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INTEGRALS OF EXPONENTIAL FUNCTIONS WITH RESPECT TO RADON MEASURE

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Abstract. Properties of sets of convergence for integrals of exponential functions in a finite-dimensional Euclidean space are studied in the paper. It is shown that these sets are always convex. In particular, these sets include the sets of absolute convergence of series of exponential functions.

A special class of convex sets is introduced and a complete description of sets of convergence is obtained for the case of open and relatively close convex sets in terms of this class.

Necessary and sufficient conditions for any set of convergence to be open and independently unbounded are formulated.

Keywords: convex sets, Radon measure, Laplace integrals, absolutely convergent series of exponentials.

1. Introduction

The first result related to the subject of the present paper can be considered to be the following property of exponential series (see [1], [2], p. 194–195).

The set of absolute convergence of the exponential series

$$\sum_{n=1}^{\infty} a_n e^{\lambda_n z},\ a_n, \lambda_n \in \mathbb{C},\ n \in \mathbb{N},\ |\lambda_n| \to \infty$$

is convex, the series converges uniformly on compacts inside the set.

The proof of convexity is carried over directly to multidimensional exponential series, whereas the second part does not follow from the cited works.

Properties of sets of absolute convergence of series of exponential monomials of one complex variable depending on the coefficients, satisfying certain conditions, has been considered in the article [3].

2. Definitions and preliminary results

Let $\mathbb{E}$ be a finite-dimensional Euclidean space over a field of real numbers. In what follows, a scalar product of elements $x,\ y \in \mathbb{E}$ is written as xy.

A typical example of such space is the space $\mathbb{R}^m$, $m \in \mathbb{N}$ with an ordinary scalar product.

Another example is a finite-dimensional Euclidean space $\mathbb{H}$ over a field of complex numbers with a scalar product zw, if the new scalar product is given as

$$\langle z, w\rangle = \operatorname{Re} zw,$$

where $z,\ w \in \mathbb{H}$.

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Denote a unit sphere, and a closed unit ball with the centre at zero of the space $\mathbb{E}$ by $\mathbb{S}$ and $\mathbb{B}$, respectively.

The support function of the set $M \subset \mathbb{E}$ is determined by the formula

$$H(\lambda, M) = \sup_{x \in M} \lambda x, \ \ \lambda \in \mathbb{E}.$$

It is a homogeneous convex function.

Due to homogeneity, it is sufficient to know the support function on the unit sphere.

Let us draw examples of support functions.

Example 1 For a vector $\alpha \in \mathbb{S}$ and a number $c \in \mathbb{R}$, the support function of the hyperplane $M = \{x \in \mathbb{E} : \alpha x = c\}$ is given by the formula

$$H(s, M) = \begin{cases} +\infty, & s \neq \pm\alpha \\ \pm c, & s = \pm\alpha \end{cases},$$

where $s \in \mathbb{S}$.

Indeed, let $s \in \mathbb{S}, \ s \neq \pm\alpha$. Manifestly, in this case the number $t = \alpha s$ satisfies the inequality $|t| < 1$. For a fixed number $r \in \mathbb{R}$, the vector

$$x = \frac{c - rt}{1 - t^2}\alpha + \frac{r - ct}{1 - t^2}s$$

obviously has the property $\alpha x = c, \ sx = r$, whence follows the unknown.

Example 2 For a vector $\alpha \in \mathbb{S}$ and a number $c \in \mathbb{R}$, the support function of the subspace $M = \{x \in \mathbb{E} : \alpha x \leqslant c\}$ is given by theformula

$$H(s, M) = \begin{cases} +\infty, & s \neq \alpha \\ c, & s = \alpha \end{cases},$$

where $s \in \mathbb{S}$.

This can be readily deduced from the above.

The closure and the interior of the set M are written as $\overline{M}$ and $\operatorname{int} M$, respectively.

Denote by $\operatorname{aff} M$ the affine hull of the set M, i.e. the set of vectors of the form $t_1x_1 + \cdots + t_kx_k$, where $x_j \in M, \ t_j \in \mathbb{R}, \ j = 1, \ldots, k$, and $t_1 + \cdots + t_k = 1$. For any vector $a \in \operatorname{aff} M$, the set $\operatorname{aff} M - a$ is a linear space.

Let us define the relative interior $\operatorname{ri} M$ of the set M as its interior in the space $\operatorname{aff} M$ with an induced topology.

If the convex set M is not empty, then the set $\operatorname{ri} M$ is not empty as well (see [4], p. 60).

Let us indicate an orthogonal projection of the space $\mathbb{E}$ to the subspace L by Π_L for the linear subspace $L \subset \mathbb{E}$.

Let S be a closed subset of the unit sphere $\mathbb{S}$.

Let us define a weakly S-convex hull of the set M as a set of points $x \in \mathbb{E}$, satisfying the condition

$$sx \leqslant \max_{1 \leqslant j \leqslant k} sx_j, \ \ s \in S, \tag{1}$$

for a certain system of vectors $x_1, \ldots, x_k \in M$. This set will be written as $\operatorname{conv}_S M$. Sets, corresponding to their weakly S-convex hull, are referred to as weakly S-convex.

Obviously, a weakly S-convex hull preserves the injections, namely:

$$M_1 \subset M_2 \Rightarrow \operatorname{conv}_S M_1 \subset \operatorname{conv}_S M_2. \tag{2}$$

One can readily observe that in case of the equality $S = \mathbb{S}$, the weakly S-convex hull coincides with an ordinary convex hull.

As it can be readily demonstrated, for any set $M \subset \mathbb{E}$, there exists the equality

$$H(s, \operatorname{conv}_S M) = H(s, M), \; s \in S. \tag{3}$$

For the number $\varepsilon > 0$, assume that

$$S_\varepsilon = \{s \in S : \exists u \in S, \; |s - u| \leqslant \varepsilon\}.$$

In what follows the following simple result will be useful.

Lemma 1. *For the set $M \subset \mathbb{E}$ and the compact*

$$K \subset \{x \in \mathbb{E} : sx < H(s, M), \; s \in S\},$$

there are a number $\varepsilon > 0$ and a system of vectors $x_1, \ldots, x_k \in M$ such that the injection

$$K \subset \operatorname{conv}_{S_\varepsilon} \{x_1, \ldots, x_k\}$$

holds.

Proof. For any fixed points $y_0 \in K$ and $s_0 \in S$, there is a vector $x_0 \in M$ with the condition $s_0 y_0 < s_0 x_0$. Due to continuity, one obtains the inequality $sy < sx_0$, where $|s - s_0| < \delta$, $|y - y_0| < \delta$, $s \in \mathbb{S}, y \in \mathbb{E}$, for a certain number $\delta > 0$.

