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BOUNDARY VALUE PROBLEMS FOR THE LOADED THIRD ORDER EQUATIONS OF THE HYPERBOLIC AND MIXED TYPES

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Abstract. In this paper, the unique solvability is proved for the solution of boundary value problems of a loaded third order differential equation with hyperbolic and parabolic-hyperbolic operators. The boundary value problems for loaded differential equations are reduced to the Volterra integral equation of the second kind. On this basis, existence and uniqueness of the solution of boundary value problems is proved by the method of integral equations.

Keywords:loaded equation, equations of the mixed type, integral equation, integral equation with a shift, Bessel's functions.

1. Introduction

In the recent years, in connection with intensive research on problems of optimal control, long-term forecasting and regulating the level of ground waters and soil moisture, it has become necessary to investigate a new class of equations called "'loaded equations"'. Such equations were investigated for the first time in works of N.N. Nazarov and N.N. Kochin. However, they did not use the term "loaded equation". For the first time, the term has been used in works of A.M. Nakhushev, where the most general definition of a loaded equation is given and various loaded equations are classified in detail, e.g., loaded differential, integral, integro-differential, functional equations etc., and numerous applications are described.

Works of A.M. Nakhushev, M.Kh. Shkhankov, A.B. Borodin, V.M. Kaziev, A.Kh. Attaev, C.C. Pomraning, E.W. Larsen, V.A. Eleev, M.T. Dzhenaliev, B. Islomov and D.M. Kuriazov, D.M. Kuriazov, K.U. Khubiev, M.I. Ramazanov et al. are devoted to loaded second-order partial differential equations.

It should be noted that boundary-value problems for loaded equations of a hyperbolic, parabolic-hyperbolic, elliptic-hyperbolic types of the third order are less well understood. We indicate only the works of V.A. Elkeev, B. Islomov and D.M. Kur'yazov, V.A. Eleev and A.V. Dzarakhokhov.

The present paper is devoted to formulation and investigation of the analogue of the Cauchy-Goursat problem for the loaded equation of a hyperbolic type

$$\frac{\partial}{\partial x}\left(u_{xx}-u_{yy}-\lambda u\right)-\mu u(x,0)=0, \tag{1}$$

and a boundary-value problem for a loaded equation of a mixed parabolic hyperbolic type

$$\frac{\partial}{\partial x}\left(Lu\right)-\mu u(x,0)=0, \tag{2}$$

where

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$$Lu = \begin{cases} L_1 u \equiv u_{xx} - u_y - \lambda u, & y > 0, \\ L_2 u \equiv u_{xx} - u_{yy} - \lambda u, & y < 0, \end{cases}$$

λ, μ are real constants, and $\lambda > 0$.

2. Analogue of the Cauchy-Goursat problem for a loaded equation of the hyperbolic type

Let D be a domain bounded by the characteristics

$$AC : x + y = 0, \qquad BC : x - y = 1$$

of Equations (1) and the segment AB of the axis $y = 0$.

Let us consider the following analogue of the Cauchy-Goursat problem for the loaded equation (1) in the domain D.

Problem A. Find a solution $u(x, y)$ to Equation (1), which is regular in the domain D, continuous in $\bar{D}$, and has continuous derivatives u_x, u_y up to $AB \cup AC$, and satisfies the boundary-value conditions

$$u_y(x, y)|_{AB} = \nu(x), \quad 0 \le x < 1, \tag{3}$$

$$u(x, y)|_{AC} = \psi_1(x), \qquad \left.\frac{\partial u(x, y)}{\partial n}\right|_{AC} = \psi_2(x), \qquad 0 \le x \le \frac{1}{2}, \tag{4}$$

where n is an inner normal, $\nu(x), \psi_1(x), \psi_2(x)$ are given functions, and $2\nu(x) = \sqrt{2}\psi_2(0) - \psi_1^{'}(0)$,

$$\nu(x) \in C[0, 1] \cap C^2(0, 1), \tag{5}$$

$$\psi_1(x) \in C^1\left[0, \frac{1}{2}\right] \cap C^3\left(0, \frac{1}{2}\right), \quad \psi_2(x) \in C\left[0, \frac{1}{2}\right] \cap C^2\left(0, \frac{1}{2}\right). \tag{6}$$

Theorem 1. If Conditions (5), (6) are satisfied, then there is a unique solution to Problem A in the domain D.

