^N\right). \tag{57}$$

The estimate (52) follows from (53), (54), (55), (57) and (3).

By virtue of the resolvent estimate for linear selfadjoint operators (see, e.g., [1, Chapter V, § 3]) we have:

$$\|\Psi_N\|_{L_2(\mathbb{R})} \leqslant \frac{\|F_N\|_{L_2(\mathbb{R})}}{\left|\lambda^{\mu,\varepsilon} - \widehat{\lambda}_N\right|}.$$

for solution of Equation (51). This estimate, Lemma 8 and (50) provide the equality

$$\left|\lambda^{\mu,\varepsilon} - \widehat{\lambda}_N\right| = O\left(\varepsilon^{\frac{N}{2}-1} + \varepsilon^{\gamma N-1}\right).$$

Whence, due to the arbitrary choice of N, we obtain that the constructed series (4) is a complete asymptotic expansion of the eigenvalue $\lambda^{\mu,\varepsilon}$.

This completes the proof of Theorem 2.

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REFERENCES

1. Kato T. *Perturbation theory for linear operators.* Moscow. Mir. 1972. 740 p. English version by Springer-Verlag, Berlin, 1966.
2. Golovaty Yu.D., Man'ko S.S. *Exact models for the Schrödinger operators with δ'-like potentials* // Ukrain'skii matematichnii vistnik. V. 6, No 2. 2009. P. 173–207. In Ukranian.
3. Vladimirov V.S., Zharinov V.V. *Equations of mathematical physics* M.: Fizmatlit, 2004. In Russian.
4. Khusnullin I.Kh. *The perturbed boundary-value problem on eigenvalues for the Schrödinger operator on an interval* // J. comput. math. and math. phys. V. 50, No. 4. 2010. P. 679–698.In Russian.
5. Il'in A.M. *Matching asymptotic expansions of solutions to boundary-value problems* M.: Nauka, 1989. In Russian.

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