However, the set $S \times K$ is a compact in the topological product $\mathbb{S} \times \mathbb{E}$ therefore, one can find a number $\varepsilon > 0$ and vectors $x_1, \ldots, x_k \in M$ such that

$$sy < \max_{1 \leqslant j \leqslant k} sx_j, \; s \in S_\varepsilon, \; y \in K,$$

whence the unknown follows.

Corollary 1. *Let us assume that the set $M \subset \mathbb{E}$ is weakly S_ε-convex for any number $\varepsilon > 0$. Then, the set* $\operatorname{int} M$ *is weakly S-convex and the equality*

$$\operatorname{int} M = \{x \in \mathbb{E} : sx < H(s, M), s \in S\} \tag{4}$$

holds.

Indeed, if the point x belongs to the left-hand side of the relation, then there is a number $\varepsilon > 0$, for which the injection $x + \varepsilon\mathbb{B} \subset M$ takes place and therefore, $sx + \varepsilon \leqslant H(s, M)$, $s \in \mathbb{S}$, so that the point lies in the right-hand side.

The reverse injection follows from the condition on the set M and the later lemma.

The right-hand side of the relation (4) is obviously weakly S-convex.

Corollary 2. *For a convex set $M \subset \mathbb{E}$ and the compact $K \subset \operatorname{ri} M$, there are vectors $x_1, \ldots, x_k \in M$ such that*

$$K \subset \operatorname{conv} \{x_1, \ldots, x_k\}.$$

One can consider that $0 \in \operatorname{ri} M$.

Let $L = \operatorname{aff} M$. Using the lemma for the Euclidean space L in the case $S = L \cap \mathbb{S}$, find the system of points $x_1, \ldots, x_k \in \operatorname{ri} M$, satisfying the relation

$$sx \leqslant \max_{1 \leqslant j \leqslant k} sx_j, \; s \in S, \; x \in K.$$

Manifestly, there are equalities $sx = \Pi_L(s)x$, $x \in M$ for an arbitrary point $s \in \mathbb{S}$, whence the unknown follows.

proposition 1. *The injection $\overline{\operatorname{conv}_S K} \subset \operatorname{ri} M$ holds for a weakly S-convex set $M \subset \mathbb{E}$ and the compact $K \subset \operatorname{ri} M$.*

Proof. According to the above proved result, there are points $x_1, \dots, x_k \in M$ such that the injection

$$K \subset \operatorname{conv}\{x_1, \dots, x_k\}$$

holds and therefore, $K \subset \operatorname{conv}_S\{x_1, \dots, x_k\}$. The latter set is weakly S-convex and closed, which provides the desired.

The set M, for which the equality

$$M = \{x \in \mathbb{E} : sx \leqslant H(s, M),\ s \in S\} \tag{5}$$

holds, is said to be S-convex (see [5], Chapter III). Such set is obviously closed.

As it can be readily demonstrated, the set

$$\{x \in \mathbb{E} : sx \leqslant H(s, D),\ s \in S\}$$

is S-convex for any set $D \subset \mathbb{E}$. A particular case of such sets is obviously the set $\operatorname{conv}_S\{x_1, \dots, x_k\}$ for an arbitrary system of vectors $x_1, \dots, x_k \in \mathbb{E}$.

Manifestly, S-convex sets are weakly S-convex, and the latter are convex.

Note, that the operations defined above are permutable with the shifts, namely:

$$\begin{gathered} \overline{x + M} = x + \overline{M},\ \operatorname{int}(x + M) = x + \operatorname{int} M, \\ \operatorname{ri}(x + M) = x + \operatorname{ri} M,\ \operatorname{aff}(x + M) = x + \operatorname{aff} M, \\ \operatorname{conv}_S(x + M) = x + \operatorname{conv}_S M,\ H(s, x + M) = sx + H(s, M), \end{gathered} \tag{6}$$

where the set $M \subset \mathbb{E},\ x \in \mathbb{E}$.

3. Properties of special convex sets

The given section presents properties of S-convex and weakly S-convex sets (see also [6], §4).

The following simple facts are true.

proposition 2. *The intersection of weakly S-convex sets is weakly S-convex.*

proposition 3. *Let $\{M_\alpha : \alpha \in A\}$ be a family of linearly ordered sets with respect to the injection $M_\alpha \subset \mathbb{E}$.*

Then,

$$\operatorname{conv}_S \bigcup_{\alpha \in A} M_\alpha = \bigcup_{\alpha \in A} \operatorname{conv}_S M_\alpha.$$

Corollary *The combination of a sequence of weakly S-convex sets increasing in injection is weakly S-convex.*

proposition 4. *Let the set $M \subset \mathbb{E}$ be weakly S-convex and not empty.*

Then, the following sets are weakly S-convex as well:

1. $M + x_0,\ x_0 \in \mathbb{E}$.
2. $tM,\ t > 0$.
3. $\operatorname{ri} M$.
4. $\overline{M}$.
5. $\operatorname{aff} M$.

Proof. This is obvious for the first two sets.

In view of the equalities (6), one can consider that $0 \in \operatorname{ri} M$, and the last three sets are weakly S-convex by virtue of the known equalities

$$\operatorname{ri} M = \bigcup_{0<t<1} tM,\ \overline{M} = \bigcap_{t>1} tM,\ \operatorname{aff} M = \bigcup_{t>0} tM,$$

and the above statements.

The following result is necessary to proceed.

proposition 5. *Let us assume that $M \subset \mathbb{E}$ is a convex set and the condition*

$$\mathrm{ri}\, M \bigcap \mathrm{ri}\,\mathrm{conv}_S M \neq \varnothing$$

holds for it.
Then,

$$\mathrm{ri}\,\mathrm{conv}_S M = \mathrm{conv}_S \mathrm{ri}\, M.$$

Proof. Without restriction of generality, one can consider that

$$0 \in \mathrm{ri}\, M \bigcap \mathrm{ri}\,\mathrm{conv}_S M.$$

Applying the proved properties of weakly S-convex hulls, one obtains

$$\mathrm{ri}\,\mathrm{conv}_S M = \bigcup_{t<1} t\,\mathrm{conv}_S M = \bigcup_{t<1} \mathrm{conv}_S tM = \mathrm{conv}_S \bigcup_{t<1} tM = \mathrm{conv}_S \mathrm{ri}\, M,$$

which proves the statement.
Corollary 1 *Let us assume that the relation*

$$U \bigcap \mathrm{ri}\,\mathrm{conv}_S U \neq \varnothing$$

holds for a convex relatively open set $U \subset \mathbb{E}$.
Then, the set $\mathrm{conv}_S U$ is relatively open as well.

Corollary 2 *Let us assume that $M \subset \mathbb{E}$ is a convex set and the injection*

$$\mathrm{ri}\,\mathrm{conv}_S M \subset M$$

holds for it.
Then, the set $\mathrm{ri}\, M$ is weakly S-convex.

