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ON NECESSARY CONDITIONS FOR EXISTENCE OF PERIODIC SOLUTIONS IN A DYNAMIC SYSTEM WITH DISCONTINUOUS NONLINEARITY AND AN EXTERNAL PERIODIC INFLUENCE

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Abstract. The system of ordinary differential equations with discontinuous nonlinearity of a non-ideal relay type and an external continuous periodic influence in the right-hand side is considered in the Euclidean space. Necessary conditions for existence of periodic solutions with given properties in problems of the specified class imposed on the coefficients of the system are obtained by means of accurate analytical methods. An approach for finding switching instants and points of the image point of the required solution is suggested in the case when the period of the solution is multiple to the period of the function describing the external perturbation.

Keywords: switching points, forced periodic oscillations, automatic control systems, discontinuous hysteresis nonlinearity.

Introduction

The key problem in the theory of nonlinear oscillations is to prove the existence of cyclic behavior in nonlinear control systems. The present paper suggests an approach to solving this problem for nonlinear systems of ordinary differential equations containing a hysteresis nonlinearity and an external perturbation imposed on the control object. Automatic systems used in seacraft, e.g., heading control systems or stabilizers can be considered as control objects. Mathematical models of such objects have been studied by a number of authors (see, e.g., [1]–[3]).

In the present paper, unlike [3], another approach to investigation of systems of the considered class is used, restrictions imposed on the considered system are relaxed, periodic solutions are sought to be not only equal, but also multiple to the period of an external perturbation. The paper [4] considers nonperiodic external influence of the amplitude changing with time. On the contrary to [5], in the present paper the emphasis is made on finding switching instants when the desired regimes change over and on the analysis of the coefficient space of the initial system.

1. Problem statement

A system of ordinary differential equations of the form

$$\dot{Y} = AY + Bu(\sigma) + Kf(t), \quad \sigma = (C, Y) \tag{1}$$

is considered in an n-dimensional Euclidean space E^n. Here A is a matrix, vectors B, K, C are real and constant, Y is a state vector of the system ($Y \in E^n$). The function $u(\sigma)$ describes nonlinearity of a non-ideal relay type with threshold numbers ℓ_1, ℓ_2 and output numbers m_1, m_2. To be specific, assume that $\ell_1 < \ell_2$ and $m_1 < m_2$. The function $u(\sigma(t))$ is defined for

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$t \geq 0$ in a class of continuous functions, it can take only two values m_1 and m_2, and is given as follows. When $\sigma(t) \leq \ell_1$, the equality $u(\sigma(t)) = m_1$ holds, and when $\sigma(t) \geq \ell_2$ the equality $u(\sigma(t)) = m_2$ holds. If $\ell_1 < \sigma(t) < \ell_2$ for all $t_1 < t < t_2$ and $\sigma(t_1) = \ell_1$ or $\sigma(t_1) = \ell_2$, then assume that $u(\sigma(t)) = u(\sigma(t_1))$. Finally, if $\ell_1 < \sigma(t) < \ell_2$ for all $0 \leq t < t_2$, then assume that $u(\sigma(t)) = u_0$, where u_0 is one of the numbers m_1 or m_2. In the latter case, the dynamics of the system differs depending on the choice of the initial state u_0 of the relay. The hysteresis loop described in the coordinates (σ, u) by equations $\sigma = \sigma(t)$, $u = u(\sigma(t))$ is run counterclockwise. The function $f(t)$ describes the external influence on the system and belongs to the class of continuous periodic functions.

The problem of existence and finding switching instants when the switch over occurs in relay for periodic oscillations to appear and be supported in the system is considered.

2. General approach to investigation of the system

A.V. Pokrovskii obtained powerful analytical results in [3] for systems of the considered class. The existence theorem is proved for at least one asymptotically stable solution with the period equal to the period of external influence. The positivity condition of the system (restrictions on the coefficients vector of the feedback C) is stipulated and the matrix A is supposed to be hurwitzean.

Another approach for investigating systems of the form (1) is suggested in the present paper. In the coefficient space of the system, it allows one to determine sets to which periodic solutions with the period multiple to the period of external influence correspond, and if the periods are equal, the above restrictions to the system are removed.