Proof of Theorem 1.

A n important part in proving Theorem 1 is played by the following lemma.

Lemma 1. Any regular solution to Equation (1) is represented in the form

$$u(x, y) = z\,(x, y) + w(x), \tag{7}$$

where $z(x, y)$ is solution of the equation

$$\frac{\partial}{\partial x}(z_{xx} - z_{yy} - \lambda z) = 0, \tag{8}$$

and $w(x)$ is the solution of the following ordinary differential equation

$$w^{'''}(x) - \lambda w^{'}(x) - \mu w(x) = \lambda z(x, 0). \tag{9}$$

Proof of Lemma 1

Let $u(x,y)$, represented by Formula (7), be the solution of Equation (1). Then, substituting (7) into (1), we have

$$\frac{\partial}{\partial x}\left(u_{xx}-u_{yy}-\lambda u\right)-\mu u(x,0)=\frac{\partial}{\partial x}\left(z_{xx}-z_{yy}-\lambda z\right)+$$

$$+w'''(x)-\lambda w'(x)-\mu w(x)-\mu\, z(x,0)=0,$$

i.e., it satisfies Equation (1).

Then, vice versa, let $u(x,y)$ be a regular solution to Equation (1), and $w(x)$ be a certain solution

$$w'''(x)-\lambda w'(x)=\mu u(x,0). \tag{10}$$

Let us prove the validity of the relation (7). Manifestly, the function

$$u(x,y)=z(x,y)+\frac{\mu}{\lambda}\int\limits_0^x\left(ch\sqrt{\lambda}(x-t)-1\right)u(t,0)dt$$

is a solution to Equation (1), where $z(x,y)-$ is a solution to Equation (8), and the function

$$u(x,y)=\frac{\mu}{\lambda}\int\limits_0^x\left(ch\sqrt{\lambda}(x-t)-1\right)u(t,0)dt$$

is a partial solution to Equation (1). Hence, (1) entails the validity of the representation (7), i.e. $u(x,y)=z(x,y)+w(x)$.

It follows from the latter representation that $u(x,0)=z(x,0)+w(x)$. Then, (10) provides

$$w'''(x)-\lambda w'(x)-\mu w(x)-\mu\, z(x,0)=0,$$

and the function $z(x,y)=u(x,y)-w(x)$ satisfies Equation (8).

Lemma 1 is proved.

Invoking that the function $a\cos\sqrt{\lambda}x+b\sin\sqrt{\lambda}x+c\,e^{\sqrt{\lambda}x}$ satisfies Equation (8), we can assume without loss of generality that

$$w(0)=w'(0)=w''(0)=0 \tag{11}$$

when studying Problem A.

Let us solve the Cauchy problem for Equation (9) with the conditions (11) with respect to $w(x)$.

The characteristic equation, corresponding to the homogeneous equation (9), has the form

$$k^3-\lambda k-\mu=0. \tag{12}$$

Let us introduce the notation $\Delta=\frac{\mu^2}{4}-\frac{\lambda^3}{27}$.

1) If $\Delta>0$ then, it is known [3] that equation (12) has one real and two complex conjugate roots in the form

$$k_1=u_1+v_1,\qquad k_{2,3}=-\frac{1}{2}(u_1+v_1)\pm\frac{\sqrt{3}}{2}i(u_1-v_1),$$

where

$$u_1=\sqrt[3]{\frac{\mu}{2}+\sqrt{\Delta}},\quad v_1=\sqrt[3]{\frac{\mu}{2}-\sqrt{\Delta}}.$$

Thus, the solution of the Cauchy problem for Equation (9) with the conditions (11) for $\Delta > 0$ has the form

$$w(x) = \int_0^x T_1(x,t) z(t,0)dt, \tag{13}$$

where

$$T_1(x,t) = \frac{\mu}{3\left(u_1^2 + u_1 v_1 + v_1^2\right)} \left\{ e^{\frac{3}{2}(u_1+v_1)(x-t)} + \frac{\sqrt{3}\,(u_1+v_1)}{u_1 - v_1} \sin\frac{\sqrt{3}}{2}(u_1 - v_1)(t-x) - \right.$$
$$\left. - \cos\frac{\sqrt{3}}{2}(u_1 - v_1)(t-x) \right\} e^{-\frac{1}{2}(u_1+v_1)(x-t)};$$