Indeed, on the basis of the condition on the set and the relation $M \subset \mathrm{conv}_S M$, one can conclude that the convex sets $\mathrm{conv}_S M$ and M are of one dimension. In this case, the injection $\mathrm{ri}\,\mathrm{conv}_S M \subset \mathrm{ri}\, M$ holds obviously, and the unknown follows from the above statement and the injection $\mathrm{conv}_S \mathrm{ri}\, M \supset \mathrm{ri}\, M$.

Denote by $\rho(x, M)$ the distance from the point x to the set M for the point $x \in \mathbb{E}$ and the set $M \subset \mathbb{E}$.
For the set $M \subset \mathbb{E}$, ∂M stands for the relative boundary of the set M.
The following result will be of use.

Lemma 2. *Let $M \subset \mathbb{E}$ be a closed convex set.*
There is a relation

$$\max_{s \in \mathbb{S}} [sx - H(s, M)] = \begin{cases} \rho(x, \partial M), & x \notin M \\ -\rho(x, \partial M), & x \in M \end{cases}.$$

Proof. Let us assume that $r = \rho(x, \partial M)$ for the point $x \in \mathbb{E}$ and denote by $x_0 \in \mathbb{E}$ the point such that the equality $|x_0 - x| = r$ holds.
If $x \notin M$, one can readily deduce the existence of the vector $s_0 \in \mathbb{S}$ from the the Khan-Banach theorem and that for any vectors $y \in \mathbb{E}$ and $w \in M$the injection

$$|y - x| \leqslant r \Rightarrow s_0 y \geqslant s_0 w$$

holds, whence the inequality $s_0 x - r \geqslant H(s_0, M)$ follows.
On the other hand, for any vector $s \in \mathbb{S}$, one obtains

$$-H(s, M) \leqslant -s x_0 = s\,[-x + (x_0 - x)] \leqslant -sx + r,$$

and the first half of the lemma is proved.

Let $x \in M$. In this case, the inequality $sx + r \leqslant H(s, M),\ s \in \mathbb{S}$ holds.

Conversely, one obtains

$$H(s_0, M) = s_0 x_0 = s_0 \left[x + (x_0 - x)\right] \leqslant s_0 x + r$$

for the point $s_0 \in \mathbb{S}$ defining a tangent hyperplane to the set M at the point x_0. This proves the lemma.

Corollary *The function*

$$f(x) = \begin{cases} \rho(x, \partial M), & x \notin M \\ -\rho(x, \partial M), & x \in M \end{cases}$$

is convex for a closed convex set $M \subset \mathbb{E}$.

Indeed, this follows from the proved lemma and Theorem 5.5 of the monograph [4].

proposition 6. *Let $L_1 \subset \mathbb{E}$ be a linear subspace, $\{0\} \neq L_1 \neq \mathbb{E},\ L_2 = L_1^{\perp}$, the set $S = \{s \in \mathbb{S} : |\Pi_{L_1}(s)| \geqslant \varepsilon\}$ for a certain number $\varepsilon,\ 0 < \varepsilon \leqslant 1$, and the convex set $U \subset L_1$ be open in the topology of the space L_1.*

Then, the relation

$$\begin{aligned} \mathrm{conv}_S U = \{& x \in \mathbb{E} : x = x_1 + x_2,\ x_1 \in U,\ x_2 \in L_2, \\ & \sqrt{1 - \varepsilon^2}|x_2| < \varepsilon \rho(x_1, \partial U)\Big\} \end{aligned} \tag{7}$$

holds.

Proof. Let us denote the right-hand side of the equality (7) by D and demonstrate that there is a relation

$$D = \{x \in \mathbb{E} : sx < H(s, U),\ s \in S\}. \tag{8}$$

Note that the equality

$$H(s, U) = H(\Pi_{L_1}(s), U)$$

holds due to the injection $U \subset L_1$.

Any vector x, belonging to the right-hand side of (8), is uniquely represented in the form $x = x_1 + x_2,\ x_1 \in L_1,\ x_2 \in L_2$. If $x_2 \neq 0$, then assume that

$$s_2 = \frac{\sqrt{1 - \varepsilon^2} x_2}{|x_2|}.$$

Otherwise, let s_2 be an arbitrary vector of the space L_2 with the condition $|s_2| = \sqrt{1 - \varepsilon^2}$.

For any point $s_1 \in L_1,\ |s_1| = \varepsilon$, the vector $s = s_1 + s_2$ belongs to the set S. Therefore, $sx = s_1 x_1 + s_2 x_2 = s_1 x_1 + \sqrt{1 - \varepsilon^2}|x_2| < H(s_1, U)$, and one can readily deduce from Lemma 2 that $x_1 \in L_1$ and $\sqrt{1 - \varepsilon^2}|x_2| < \varepsilon \rho(x_1, \partial U)$.

Conversely, let the point x belong to the left-hand side of (8). Any vector $s \in S$ can be represented in the form $s = s_1 + s_2,\ s_1 \in L_1,\ s_2 \in L_2$ and, by condition, the inequalities $|s_1| \geqslant \varepsilon,\ |s_2| \leqslant \sqrt{1 - \varepsilon^2}$ hold. Using Lemma 2, one obtains

$$sx = s_1 x_1 + s_2 x_2 < H(s_1, U) - |s_1| \rho(x_1, \partial U) + \varepsilon \rho(x_1, \partial U) \leqslant H(s_1, U),$$

which proves the equality (8).

Corollary 1 of Lemma 1 entails the equality

$$\mathrm{int}\, \mathrm{conv}_S U = D.$$

Since the set D is obviously open and $U \subset D$, the unknown result is finally obtained from Corollary 1 of the Proposition 5.

proposition 7. *Let us assume that the hyperplane $L = \{x \in \mathbb{E} : \alpha x = 0\}$ for a certain vector $\alpha \in \mathbb{S}$, the set $S = \{s \in \mathbb{S} : \alpha s \leqslant \varepsilon\}$ for a fixed number ε, $0 \leqslant \varepsilon < 1$, and that the convex set $U \subset L$ is open in the topology of the space L.*

Then, the relation

$$\begin{aligned} \operatorname{ri}\operatorname{conv}_S U = \{x \in \mathbb{E} : x = x_1 + t\alpha,\ x_1 \in U,\\ t > 0,\ \varepsilon t < \sqrt{1-\varepsilon^2}\rho(x_1, \partial U)\Big\} \end{aligned} \tag{9}$$

holds.

Proof. Similarly to the above, denote the right-hand side of the equality (9) by D and demonstrate that the relation (8) holds true.

Any vector x, belonging to the right-hand side of (8), is uniquely represented in the form $x = x_1 + t\alpha$, $x_1 \in L$, $t \in \mathbb{R}$, and since the point $-\alpha$ obviously belongs to the set S, the inequality

$$-\alpha x = -t < H(\alpha, U) = 0$$

should hold, i.e. $t > 0$.