The present approach is based on exact analytical methods of investigation namely, methods of the theory of canonical transformations of systems, results by V.I. Zubov [1] built upon the idea of constructing an auxiliary system in view of the periodicity property of a solution to autonomous systems, and the method of section for the system parameter space suggested by R.A. Nelepin [2].

In a phase n-dimensional space the trajectory of any solution of the system (1) can be composed of pieces of trajectories due to linear systems of the following form:

$$\dot{Y} = AY + Bm_1 + Kf(t), \quad \dot{Y} = AY + Bm_2 + Kf(t). \tag{2}$$

Pieces of trajectories on continuity are "joined together" at the points lying in the hyperplanes of the form $(C, Y) = \ell_i$ $(i = 1, 2)$.

To be specific, let us find solutions to the system (1) in the class of continuous, periodic functions with two switching points that will be referred to as "join" points in what follows. Closed bounded trajectories correspond to periodic solutions of the system (1) in an n-dimensional phase space. In an expanded $(n+1)$-dimensional space (Y, t) an integral curve consisting of several integral curves corresponds to the periodic solution of the system (1) by virtue of various systems of the form (2). The curves repeat themselves with a period T_B, which is further referred to as the period of forced oscillations of the system (1). The switching points Y^1, Y^2 of the periodic solution (the join points of pieces of trajectories) have the following properties:

$$Y^i = Y(t_0, m_j, t_0) = Y(t_0, m_j, t_0 + T_B), \ \ (C, Y^i) = \ell_k \ \ \forall i, j, k = 1, 2,$$

i. e. one can write out 8 various systems depending on the chosen sequence of motions of the image point of the periodic solution from one hyperplane to another one.

Consider the solution of the system (1) in the Cauchy form

$$Y(t) = e^{A(t-t_0)} Y(t_0) + \int\limits_{t_0}^{t} e^{-A(\tau - t)} \left(Bm_i + Kf(\tau) \right) d\tau \ \ (i = 1, 2).$$

Let us assume that (1) has at least one periodic solution with the period T_B. Let the image point of the desired periodic solution to the system (1) begin its motion at the point Y^1 on the hyperplane $\sigma = \ell_1$ at the instant $t_0 = 0$ and reach the point Y^2 on the hyperplane $\sigma = \ell_2$ at the instant t_1 by virtue of the system (2) provided that $m_i = m_1$. Then it returns to the point Y^1 on the hyperplane $\sigma = \ell_1$ at the instant T_B by virtue of the system (2) provided that $m_i = m_2$.

Let us set up a system of transcendental equations with respect to the switching points and switching instants on the basis of the periodicity property of the desired solution and taking into account that the switching points belong to the hyperplanes and the image point of the solution is moving along the trajectory according to the sequence prescribed above. One has

$$\ell_1 = \left(C, Y^1\right), \quad \ell_2 = \left(C, Y^2\right), \tag{3}$$

where

$$Y^2 = e^{At_1}Y^1 + \int\limits_0^{t_1} e^{A(t_1-\tau)}(Bm_1 + Kf(\tau))d\tau,$$

$$Y^1 = e^{A(T_B-t_1)}Y^2 + \int\limits_{t_1}^{T_B} e^{A(T_B-\tau)}(Bm_2 + Kf(\tau))d\tau.$$

The resulting system of four equations can be solved with respect to t_1, T_B, Y^1, Y^2 by numerical methods. In order to solve the system (3) in an analytical form let us transform the initial system.

To be specific, let us assume that the matrix A has only simple, nonzero, real eigenvalues λ_i $(i = \overline{1,n})$, the system (1) is completely controllable with respect to the input $u(\sigma)$, i. e. the inequality

$$\det \|B, AB, A^2B, \ldots, A^{n-1}B\| \neq 0$$

holds. In this case the system (1) can be reduced to the canonical form by a nonsingular transformation $Y = SX$:

$$\dot{X} = A_0X + B_0u(\sigma) + K_0f(t), \quad \sigma = (\Gamma, X), \tag{4}$$

where

$$A_0 = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}, \quad B_0 = S^{-1}B = \begin{pmatrix} 1 \\ \ldots \\ 1 \end{pmatrix}, \quad K_0 = S^{-1}K, \quad \Gamma = \begin{pmatrix} \gamma_1 \\ \ldots \\ \gamma_n \end{pmatrix}.$$