2) If $\Delta = 0$, Equation (12) has three real roots, two of them being equal to

$$k_1 = \frac{3\mu}{\lambda}, \qquad k_2 = k_3 = -\frac{3\mu}{2\lambda}.$$

Solution to the Cauchy problem for Equation (9) with the conditions (11) for $\lambda = -3\,(\mu/2)^{\frac{2}{3}}$ has the form

$$w(x) = \int_0^x T_2(x,t) z(t,0)dt, \tag{14}$$

where

$$T_2(x,t) = \frac{2}{9}\left(\frac{\mu}{2}\right)^{\frac{1}{3}} e^{\sqrt[3]{\frac{\mu}{2}}(x-t)} \left(e^{\sqrt[3]{\frac{\mu}{2}}(x-t)} - 3\left(\frac{\mu}{2}\right)^{\frac{1}{3}}(x-t) - 1 \right).$$

3) If $\Delta < 0$, Equation (12) has three various real roots, in the form [3]

$$k_1 = 2\left|\sqrt[3]{r}\right| \cos\frac{\varphi}{3}, \quad k_2 = 2\left|\sqrt[3]{r}\right| \cos\frac{\varphi + 2\pi}{3}, \quad k_3 = 2\left|\sqrt[3]{r}\right| \cos\frac{\varphi + 4\pi}{3},$$

where

$$r = \left|\left(\frac{\lambda}{3}\right)^{\frac{3}{2}}\right|, \quad \cos\varphi = \frac{\mu}{2}\left|\left(\frac{\lambda}{3}\right)^{\frac{3}{2}}\right|^{-1}.$$

Correspondingly, the solution of the Cauchy problem for equation (9) with the conditions (11) for $\Delta < 0$ has the form

$$w(x) = \int_0^x T_3(x,t) z(t,0)dt, \tag{15}$$

where

$$T_3(x,t) = \frac{\mu}{(k_2 - k_1)(k_3 - k_1)(k_2 - k_3)} \left\{ (k_2 - k_3)\, e^{k_1(x-t)} + \right.$$
$$\left. + (k_3 - k_1)\, e^{k_2(x-t)} - (k_2 - k_1)\, e^{k_3(x-t)} \right\}.$$

By virtue of the representation (7), Problem A is reduced to Problem A* of finding a solution $z(x,y)$ of Equation (8), which is regular in the domain D and satisfies the conditions

$$z_y(x,y)\big|_{AB} = \nu(x), \quad 0 < x < 1, \tag{16}$$

$$z(x,y)\big|_{AC} = \psi_1(x) - w(x), \quad \left.\frac{\partial z(x,y)}{\partial n}\right|_{AC} = \psi_2(x) - \frac{1}{\sqrt{2}}w'(x), \;\; 0 \le x \le \frac{1}{2}, \tag{17}$$

where

$$w(x) = \int\limits_0^x T_i(x,t)z(t,0)dt\,, \qquad (i = 1,3). \tag{18}$$

Similarly to [4] and [5], we can write out the solution to Equation (8) in D with the conditions (16), (17) by means of the general representation in view of (5), (6) and [6]:

$$z(x,y) = \int\limits_0^{x+y} \nu(t)I_0\Big[\sqrt{\lambda(x+y-t)(x-y-t)}\Big]dt - \psi_1^*(0)I_0\Big[\sqrt{\lambda\,(x^2-y^2)}\Big] +$$

$$+\psi_1^*\left(\frac{x+y}{2}\right) + \psi_1^*\left(\frac{x-y}{2}\right) + \frac{1}{\sqrt{\lambda}}\int\limits_0^{x+y} \left(\lambda\psi_1^*(t) - \sqrt{2}{\psi_2^*}'(t)\right)\sin\sqrt{\lambda}\left(t - \frac{x+y}{2}\right)dt +$$