The vector $s = \sqrt{1-\varepsilon^2}s_1 + \varepsilon\alpha$ belongs to the set S for any vector $s_1 \in L$, $|s_1| = 1$. Therefore, $sx = s_1x_1 + \varepsilon t < H(s_1, U)$, and Lemma 2 readily provides that $x_1 \in L_1$ and $t\varepsilon < \sqrt{1-\varepsilon^2}\rho(x_1, \partial U)$.

The reverse injection is proved similarly to the above and the unknown quantity follows from the relation (3).

Let us draw several results on the S-convexity of weakly S-convex sets.

proposition 8. *The closed weakly S-convex set $M \subset \mathbb{E}$ with a nonempty interior is S-convex.*

Proof. One can assume that $0 \in \operatorname{int} M$.

It can be inferred from Proposition 4 and corollary 1 of Lemma 1 that

$$\operatorname{int} M = \{x \in \mathbb{E} : sx < H(s, M), s \in S\}.$$

One can readily see that this representation entails the relation

$$M \subset \{x \in \mathbb{E} : sx \leqslant H(s, M), s \in S\}.$$

Conversely, let $x \in \mathbb{E}$ and $sx \leqslant H(s, M), s \in S$. Manifestly,

$$H(s, M) > 0,\ s \in \mathbb{S}.$$

Therefore,

$$sy < H(s, M), s \in S,$$

where $y = tx$, holds for the number t, $0 < t < 1$ so that $tx \in \operatorname{int} M$ and hence, $x \in M$.

proposition 9. *The equality*

$$\overline{\operatorname{conv}_S K} = \{x \in \mathbb{E} : sx \leqslant H(s, K), s \in S\} \tag{10}$$

holds for a convex compact $K \subset \mathbb{E}$.

Proof. Without restriction of generality, one can consider that $0 \in \operatorname{ri} K$.

Manifestly, the left-hand side of the relation (10) is a subset of the right-hand one.

Conversely, let us assume that the inequality

$$sx \leqslant H(s, K), s \in S$$

holds for the point $x \in \mathbb{E}$. Then, one obtains

$$sy \leqslant H(s, tK), s \in S,$$

where $y = tx$, for the number t, $0 < t < 1$.

Evidently, the compact tK is a subset of $\mathrm{ri}\, K$. Therefore, Corollary 2 of Lemma 1 leads to the conclusion that

$$H(s, tM) \leqslant \max_{1 \leqslant j \leqslant k} sx_j, \ s \in \mathbb{S}$$

for some vectors $x_1, \ldots, x_k \in K$.

Thus, $tx \in \mathrm{conv}_S K$ for any number t, $0 < t < 1$, whence the unknown quantity follows.

As it is known, the convex hull of the compact is a compact. Let us demonstrate that a weakly S-convex hull of a compact is not closed in general.

Example 3 Let

$$\mathbb{E} = \mathbb{R}^3, \ s_0 = \frac{\sqrt{2}}{2}(1, 0, 1), \ s_n = \frac{\sqrt{2}}{2}\left(\cos\frac{1}{n}, \sin\frac{1}{n}, 1\right), \ n \in \mathbb{N},$$
$$S = \{s_0, s_1, s_2, \ldots, \}, \ K = \left\{(a, b, 0) \in \mathbb{R}^3 : a^2 + b^2 \leqslant 1\right\}.$$

Clearly, S is a closed subset of a unit sphere and K is a convex compact.

Let us prove that the point $y = (0, 0, 1)$ does not belong to the set $\mathrm{conv}_S K$.

Indeed, suppose there are points $x_1, \ldots, x_k \in K$, and the relations

$$s_n y \leqslant \max_{1 \leqslant j \leqslant k} s_n x_j, \ n \in \mathbb{N} \tag{11}$$

hold for them.

Let $x_j = r_j(\cos t_j, \sin t_j, 0)$, $0 \leqslant r_j \leqslant 1$, $0 \leqslant t_j < 2\pi$, $j = 1, \ldots, k$, and $j(n)$ be the index for which the maximum is reached in the formula (11).

In this case

$$\frac{\sqrt{2}}{2} \leqslant r_{j(n)} \frac{\sqrt{2}}{2} \cos\left(\frac{1}{n} - t_{j(n)}\right),$$

or $r_{j(n)} = 1$, $t_{j(n)} = \frac{1}{n}$. The latter equality is not realizable for all n. We have arrived to a contradiction.

On the other hand $H(s_n, K) = \sqrt{2}/2 = s_n(0, 0, 1)$, $n \in \mathbb{N}_0$, and from Proposition 9 one concludes that $(0, 0, 1) \in \overline{\mathrm{conv}_S K}$. Hence, a weakly S-convex hull of the compact K is not closed.

proposition 10. *The hyperplane $M = \{x \in \mathbb{E} : \alpha x = c\}$ is weakly S-convex for the vector $\alpha \in \mathbb{S}$ and the number $c \in \mathbb{R}$ if and only if $\pm\alpha \in S$.*

In this case, it is S-convex.

Proof. Let us assume that $c = 0$ and

$$K = M \cap \mathbb{S}.$$

If $\pm\alpha \in S$ then, according to Example 1, the set M satisfies the relation (5) so that it becomes S-convex.

Conversely, assume that the hyperplane M is weakly S-convex, but $\alpha \notin S$. Since the set S is closed, there is a number $\varepsilon, 0 < \varepsilon < 1$ such that the condition

$$|s - \alpha| \geqslant \varepsilon, \ s \in S$$

holds.

The set M is closed as well and applying the statement 9 and the property (2), one obtains

$$\{x \in \mathbb{E} : sx \leqslant H(s, K), \ s \in S\} \subset M. \tag{12}$$

Any vector $s \in \mathbb{S}$ is uniquely representable in the form

$$s = t\alpha + u, \ t \in \mathbb{R}, \ u \in \mathbb{E}, \ \alpha u = 0.$$

Hence, $1 = t^2 + |u|^2$. One has

$$|s-\alpha| = 1 - 2t + t^2 + |u|^2 = 2 - 2t \geqslant \varepsilon$$

for the points $s \in S$.

Evidently, the equality

$$H(s,K) = |u| = \sqrt{1-t^2}$$

holds for the support function of the compact K and the previous calculations readily provide the injection of the vector

$$\frac{2\varepsilon^2 - \varepsilon^4}{4-\varepsilon^2}\alpha$$

into the left-hand side of the relation (12), but this vector is not included into the right-hand side.

The resulting contradiction demonstrates that $\alpha \in S$. The situation is similar for the point $-\alpha$ and the statement is proved.

Let us demonstrate now that classes of S-convex and weakly S-convex sets differ.