The coefficients γ_i $(i = \overline{1,n})$ are calculated by the formula:

$$\gamma_i = \frac{-1}{D'(\lambda_i)} \sum_{k=1}^{n} c_k N_k(\lambda_i), \tag{5}$$

where $D'(\lambda_i) = \left.\frac{dD(p)}{dp}\right|_{p=\lambda_i}$, c_k are elements of the vector C, $N_k(\lambda_i) = \sum\limits_{j=1}^{n} b_j D_{jk}(\lambda_i)$. Here b_j is an element of the vector B, D_{jk} is a cofactor of the element a_{jk} of the matrix A, λ_i are roots of the algebraic equation $D(p) = \det\left[a_{k\alpha} - \delta_{k\alpha}p\right] = 0$, $a_{k\alpha}$ are elements of the matrix A, $\delta_{k\alpha}$ is the Kronecker symbol. The transformation matrix S has the form

$$S = -\begin{pmatrix} \dfrac{N_1(\lambda_1)}{D'(\lambda_1)} & \cdots & \dfrac{N_1(\lambda_n)}{D'(\lambda_n)} \\ \vdots & \ddots & \vdots \\ \dfrac{N_n(\lambda_1)}{D'(\lambda_1)} & \cdots & \dfrac{N_n(\lambda_n)}{D'(\lambda_n)} \end{pmatrix}.$$

Following [2], we suppose that $(n-1)$ roots of the equation $D(p)=0$ coincide with $(n-1)$ roots of the equation $\sum\limits_{k=1}^{n} c_k N_k(p) = 0$. Then, $(n-1)$ values of γ_i defined by the formula (5) vanish and one value of γ_i is not equal to zero. Denote the index for which $\gamma_i \neq 0$ by s, i. e. $\gamma_s \neq 0$.

Thus, the system of the n-th order splits into lower-order systems that can be integrated successively. This simplifies the system of transcendental equations (3).

Provided that $\gamma_i = 0$ $(i \neq s)$, the function $\sigma(t) = (\Gamma, X(t))$ is determined from the first-order system

$$\sigma(t) = \gamma_s x_s, \quad \dot{x}_s = \lambda_s x_s + u(\sigma) + k_s^0 f(t). \tag{6}$$

The remaining variables x_i $(i \neq s)$ are defined from nonhomogeneous linear equations of the first order

$$\dot{x}_i = \lambda_i x_i + u(\sigma) + k_i^0 f(t), \quad i \neq s. \tag{7}$$

Let us write out a differential equation with respect to the function $\sigma(t)$:

$$\dot{\sigma}(t) = \lambda_s \sigma(t) + \gamma_s (u(\sigma(t)) + k_s^0 f(t)). \tag{8}$$

Since periodic solutions of the system (1) and $\sigma(t) = \sigma(x_s(t))$ are sought for, the function $\sigma(t)$ is supposed to belong to the class of continuous periodic functions. By means of the solution of Equation (8), one can define periodicity conditions for the function $\sigma(t)$ and its properties (the period T_B and the switching instant t_1).

Solution of the system of equations (6), (7) has the following form:

$$x_i(t) = x_i(0) e^{\lambda_i t} + e^{\lambda_i t} \int\limits_0^t (u(\sigma(\tau)) + k_i^0 f(\tau)) e^{-\lambda_i \tau} d\tau,$$

$$x_s(t) = \sigma(t)/\gamma_s = (\sigma_0/\gamma_s) e^{\lambda_s t} + e^{\lambda_s t} \int\limits_0^t (u(\sigma(\tau) + k_s^0 f(\tau)) e^{-\lambda_s \tau} d\tau. \tag{9}$$