$$+\frac{1}{\sqrt{\lambda}}\int\limits_0^{\frac{x-y}{2}} \left(\lambda\psi_1^*(t) - \sqrt{2}{\psi_2^*}'(t)\right)\sin\sqrt{\lambda}\left(t - \frac{x-y}{2}\right)dt - 2\int\limits_0^{\frac{x-y}{2}} \psi_1^*(t)\times \tag{19}$$

$$\times B_t\left(0,2t;x+y,x-y\right)dt + \frac{1}{\sqrt{\lambda}}\int\limits_0^{y} \left(\lambda\psi_1^*(-t) - \sqrt{2}{\psi_2^*}'(-t)\right)\sin\sqrt{\lambda}(y-t)dt -$$

$$-\frac{2}{\sqrt{\lambda}}\int\limits_0^{\frac{x-y}{2}} B_t\left(0,2t;x+y,x-y\right)dt\int\limits_0^{t} \left(\lambda\psi_1^*(z) - \sqrt{2}{\psi_2^*}'(z)\right)\sin\sqrt{\lambda}\left(-t+z\right)dz,$$

where

$$\psi_1^*(x) = \psi_1(x) - w(x), \qquad \psi_2^*(x) = \psi_2(x) - \frac{1}{\sqrt{2}}w'(x),$$

$B(t,z;x+y,x-y)$ is the Riemann-Hadamard function [6], $I_0[z]$ is the modified Bessel function [7].

Assuming that $y = 0$ in (19) and invoking (18), we obtain the following functional correlation, transferred from the domain D onto AB:

$$\tau(x) + \int\limits_0^x K(x,t)\tau\left(\frac{t}{2}\right)dt = \Phi(x), \quad 0 \le x \le 1, \tag{20}$$

where

$$\tau(x) = z(x,0),$$

$$K(x,t) = T_i\left(\frac{x}{2},\frac{t}{2}\right) + \frac{\lambda x}{2}\int\limits_t^x T_i\left(\frac{s}{2},\frac{t}{2}\right)\bar{I}_1\Big[\sqrt{\lambda x(x-s)}\Big]ds +$$

$$+\frac{1}{2\sqrt{\lambda}}\int_t^x K^*\left(\frac{x}{2},\frac{t}{2}\right)\left(\lambda T_i\left(\frac{s}{2},\frac{t}{2}\right)-\frac{1}{4}T_i'\left(\frac{s}{2},\frac{t}{2}\right)\right)ds\,, \tag{21}$$

$$\Phi(x)=2\psi_1\left(\frac{x}{2}\right)-\psi_1(0)I_0\left[\sqrt{\lambda}x\right]+\int_0^x \nu(t)I_0\left[\sqrt{\lambda}(x-t)\right]dt+$$

$$+\lambda x\int_0^x \bar{I}_1\left[\sqrt{\lambda x(x-t)}\right]\psi_1\left(\frac{t}{2}\right)dt+ \tag{22}$$

$$+\frac{1}{\sqrt{\lambda}}\int_0^x K^*\left(\frac{x}{2},\frac{t}{2}\right)\left(\lambda\psi_1\left(\frac{t}{2}\right)-\frac{\sqrt{2}}{2}\psi_2'\left(\frac{t}{2}\right)\right)dt,$$

$$K^*(x,t)=\sin\sqrt{\lambda}(t-x)+\int_t^x \lambda x\bar{I}_1\left[\sqrt{\lambda x(x-2s)}\right]\sin\sqrt{\lambda}\,(t-s)ds,$$

$\bar{I}_1(x)=I_1(x)/x,\ I_0(x),\ I_1(x)$ are the modified Bessel functions [7].

Whence, we conclude that the integral equation (20) always has a solution, which is unique [8].

Thus, it is proved that Problem A is uniquely solvable.

Theorem 1 is proved.

3. Investigation of Problem C for Equation (2)

3.1. Formulation of Problem C for Equation (2).

Let Ω_1 be a domain bounded by the segments AB, BB_0, AA_0, A_0B_0 of the straight lines $y=0,\ x=1,\ x=0,\ y=h$, respectively when $y>0$. Ω_2 is a characteristic triangle bounded by the segment AB of the axis OX and two characteristics

$$AC: x+y=0,\quad BC: x-y=1$$

of Equation (2) for $y<0$.