Example 4 Let

$$\mathbb{E} = \mathbb{R}^3,\ M = \left\{\left(x^1, x^2, 0\right) \in \mathbb{R}^3 : x^1 \leqslant 0\right\},$$
$$\varphi(t) = \left(\sin t \cos t, \cos^2 t, \sin t\right),\ S = \{\varphi(t) : -\pi \leqslant t \leqslant \pi\}.$$

Let us suppose that the relation (1) holds for the points $x \in \mathbb{R}^3$, $x_1, \ldots, x_k \in M$:

$$x^1 \sin t \cos t + x^2 \cos^2 t + x^3 \sin t \leqslant \max_{1 \leqslant j \leqslant k} \left(x_j^1 \sin t \cos t + x_j^2 \cos^2 t\right),\ -\pi \leqslant t \leqslant \pi.$$

Substituting the values $t = \pm\pi$ into the inequalities, one obtains $x^3 = 0$.

Cancelling out entails

$$x^1 \sin t + x^2 \cos t \leqslant \max_{1 \leqslant j \leqslant k} \left(x_j^1 \sin t + x_j^2 \cos t\right),\ -\pi < t < \pi.$$

Whence, the inequality $x^1 \leqslant 0$ follows. Thus, the set M is weakly S-convex.

Let us determine the points $x \in \mathbb{R}^3$ such that the inequality

$$sx \leqslant H(s, M),\ s \in S \tag{13}$$

holds. The support function of the set M obviously satisfies the relation

$$H(\varphi(t), M) = \begin{cases} +\infty, & t \neq \pm\pi \\ 0, & t = \pm\pi. \end{cases}$$

Therefore, the inequality (13) is equivalent to the equality $x^3 = 0$ so that the set M is not S-convex.

Let us demonstrate how the affine and weakly S-convex hulls are connected.

proposition 11. *The equality*

$$\operatorname{aff}\operatorname{conv}_S M = \operatorname{aff}\operatorname{conv}_S \operatorname{aff} M$$

holds for a convex set $M \subset \mathbb{E}$.

Proof. The left-hand side of the relation obviously lies in the right-hand one.

In order to prove the reverse injection, it suffices to demonstrate that

$$\operatorname{aff}\operatorname{conv}_S M \supset \operatorname{conv}_S \operatorname{aff} M,$$

and the condition $0 \in \operatorname{ri} M$ can be assumed to hold true.

Let the point x belong to the right-hand side of the latter formula:

$$sx \leqslant \max_{1 \leqslant j \leqslant k} sx_j,\ s \in S,$$

for some points $x_1, \dots, x_j \in \operatorname{aff} M$.

The set M is the neighborhood of zero in a linear topological space aff M. Therefore, there is a number $t > 0$ such that $tx_j \in M,\ j = 1, \dots, k$ and hence, $tx \in \operatorname{conv}_S M$.

The statement is proved.

Let us provide a result for a particular case of spherical sets.

Let us term the set S as a spherically convex one if the cone

$$\{ts : t \geqslant 0,\ s \in S\}$$

is convex.

proposition 12. *Let the set $M \subset \mathbb{E}$ be convex, the set $S \subset \mathbb{S}$ be spherically convex, and $V = \operatorname{conv}_S \{0\}$.*

Then,

$$\operatorname{conv}_S M = M + V.$$

Proof. There are vectors $x_1, \dots,\ x_k \in M$ with the condition

$$sx_0 \leqslant \max_{1 \leqslant j \leqslant k} sx_j,\ \ s \in S$$

for the point $x_0 \in \operatorname{conv}_S M$. Suppose that $K = \operatorname{conv}\{x_1, \dots, x_k\}\,,\ C = \operatorname{conv} S$. The spherical convexity of the set S entails the inequality

$$0 \leqslant \max_{x \in K} s(x - x_0),\ \ s \in C,$$

and applying the minimax theorem (see [4], p. 404), one obtains

$$0 \leqslant \max_{x \in K} \min_{s \in C} s(x - x_0).$$

Thus, there is a point $y \in K$, for which $0 \leqslant s(y - x_0),\ \ s \in C$. Hence, $x_0 - y \in V$, so that the injection $x_0 \in M + V$ holds true.

Conversely, let the vector $x_0 \in M + V$. Then, obviously, $x_0 \in \operatorname{conv}_S \{x_1\}$ for a certain point $x_1 \in M$, and the statement is proved.

Corollary *Let us assume that the conditions of Proposition 12 are satisfied.*

The set M is weakly S-convex if and only if the relation

$$M + V = M$$

holds.

The statement 12 admits the conversion and the following result are necessary to prove this.

Lemma 3. *The equality*

$$\mathbb{S} \setminus \operatorname{int} \operatorname{conv}_S \mathbb{B} = S \tag{14}$$

holds for the set S and a sphere $\mathbb{B}$.

Proof. Let the point x belong to the left-hand side of the formula (14). According to Corollary 1 of Lemma 1 it means that the inequality $sx \geqslant 1$ holds for some vector $s \in S$. But since the points s and x belong to the unit sphere, the equality $x = s$ holds. Therefore, $x \in S$.

Conversely, assume that $x \in S$. Obviously, the relation $sx < 1,\ s \in S$ does not hold. Hence, the point x belongs to the left-hand side of the equality (14).

The lemma is proved.

Corollary. *The solid sphere $\mathbb{B}$ is weakly S-convex if and only if the equality $S = \mathbb{S}$ holds.*

proposition 13. *Let the equality*

$$\mathrm{conv}_S \mathbb{B} = \mathbb{B} + V,$$

where $V = \mathrm{conv}_S \{0\}$, *hold for the set* S.
Then, the set S *is spherically convex.*

Proof. Denote by S_1 the set

$$\mathbb{S} \cap \mathrm{conv} \{ts : t \geqslant 0,\ s \in S\}.$$

Manifestly, the set is closed, spherically convex, and the equality $\mathrm{conv}_{S_1} \{0\} = V$ holds. Therefore, Proposition 12 entails the equality $\mathrm{conv}_{S_1} \mathbb{B} = \mathbb{B} + V$. In this case, the equality 14 entails the equality $S = S_1$, which completes the proof.

4. Exponential integral

In this section we demonstrate that the convergence sets for exponential integrals are closely connected to weakly S-convex sets.

Let Λ be an arbitrary closed subset of the set $\mathbb{E}$. The set of points $x \in \mathbb{E}$, where the integral

$$\int_\Lambda e^{\lambda x}\, d\mu(\lambda) \tag{15}$$

is determined, will be investigated for the complex Radon measure μ on the set Λ.

As it is known, the integral $\int_\Lambda f(\lambda)\, d\mu(\lambda)$ of a continuous function f is determined if and only if the integral $\int_\Lambda |f(\lambda)|\, d|\mu|(\lambda)$ is defined.

A significant example of the integral (15) is the series

$$\sum_{n=1}^{\infty} a_n e^{\lambda_n x},\ x \in \mathbb{E},\ a_n \in \mathbb{C},\ \lambda_n \in \Lambda,\ n \in \mathbb{N}, \tag{16}$$

for which the condition

$$\forall R \in \mathbb{R} \sum_{|\lambda_n| \leqslant R} |a_n| < \infty$$

is satisfied.