The system of equations (9) determines the pointwise mapping of one switching plane into another one. Let us write out the solution of Equation (8) in the general form

$$\sigma(t) = \sigma_0 e^{\lambda_s (t - t_0)} + \gamma_s e^{\lambda_s t} \left(m_i \int\limits_{t_0}^{t} e^{-\lambda_s \tau} d\tau + k_s^0 \int\limits_{t_0}^{t} e^{-\lambda_s \tau} f(\tau) d\tau \right)$$

with the initial and boundary-value conditions

$$\ell_1 = \sigma(\ell_1, 0, m_1, 0), \quad \ell_2 = \sigma(\ell_1, 0, m_1, t_1), \quad \ell_1 = \sigma(\ell_2, t_1, m_2, T_B).$$

The system of transcendental equations for finding only switching instants t_1, T_B has the following form:

$$\ell_2 = \left(\ell_1 + \frac{\gamma_s m_1}{\lambda_s} \right) e^{\lambda_s t_1} - \frac{\gamma_s m_1}{\lambda_s} + \gamma_s k_s^0 \int\limits_0^{t_1} e^{\lambda_s (t_1 - \tau)} f(\tau) d\tau,$$

$$\ell_1 = \left(\ell_2 + \frac{\gamma_s m_2}{\lambda_s} \right) e^{\lambda_s (T_B - t_1)} - \frac{\gamma_s m_2}{\lambda_s} + \gamma_s k_s^0 \int\limits_{t_1}^{T_B} e^{\lambda_s (T_B - \tau)} f(\tau) d\tau. \tag{10}$$

The switching points X^1, X^2 of the transformed system (4) are determined by the following formulae:

$$X^1 = \left(E - e^{A_0 T_B}\right)^{-1} \left(\int\limits_{t_1}^{T_B} e^{A_0 (T_B - \tau)} \left(B_0 m_2 + K_0 f(\tau)\right) d\tau + \right.$$

$$\left. \int\limits_{0}^{t_1} e^{A_0 (T_B - \tau)} \left(B_0 m_1 + K_0 f(\tau)\right) d\tau \right),$$

$$X^2 = \left(E - e^{A_0 T_B}\right)^{-1} \left(\int\limits_{0}^{t_1} e^{A_0 (t_1 - \tau)} \left(B_0 m_1 + K_0 f(\tau)\right) d\tau + \right.$$

$$\left. e^{A_0 t_1} \int\limits_{t_1}^{T_B} e^{A_0 (T_B - \tau)} \left(B_0 m_2 + K_0 f(\tau)\right) d\tau \right).$$

3. Main result

Consider a model of external perturbation of the following form:

$$f(t) = f_0 + f_1 \sin(\omega t + \varphi_1) + f_2 \sin(2\omega t + \varphi_2), \tag{11}$$

where f_0, f_1, f_2, φ_1, φ_2, ω are real constants.

The function $f(t)$ of the form (11) can be considered as a truncated Fourier series. Since any periodic function satisfying the Dirichlet principle can be represented in the form of a converging Fourier series, the representation (11) is an approximation of an arbitrary periodic external influence.

The main result of the present paper is the following theorem.

Theorem. *Let the function $f(t)$ have the form (11). Let the system (1) have a periodic solution with the period $T_B = kT$, where $k \in \mathbf{N}$, $T = 2\pi/\omega$, $\omega > 0$. Let us assume that all eigenvalues of the matrix A are prime, real, and at least one of them is positive ($\lambda_s > 0$), and the element γ_s of the transformed vector of the feedback Γ is other than zero. Finally, let the following inequalities hold:*

1)

$$m_2 - m_1 e^{\lambda_s kT} + \lambda_s (1 - e^{\lambda_s kT})(\ell_1/\gamma_s + k_s^0 L) > 0,$$

$$m_1 < -\lambda_s \left(\frac{\ell_1}{\gamma_s} + k_s^0 L \right) < m_2,$$

where

$$L = \frac{f_0}{\lambda_s} + \frac{f_1 \sin(\varphi_1 + \delta_1)}{\sqrt{\lambda_s^2 + \omega^2}} + \frac{f_2 \sin(\varphi_2 + \delta_2)}{\sqrt{\lambda_s^2 + 4\omega^2}},$$

$$\delta_1 = \operatorname{arctg}(\omega/\lambda_s), \delta_2 = \operatorname{arctg}(2\omega/\lambda_s);$$