Let us introduce the following notation:

$$I=\{(x,y): 0<x<1,\ \ y=0\},\quad \Omega=\Omega_1\cup\Omega_2\cup I.$$

Let us term the function $u(x,y)\in C(\bar{\Omega})\bigcap C^1(\Omega)\bigcap C^{3,1}(\Omega_1)\bigcap C^{3,2}(\Omega_2)$, satisfying Equation (2) in Ω_1 and Ω_2, as a regular solution of Equation (2).

Problem C. Find the function $u(x,y)$, possessing the following properties:

1) $u(x,y)\in C(\bar{\Omega})$;

2) $u_x(u_y)$ is continuous up to $AA_0\ \cup AC\ \ (AB\ \cup AC)$;

3) $u(x,y)$ is a regular solution of Equation (2) in the domains Ω_1 and Ω_2;

4) the sewing conditions

$$u_y(x,-0)=u_y(x,+0),\quad (x,0)\in I$$

are satisfied on AB;

5) $u(x,y)$ satisfies the boundary-value conditions

$$u(x,y)|_{AA_0}=\varphi_1(y),\ \ u(x,y)|_{BB_0}=\varphi_2(y),\ \ u_x(x,y)|_{AA_0}=\varphi_3(y),\ \ 0\le y\le h, \tag{23}$$

$$u(x,y)|_{AC} = \psi_1(x), \quad \left.\frac{\partial u(x,y)}{\partial n}\right|_{AC} = \psi_2(x), \quad 0 \le x \le \frac{1}{2}, \tag{24}$$

where n is the inner normal, $\varphi_1(y)$, $\varphi_2(y)$, $\varphi_3(y)$, $\psi_1(x)$ and $\psi_2(x)$ are given functions and

$$\varphi_1(0) = \psi_1(0), \;\; \varphi_j(y) \in C^1[0,1], \;\; (j = 1,2), \quad \varphi_3(y) \in C[0,1] \cap C^1(0,1), \tag{25}$$

$$\psi_1(x) \in C^1\left[0, \frac{1}{2}\right] \cap C^3\left(0, \frac{1}{2}\right), \;\; \psi_2(x) \in C\left[0, \frac{1}{2}\right] \cap C^2\left(0, \frac{1}{2}\right). \tag{26}$$

Theorem 2. If $\lambda > 0$ and the conditions (25) and (26) are satisfied, then there exists a unique solution to the problem C in the domain Ω.

Proof of Theorem 2.

The following theorem holds.

Lemma 2. Any regular solution of Equation (2) (when $y \ne 0$) is represented in the form

$$u(x,y) = z(x,y) + w(x), \tag{27}$$

where $z(x,y)$ is a solution to the equation

$$0 = \frac{\partial}{\partial x}\begin{cases} z_{xx} - z_y - \lambda z, \;\; y > 0, \\ z_{xx} - z_{yy} - \lambda z, \;\; y < 0, \end{cases} \tag{28}$$

$w(x)$ is a solution of the following ordinary differential equation

$$w^{'''}(x) - \lambda w^{'}(x) - \mu w(x) = \lambda z(x,0). \tag{29}$$

The lemma is proved similarly to Lemma 1.

Invoking that the function $ae^{\sqrt{\lambda}x} + be^{-\sqrt{\lambda}x} + c$ satisfies Equation (28), we can subordinate the function $w(x)$ to the conditions

$$w(0) = w'(0) = w''(0) = 0. \tag{30}$$

Solution of the Cauchy problem for Equation (29) with the conditions (30) can be represented correspondingly in the form (13), (14), (15) considering $\Delta > 0, \Delta = 0$ and $\Delta < 0$, while

$$T_i(x,x) = T_i'(x,x) = 0, \quad T_i''(x,x) = \mu, \;\; (i = 1,3).$$

By virtue of the representation (27), Equation (2) and the boundary-value conditions (23), (24), in view of (30), are reduced to the form (28)

$$z(x,y)|_{AA_0} = \varphi_1(y), \;\; z(x,y)|_{BB_0} = \varphi_2(y) - w(1), \;\; \left.\frac{\partial z(x,y)}{\partial x}\right|_{AA_0} = \varphi_3(y), \, 0 \le y \le h, \tag{31}$$

$$z(x,y)|_{AC} = \psi_1(x) - w(x), \quad 0 \le x \le \frac{1}{2}, \tag{32}$$

$$\left.\frac{\partial z(x,y)}{\partial n}\right|_{AC} = \psi_2(x) - \frac{1}{\sqrt{2}}w'(x), \quad 0 \le x \le \frac{1}{2}. \tag{33}$$