Note, that integrals of the form (15) take part in the description of solutions to partial differential equations with constant coefficients.

proposition 14. *The set* $M \subset \mathbb{E}$ *of existence of the integral* (15) *is convex, and the integral itself converges uniformly on compacts of the set* $\mathrm{ri}\, M$ *to a continuous function.*

Proof. A function of the form

$$f_n(x) = \int_{\{\lambda \in \Lambda : |\lambda| \leqslant n\}} e^{\lambda x}\, d\mu(\lambda),\ n \in \mathbb{N},$$

is represented by the Taylor series convergent everywhere:

$$f_n(x) = \sum_{k=0}^{\infty} \left(\int_{\{\lambda \in \Lambda : |\lambda| \leqslant n\}} \lambda^k\, d\mu(\lambda) \right) \frac{x^k}{k!}.$$

Therefore, it is a real analytic function. Like in Theorem 3.1.1 of the monograph [2], it is demonstrated that the integral

$$\int_\Lambda e^{\lambda x}\, d|\mu|(\lambda)$$

converges on some convex set $M \subset \mathbb{E}$ and, evidently, it defines the function convex on it. In this case, the function is continuous on the set $\operatorname{ri} M$ (see [4]), and by the Dini theorem

$$\lim_{n\to\infty} \int_{\{\lambda\in\Lambda:|\lambda|\leqslant n\}} e^{\lambda x}\, d|\mu|(\lambda) = \int_\Lambda e^{\lambda x}\, d|\mu|(\lambda)$$

uniformly on any compact of the set $\operatorname{ri} M$.

Corollary *Let the integral*

$$\int_\Lambda e^{\langle\lambda,z\rangle}\, d\mu(\lambda)$$

in the space $\mathbb{H}$ converge on the set M, and the set $\operatorname{ri} M$ have a complex structure. Then, the integral is holomorphic on this set.

Let us term the nonempty set $M \subset \mathbb{E}$ as the set of Λ-integrability, if there is a positive Radon measure μ on the set Λ and the set of points $x \in \mathbb{E}$ of existence of the integral (15) coincides with the set M. As it has been demonstrated above, the set of Λ-integrability is convex.

In what follows, μ is considered to be the positive Radon measure on the set Λ. A simple, but important result occurs.

Lemma 4. *Let us assume that the condition*

$$\lambda \in \Lambda,\ |\lambda| > R \Rightarrow \frac{\lambda}{|\lambda|} \in S \tag{17}$$

is satisfied for the set $S \subset \mathbb{S}$ and the number $R \geqslant 0$.

Then, the following inequalities hold for the points $x, x_1, \dots, x_k \in \mathbb{E}$, $x \in \operatorname{conv}_S \{x_1, \dots, x_k\}$:

$$f(x) \leqslant e^{R|x-x_1|} f(x_1) + \sum_{j=2}^k f(x_j),$$

where

$$f(x) = \int_\Lambda e^{\lambda x}\, d\mu(\lambda). \tag{18}$$

Proof. Evidently, the inequality

$$\lambda x \leqslant \max_{1\leqslant j\leqslant k} \lambda x_j$$

holds for coefficients with the condition $|\lambda| > R$. Whence,

$$e^{\lambda x} \leqslant \sum_{j=1}^k e^{\lambda x_j}$$

for such coefficients.

Let us estimate the function f at the point x:

$$\begin{aligned} f(x) &= \int_{\{\lambda\in\Lambda:|\lambda|\leqslant R\}} e^{\lambda x}\, d\mu(\lambda) + \int_{\{\lambda\in\Lambda:|\lambda|> R\}} e^{\lambda x}\, d\mu(\lambda) \leqslant \\ &\leqslant \int_{\{\lambda\in\Lambda:|\lambda|\leqslant R\}} e^{\lambda(x-x_1)} e^{\lambda x_1}\, d\mu(\lambda) + \sum_{j=2}^k \int_{\{\lambda\in\Lambda:|\lambda|> R\}} e^{\lambda x}\, d\mu(\lambda) \leqslant \\ &\leqslant e^{R|x-x_1|} f(x_1) + \sum_{j=2}^k f(x_j). \end{aligned}$$

Let us introduce a set of limiting directions $P(\Lambda)$ for the set Λ as a set of points $s \in \mathbb{S}$, for which there is a sequence of elements $\{\lambda_n \in \Lambda,\ n \in \mathbb{N}\}$, such that

$$\lim_{n\to\infty} \frac{\lambda_n}{|\lambda_n|} = s,\ \lim_{n\to\infty} |\lambda_n| = \infty.$$

This set is obviously closed.

Let us demonstrate that the condition of the previous lemma holds for swelling of the set of limiting directions of the set Λ.

proposition 15. *There is a number $R = R(\varepsilon) \geqslant 0$ for any number $\varepsilon > 0$ such that the set $S = P(\Lambda)_\varepsilon$ satisfies the relation* (17)*.*

Proof. Let us assume that such R does not exist for a certain number $\varepsilon > 0$. Hence, there is a sequence $\{\lambda_n : n \in \mathbb{N}\}$ of elements Λ with the property

$$|\lambda_n| \geqslant n,\ \frac{\lambda_n}{|\lambda_n|} \notin S. \tag{19}$$

One can choose a sequence $\{\lambda_{n_k} : k \in \mathbb{N}\}$ from this sequence such that

$$\lim_{k\to\infty} \frac{\lambda_{n_k}}{|\lambda_{n_k}|} = s$$

for a certain point $s \in \mathbb{S}$.

By virtue of the relation (19), the point s, does not belong to the set $P(\Lambda)$, which contradicts the definition of the set. This completes the proof.

proposition 16. *Let the integral* (15) *be defined on the set $M \subset \mathbb{E}$ and S be equal to $P(\Lambda)$.*

Then, the integral is determined on the set $\operatorname{int}\operatorname{conv}_S M$ *as well, and there are a number $c > 0$ and points $x_1 \dots, x_k \in M$, depending only on the set Λ for any its compact K such that the inequality*

$$\max_{x\in K} f(x) \leqslant c \max\{f(x_1), \dots, f(x_k)\}$$

holds for the function (18)*.*

Proof. Denote by D the set of existence points of the integral (15). It follows from Proposition 15, Lemma 4, and Corollary 1 of Lemma 1 that the set $\operatorname{int} D$ is weakly S-convex.

As it has been demonstrated, the set $\overline{D}$ is weakly S-convex as well. Therefore, $\operatorname{conv}_S M \subset \overline{D}$ and hence, $\operatorname{int}\operatorname{conv}_S M \subset \operatorname{int} D$, and the statement follows from the above references.

Let us demonstrate that such result is untrue for the relative interior.