2)

$$\left(\ell_1 + \frac{\gamma_s}{\lambda_s}(m_1 + k_s^0 f_0) \right) (e^{\lambda_s kT} H - 1) +$$

$$\frac{\gamma_s k_s^0 f_1}{\sqrt{\lambda_s^2 + \omega^2}} \left(\sin(\varphi_1 + \delta_1) e^{\lambda_s kT} H - \sin\left(\frac{\omega}{\lambda_s} \ln H + \varphi_1 + \delta_1 \right) \right) +$$

$$\frac{\gamma_s k_s^0 f_2}{\sqrt{\lambda_s^2 + 4\omega^2}} \left(\sin(\varphi_2 + \delta_2) e^{\lambda_s kT} H - \sin\left(\frac{2\omega}{\lambda_s} \ln H + \varphi_2 + \delta_2 \right) \right) > 0,$$

where

$$H = \frac{m_2 - m_1}{\lambda_s(1 - e^{\lambda_s kT})(\ell_1/\gamma_s + k_s^0 L) + m_2 - m_1 e^{\lambda_s kT}};$$

and the equality
3)

$$\ell_2 = \ell_1 e^{\lambda_s kT} H + \frac{\gamma_s}{\lambda_s}(m_1 + k_s^0 f_0)(e^{\lambda_s kT} H - 1) +$$
$$\frac{\gamma_s k_s^0 f_1}{\sqrt{\lambda_s^2 + \omega^2}} \left(\sin(\varphi_1 + \delta_1) e^{\lambda_s kT} H - \sin\left(\frac{\omega}{\lambda_s} \ln H + \varphi_1 + \delta_1 \right) \right) +$$
$$\frac{\gamma_s k_s^0 f_2}{\sqrt{\lambda_s^2 + 4\omega^2}} \left(\sin(\varphi_2 + \delta_2) e^{\lambda_s kT} H - \sin\left(\frac{2\omega}{\lambda_s} \ln H + \varphi_2 + \delta_2 \right) \right)$$

is true.
Then the system (10) has a unique solution $t_1 \in (0, kT)$ defined by the formula $t_1 = kT + \frac{1}{\lambda_s} \ln H$.

Proof.

The system of transcendental equations (10) takes the form

$$\ell_2 = \left(\ell_1 + \frac{\gamma_s}{\lambda_s}(m_1 + k_s^0 f_0) + \frac{\gamma_s k_s^0 f_1}{\sqrt{\lambda_s^2 + \omega^2}} \sin(\varphi_1 + \delta_1) + \right.$$
$$\left. \frac{\gamma_s k_s^0 f_2}{\sqrt{\lambda_s^2 + 4\omega^2}} \sin(\varphi_2 + \delta_2) \right) e^{\lambda_s t_1} - \frac{\gamma_s}{\lambda_s}(m_1 + k_s^0 f_0) -$$
$$\frac{\gamma_s k_s^0 f_1}{\sqrt{\lambda_s^2 + \omega^2}} \sin(\omega t_1 + \varphi_1 + \delta_1) - \frac{\gamma_s k_s^0 f_2}{\sqrt{\lambda_s^2 + 4\omega^2}} \sin(2\omega t_1 + \varphi_2 + \delta_2),$$
$$\ell_1 = \left(\ell_2 + \frac{\gamma_s}{\lambda_s}(m_2 + k_s^0 f_0) + \frac{\gamma_s k_s^0 f_1}{\sqrt{\lambda_s^2 + \omega^2}} \sin(\omega t_1 + \varphi_1 + \delta_1) + \right.$$
$$\left. \frac{\gamma_s k_s^0 f_2}{\sqrt{\lambda_s^2 + 4\omega^2}} \sin(2\omega t_1 + \varphi_2 + \delta_2) \right) e^{\lambda_s (T_B - t_1)} - \frac{\gamma_s}{\lambda_s}(m_2 + k_s^0 f_0) -$$
$$\frac{\gamma_s k_s^0 f_1}{\sqrt{\lambda_s^2 + \omega^2}} \sin(\omega T_B + \varphi_1 + \delta_1) - \frac{\gamma_s k_s^0 f_2}{\sqrt{\lambda_s^2 + 4\omega^2}} \sin(2\omega T_B + \varphi_2 + \delta_2), \tag{12}$$