3.2. Derivation of basic functional relations

As it is known from Problem A, solution to Equation (28) with the boundary-value conditions (32), (33) and

$$\left.\frac{\partial z(x,y)}{\partial y}\right|_{y=0} = \nu(x), \quad 0 < x < 1 \tag{34}$$

is given by the formula (19).

Assuming that $y = 0$ in (19), in view of (30), and

$$z(x,0) = \tau(x), \quad 0 \le x \le 1, \tag{35}$$

we obtain the functional relation, transferred from the domain Ω_2 onto AB:

$$\tau(x) + \int_0^x K(x,t)\tau\left(\frac{t}{2}\right)dt - \int_0^x I_0\left[\sqrt{\lambda}(x-t)\right]\nu(t)dt = f_1(x), \tag{36}$$

where

$$f_1(x) = 2\psi_1\left(\frac{x}{2}\right) - \psi_1(0)J_0\left[\sqrt{\lambda}x\right] + \lambda x\int_0^x \bar{I}_1\left[\sqrt{\lambda x(x-t)}\right]\psi_1\left(\frac{t}{2}\right)dt+$$

$$+\frac{1}{\sqrt{\lambda}}\int_0^x K^*\left(\frac{x}{2},\frac{t}{2}\right)\left(\lambda\psi_1\left(\frac{t}{2}\right) - \sqrt{2}\psi_2'\left(\frac{t}{2}\right)\right)dt, \tag{37}$$

where $K(x,t)$ can be represented in the form (21).

Denoting

$$\tilde{f}_1(x) = \tau(x) + \int_0^x K(x,t)\tau\left(\frac{t}{2}\right)dt - f_1(x), \tag{38}$$

from (37), and using the inversion formula for such equations [9]:

$$\nu(x) = C_{OX}^{0,\sqrt{\lambda}}\left[\tilde{f}_1(x)\right] \equiv \tilde{f}_1'(x) - \lambda\int_0^x \tilde{f}_1(t)\bar{I}_1\left[\sqrt{\lambda}(x-t)\right]dt,$$

in view of (26) and (38), we obtain $\nu(x)$ with respect to $\tau(x)$ in the form

$$\nu(x) = \tau'(x) - \lambda\int_0^x \tau(t)\bar{I}_1\left[\sqrt{\lambda}(x-t)\right]dt+$$

$$+\int_0^x \tau\left(\frac{t}{2}\right)\left(K'(x,t) - \lambda\int_t^x K(s,t)\bar{I}_1\left[\sqrt{\lambda}(x-s)\right]ds\right)dt- \tag{39}$$

$$-f_1'(x) + \lambda\int_0^x f_1(t)\bar{I}_1\left[\sqrt{\lambda}(x-t)\right]dt.$$

Due to the property of Problem C and in view of (34), (35), we obtain [4]

$$\tau''(x) - \lambda\tau(x) = k + \nu(x) \tag{40}$$

from Equation (28) in Ω_1, tending $y \to -0$.

Here k is an unknown constant to be defined.

The equality (40) is a second functional relation between $\tau(x)$ and $\nu(x)$, transferred from the domain Ω_1 to AB.