Example 5 Let

$$\mathbb{E} = \mathbb{R}^2,\ \lambda_{2n-1} = (n, n^2), \lambda_{2n} = (n, -n^2), n \in \mathbb{N},\ \Lambda = \{\lambda_n\}.$$

The set M of convergence of the series

$$\sum_{n=1}^{\infty} e^{\lambda_n x},\ x \in \mathbb{R}^2,$$

obviously coincides with the set $\{(x^1, 0) : x^1 < 0\}$, and the set S of limiting directions of the sequence Λ equals $\{(0, 1), (0, -1)\}$.

Manifestly, $\operatorname{conv}_S M = \{(x^1, 0) : x^1 \in \mathbb{R}\}$ so that

$$\operatorname{ri}\operatorname{conv}_S M = \operatorname{conv}_S M \neq M.$$

Let us show that one can limit the consideration to measures on subspaces to investigate incomplete-dimensional sets of Λ-integrability.

proposition 17. *Let the constant functions be integrable with respect to the measure μ and L be a subspace of the space $\mathbb{E}$.*

Then, there is a positive Radon measure μ_1 on the set $\Lambda_1 = \overline{\Pi_L(\Lambda)}$ such that the function $\exp\lambda_1 x,\ \lambda_1 \in \Lambda_1$ *is μ_1-integrable, and the equality*

$$\int_\Lambda e^{\lambda x}\, d\mu(\lambda) = \int_{\Lambda_1} e^{\lambda_1 x}\, d\mu_1(\lambda_1) \tag{20}$$

holds for the points $x \in L$, where the function $\exp\lambda x,\ \lambda \in \Lambda$ *is μ-integrable.*

Proof. Let us determine the functional F on the set $C_0(\Lambda_1)$ of continuous functions on the set Λ_1 with a compact carrier according to the formula

$$\langle F, f\rangle = \int_\Lambda f(\Pi_L(\lambda))\, d\mu(\lambda).$$

The inequality

$$\langle F, f\rangle \leqslant \max_{\Lambda_1} |f(\lambda)| \int d\mu(\lambda)$$

leads to the conclusion that the functional provides the positive Radon measure μ_1 on the set Λ_1. The equality (20) follows from the elementary properties of an integral.

The following construction, correlating the closed subsets of a unit sphere to linear subspaces of the space $\mathbb{E}$, will be of use to describe properties of incomplete-dimensional sets of Λ-integrability.

Let $L \subset \mathbb{E}$ be a linear space. Let us introduce the following set of objects by induction.

Suppose that

$$S_0 = \varnothing,\ L_1 = \mathbb{E},\ \Lambda_1 = \Lambda,\ \widetilde{S}_j = P(\Lambda_j),\ S_j = S_{j-1} \cup \widetilde{S}_j,$$
$$L_{j+1} = \operatorname{aff}\operatorname{conv}_{S_j} L,\ \Lambda_{j+1} = \overline{\Pi_{L_{j+1}}(\Lambda)},\ j \in \mathbb{N}.$$

Manifestly, the sets S_j are closed subsets of a unit sphere, and L_j are linear spaces, decreasing in injection, $j \in \mathbb{N}$. Therefore, one obtains the equality $L_m = L_{m+1}$ for a certain number $m \in \mathbb{N}$.

The set S_m will be correlated to the equality L. Denote the set by $T(L, \Lambda)$.

Note, that the sets $\widetilde{S}_2, \ldots, \widetilde{S_{m-1}}$ are nonempty. Otherwise, the sequence would be already stabilized.

Let us term the set of objects as a (Λ, L)-chain.

The (Λ, L)-chain is said to be exact if the $\left(\Lambda, \widetilde{L}\right)$-chain for any linear subspace $\widetilde{L} \subset L,\ \widetilde{L} \neq L$ stabilizes on the linear space other than L_m.

Clearly, there is always a linear space $\widetilde{L} \subset L$ such that the $\left(\Lambda, \widetilde{L}\right)$-chain is exact and its last linear space coincides with L_m.

Theorem 1. *Let us assume that the integral* (15) *is determined on the set $M \subset \mathbb{E}$, and the linear space $L \subset \mathbb{E}$ is parallel to the space* $\operatorname{aff} M$.

Then, the integral exists on the set $D = \operatorname{ri}\operatorname{conv}_S M$, where $S = T(L, \Lambda)$.

Proof. Without restriction of generality, one can consider the set M to be convex and $0 \in \operatorname{ri} M$.

Evidently, the set D lies in the linear space L_m and by the statement 11, it is open in its topology.

As it has been demonstrated, the integral (15) can be written for the points $x \in L_m$ in the form

$$\int_{\Lambda_m} e^{\lambda x}\, d\mu_m(\lambda),\ x \in L_m,$$

for a positive Radon measure μ_m on the set Λ_m.

Thus, this is an integral in the Euclidean space L_m, determined on the set $M \subset L_m$. The statement 16 entails that the integral is defined on the set $\operatorname{ri}\operatorname{conv}_{\widetilde{S_m}} M$ as well. However, the set $\widetilde{S_m}$ is involved in the set S by construction. Hence, the existence set of the integral includes the set D.

Corollary. *Let $M \subset \mathbb{E}$ be a set of Λ-integrability, the linear space $L \subset \mathbb{E}$ be parallel to the space* $\operatorname{aff} M$, *and the set S be equal to $T(L, \Lambda)$.*
Then, the set $\operatorname{ri} M$ *is weakly S-convex.*

Indeed, the above proved theorem entails the injection

$$\operatorname{ri}\operatorname{conv}_S M \subset M,$$

and the unknown quantity follows from Corollary 2 of Proposition 5.

Let us demonstrate now that any chain of spaces can occur in the above construction.

proposition 18. *Let $\mathbb{E} = L_1 \supset \cdots \supset L_m \supset L$ be linear spaces, $L_j \neq L_{j+1},\ j = 1, \ldots, m-1$.*
Then, there is a sequence $\Lambda = \{\lambda_n \in \mathbb{E} : n \in \mathbb{N}\}$ tending to infinity such that the (Λ, L)-chain is exact and its linear spaces coincide with the sequence $\{L_1, \ldots, L_m\}$.