provided that $\lambda_s > 0$. If the periodic solution to the system (1), (11) is sought for with a prescribed period, namely $T_B = kT$, $k \in \mathbf{N}$, $T = 2\pi/\omega$, the system of transcendental equations (12) depends only on one variable t_1, and as a result of the above choice of the feedback coefficients γ_i $(i = \overline{1, n})$, it is analytically solvable with respect to this variable.

Let us substitute the first equation of the system (12) into the second one. Upon transformation, one has

$$(1 - e^{\lambda_s kT}) \left(\ell_1 + \gamma_s k_s^0 \left(\frac{f_0}{\lambda_s} + \frac{f_1}{\sqrt{\lambda_s^2 + \omega^2}} \sin(\varphi_1 + \delta_1) + \right.\right.$$
$$\left.\left. \frac{f_2}{\sqrt{\lambda_s^2 + 4\omega^2}} \sin(\varphi_2 + \delta_2) \right) \right) + \frac{\gamma_s}{\lambda_s}(m_2 - m_1 e^{\lambda_s kT}) = \frac{\gamma_s}{\lambda_s}(m_2 - m_1) e^{\lambda_s (kT - t_1)},$$

whence one obtains a formula for determining the variable t_1.

Then let us define the conditions on parameters that provide the existence of the solution t_1.

The expression under the logarithm in the denominator of the formula defining the variable t_1 should be positive because $m_2 > m_1$ by assumption. This entails the first inequality of Condition 1) of the theorem.

Since the variable t_1 is defined as the first switching instant, it obviously should belong to the interval $(0, kT)$, where $k \in \mathbf{N}$. This is possible provided that the following inequalities hold:

$$m_2 - m_1 < \lambda_s(1 - e^{\lambda_s kT}) \left(\ell_1/\gamma_s + k_s^0 L\right) + m_2 - m_1 e^{\lambda_s kT},$$

$$m_2 - m_1 > \lambda_s \frac{(1 - e^{\lambda_s kT})}{e^{\lambda_s kT}} \left(\ell_1/\gamma_s + k_s^0 L\right) + \frac{m_2}{e^{\lambda_s kT}} - m_1.$$

Upon transformation, the latter inequalities take the form of the second inequality of Condition 1) of the theorem.

Condition 2) of the theorem follows from the assumption that $\ell_2 > \ell_1$.

The solution t_1 is a solution of the system of transcendental equations if it satisfies the first equation of the system (12). This provides Condition 3) of the theorem.

The theorem is proved completely.

Remark 1. The system of inequalities and equalities in Conditions 1)–3) of the proved theorem is consistent, because it is determined by strict analytical calculation with the use of equivalent passages and properties of the logarithmic function. Therefore, one can make an example of existence of a kT-periodic solution. Indeed, e.g., if $f(t) = 1 + 2\sin(t + \frac{\pi}{3}) + 5\sin(2t)$, $T_B = 2\pi$, $\lambda_s = 0,2$, $\gamma_s = -0,5$, the mentioned system of inequalities and equalities holds when $m_1 = -5$, $m_2 = 15,73$, $\ell_1 = -6$, $k_s^0 = -2$, and the system (10) has a unique solution $t_1 = 3,51$.

Remark 2. The theorem formulates the necessary conditions for existence of a periodic solution to a canonical system of equations and, due to a nonsingular transformation, to the initial system. Moreover, properties of the desired periodic solution for the prescribed period $T_B = kT$ are defined, namely the instant of the first switch t_1 and two switching points $Y^1 = SX^1$, $Y^2 = SX^2$.

Remark 3. The system of transcendental equations is composed of the necessary conditions for existence of at least one periodic solution with given properties. Therefore, when the system of transcendental equations does not have the solution t_1, conditions on the canonical system coefficients define the sets in the coefficient space of the canonical system (by virtue of the nonsingular transformation in the space of the initial system), where the desired periodic solutions cannot occur.

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