3.3. Existence of solution to Problem C

Solving Equation (40) with respect to $\tau(x)$ with the conditions

$$\tau(0) = \varphi_1(0), \quad \tau'(0) = \varphi_3(0), \tag{41}$$

we have

$$\begin{gathered}\tau(x) = \tfrac{1}{\sqrt{\lambda}}\int\limits_0^x sh\sqrt{\lambda}(x-t)\nu(t)dt - \tfrac{k}{\lambda}(1 - ch\sqrt{\lambda}x) + \\ + \varphi_1(0)ch\sqrt{\lambda}x + \tfrac{1}{\sqrt{\lambda}}\varphi_3(0)sh\sqrt{\lambda}x.\end{gathered} \tag{42}$$

Omitting the function $\nu(x)$ in (39) and (42), in view of the sewing condition, we obtain an integral equation with a shift with respect to $\tau(x)$:

$$\begin{gathered}\tau(x) - \int\limits_0^x K_1(x,t)\tau(t)dt - \int\limits_0^x K_2(x,t)\tau\left(\tfrac{t}{2}\right)dt = \\ = -\tfrac{k}{\lambda}\left(1 - ch\sqrt{\lambda}x\right) + \Phi_1(x),\end{gathered} \tag{43}$$

where

$$K_1(x,t) = ch\sqrt{\lambda}(x-t) - \sqrt{\lambda}\int\limits_t^x sh\sqrt{\lambda}(x-s)\bar{I}_1\left[\sqrt{\lambda}(s-t)\right]ds,$$

$$K_2(x,t) = -\frac{1}{\sqrt{\lambda}}\int\limits_t^x sh\sqrt{\lambda}(x-s)\left(K'(s,t) - \lambda\int\limits_t^x K(z,t)\bar{I}_1\left[\sqrt{\lambda}(s-z)\right]dz\right)ds,$$

$$\Phi_1(x) = \varphi_1(0)ch\sqrt{\lambda}x + \frac{1}{\sqrt{\lambda}}\varphi_3(0)sh\sqrt{\lambda}x - \int\limits_0^x K_1(x,t)f_1(t)dt. \tag{44}$$

Assuming that

$$\alpha(x) = \Phi_1(x) - \frac{k}{\lambda}\left(1 - ch\sqrt{\lambda}x\right) + \int\limits_0^x K_2(x,t)\tau\left(\frac{t}{2}\right)dt, \tag{45}$$

we write Equation (43) in the form

$$\tau(x) - \int\limits_0^x K_1(x,t)\tau(t)dt = \alpha(x), \quad 0 \le x \le 1. \tag{46}$$

Hence, Equation (43) is an integral Volterra equation of the second kind, which is unconditionally and uniquely solvable in the class $C(0 \leqslant x \leqslant 1)$. Thus, solution of Equation (46) has the from

$$\tau(x) = \alpha(x) + \int\limits_0^x R_1(x,t)\alpha(t)dt, \tag{47}$$

where $R_1(x,t)$ is the resolvent of the kernel $K_1(x,t)$.

In view of (45) and the Dirichlet formula, Equation (47) has the from

$$\tau(x) - \int\limits_0^x K_2^*(x,t)\tau\left(\frac{t}{2}\right)dt = \Phi_2(x), \tag{48}$$

where

$$K_2^*(x,t) = K_2(x,t) + \int\limits_t^x K_2(s,t)R_1(x,s)ds,$$

$$\Phi_2(x) = \Phi_1(x) + \int\limits_0^x R_1(x,t)\Phi_1(t)dt - \frac{k}{\lambda}\left[1 - ch\sqrt{\lambda}x + \int\limits_0^x \left(1 - ch\sqrt{\lambda}t\right)R_1(x,t)dt\right].$$

Whence, we conclude that Equation (48) always has a solution that is unique and can be represented in the form [8]

$$\tau(x) = \Phi_2(x) + \int\limits_0^x R_2(x,t)\Phi_2\left(\frac{t}{2}\right)dt, \quad 0 \le x \le 1, \tag{49}$$

where $R_2(x,t)$ is the resolvent of the kernel $K_2^*(x,t)$.

Whence, by virtue of the condition $\tau(1) = \varphi_2(0) - w(1)$, k are determined uniquely.

Upon determining $\tau(x)$, we find the functions $\nu(x)$ and $w(x)$ from (39) and (18).

Thus, solution of Problem C in the domain Ω_2 in view of (18) and (19) is determined uniquely according to the formula (27), and in the domain Ω_1 we arrive to the problem for an nonloaded equation of the third order [4].

Thus, solution of problem C in the domains Ω_1 and Ω_2 can be constructed from (27) in view of (18), (19) and Problem D $_{11}$ [4].

Thus, Problem C is uniquely solvable.

Theorem 2 is proved.

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