Proof. Let us construct an orthonormal system of vectors

$$e_1, \ldots, e_{k_1}, \ldots, e_{k_{m-1}} \in \mathbb{E},$$

for which the vectors $e_1, \ldots, e_{k_1}, \ldots, e_{k_j}$ make up the basis in the space $L_{j+1}^{\perp},\ j = 1, \ldots, m-1$, and the sequence $\{s_n \in L \cap \mathbb{S} : n \in \mathbb{N}\}$ is dense everywhere on the set $L \cap \mathbb{S}$.
Let us take the set

$$\bigcup_{j_1=1}^{k_1} \bigcup_{j_2=k_1+1}^{k_2} \cdots \bigcup_{j_{m-1}=k_{m-2}-1}^{k_{m-1}} \Big\{\pm n^m e_{j_1} \pm n^{m-1} e_{j_2} \pm \ldots \pm n^2 e_{j_{m-1}} \pm n s_n : n \in \mathbb{N}\Big\}$$

as a sequence of Λ.
As one can readily see, the set $S_1 = P(\Lambda)$ equals $\{\pm e_1, \ldots, \pm e_{k_1}\}$. Let us demonstrate that the equality

$$\operatorname{conv}_{S_1} L = L_2$$

holds.
Indeed, for any vectors $x \in \mathbb{E}$ и $x_1, \ldots, x_k \in L$, the conditions

$$sx \leqslant \max_{1 \leqslant j \leqslant k} sx_j,\ s \in S_1,$$

are obviously equivalent to the equalities $e_1 x = 0, \ldots, e_{k_1} x = 0$, which is equivalent to the injection $x \in L_2$ in its turn.
Manifestly, the orthogonal projection of the sequence Λ onto the set L_2 coincides with the closed set

$$\bigcup_{j_2=k_1+1}^{k_2} \cdots \bigcup_{j_{m-1}=k_{m-2}-1}^{k_{m-1}} \left\{\pm n^{m-1} e_{j_2} \pm \cdots \pm n^2 e_{j_{m-1}} \pm n s_n : n \in \mathbb{N}\right\},$$

so that one has the equalities $S_2 = \{\pm e_1, \ldots, \pm e_{k_1}, \ldots, \pm e_{k_2}\}$ and $\operatorname{conv}_{S_2} L = L_3$.
Finally, let us obtain

$$S_{m-1} = \left\{\pm e_1, \ldots, \pm e_{k_1}, \ldots, \pm e_{k_{m-1}}\right\},$$
$$\operatorname{conv}_{S_{m-1}} L = L_m,\ S_m = S_{m-1} \cup (L \cap \mathbb{S}).$$

Demonstrate that $\operatorname{conv}_{S_m} L = L_m$.

Evidently, the left-hand side of the relation is involved in the right-hand one. To prove the converse, let us assume that the vector $x \in L_m$. In this case, there are elements $x_1 \in L$ and $y \in L_m \cap L^{\perp}$ with the condition $x = x_1 + y$. Moreover, the following relation holds:

$$sx = sx_1, \ s \in S_m.$$

The desired injection is proved.

Let us assume now that $\widetilde{L} \subset L$ is a linear eigensubspace. The beginning of the chain of linear spaces for it coincides with such chain for the space L.

There is a vector $x \in L$ with the conditions $|x| = 1, \ x \perp \widetilde{L}$. Evidently, the relation $x \notin \mathrm{conv}_{S_m} \widetilde{L}$ holds for it. This proves the theorem.

It follows from the above that relatively open and closed sets of Λ-integrability, whose affine hull is parallel to the linear space $L \subset \mathbb{E}$, are weakly S-convex, where $S = T(L, \Lambda)$.

To prove the converse, several properties of exponential series and their coefficients will be useful.

Lemma 5. *Let us assume that the series* (16) *with non-negative coefficients converges uniformly on the set $M \subset \mathbb{E}$, and the common term of the series is unbounded on the compact $K \subset \mathbb{E}$.*

Then, for any numbers $N \in \mathbb{N}$ and $c, \ 0 < c < 1$, there are numbers $p, q \in \mathbb{N}, \ N \leqslant p \leqslant q$, such that the equalities

$$\max_{x \in M} \sum_{n=p}^{q} a_n e^{\lambda_n x} \leqslant c, \ \min_{x \in K} \max_{p \leqslant n \leqslant q} a_n e^{\lambda_n x} \geqslant c^{-1}$$

hold.

Proof. The uniform convergence of the series (16) on the set M entails existence of the number $p \in \mathbb{N}, \ p \geqslant N$ such that for any number $m \in \mathbb{N}, \ m \geqslant p$ the inequality

$$\max_{x \in M} \sum_{n=p}^{m} a_n e^{\lambda_n x} \leqslant c$$

holds.

On the other hand, the following relation holds for any point $x \in K$:

$$a_r e^{\lambda_r x} > c^{-1}$$

for a certain number $r \in \mathbb{N}, \ r \geqslant p$. Due to continuity, the inequality holds in the vicinity of the point x. Taking into account that the set K is compact, one can readily prove that the number $q \in \mathbb{N}$ exists with the necessary property.

Lemma 6. *Let the set S of limit directions of the set Λ be nonempty.*

Then, there is a sequence $\{\lambda_n \in \Lambda : n \in \mathbb{N}\}$ such that its set of limit directions coincides with S and

$$\sum_{n=1}^{\infty} \frac{1}{|\lambda_n|} < \infty.$$

Proof. There is a sequence $\{s_n \in S : n \in \mathbb{N}\}$, with the set of terms dense everywhere on the set S, where every term is repeated infinitely many times. By condition, one can find a point $\lambda_n \in \Lambda$ for any index $n \in \mathbb{N}$ such that the conditions

$$\left| \frac{\lambda_n}{|\lambda_n|} - s_n \right| \leqslant \frac{1}{n}, \ |\lambda_n| \geqslant 2^n$$

hold.

As one can readily demonstrate, the sequence $\{\lambda_n : n \in \mathbb{N}\}$ pertains the desired property.

Let us prove the result, illustrating that Theorem 1 is exact.

proposition 19. *Let us assume that a linear space $L \subset \mathbb{E}$ is parallel to the space* aff M *for the convex compact $M \subset \mathbb{E}$, and the set S equals $T(L, \Lambda)$.*

Then, there is a series (16), uniformly converging on the set M, and its common term is unbounded outside the set $\overline{\mathrm{conv}_S M}$.

Proof. Obviously, one can assume without restriction of generality that $0 \in M$.

The sets $\widetilde{S_1}, \ldots, \widetilde{S_{m-1}}$ are nonempty in the (Λ, L)-chain. To be specific, let the set $\widetilde{S_m} \neq \varnothing$ as well.

Applying Lemma 6, one finds the sequences $\Lambda^j = \{\lambda_n^j \in \Lambda : n \in \mathbb{N}\}$ with the properties

$$P(\Lambda^j) = \widetilde{S}_j, \ \sum_{n=1}^{\infty} \frac{1}{|\Pi_{L_j}(\lambda_n^j)|} < \infty,$$

$j = 1, \ldots, m$. Generally speaking, these sets can have common terms.

Let us compose the subsequence $\Gamma = \{\gamma_n : n \in \mathbb{N}\}$ from various elements of the sequences $\Lambda^1, \ldots, \Lambda^m$.

Consider the following series:

$$\sum_{n=1}^{\infty} a_n^j e^{\lambda_n^j x}, \ a_n^j = \frac{e^{-H\left(\Pi_{L_j}(\lambda_n^j), M\right)}}{\left|\Pi_{L_j}(\lambda_n^j)\right|}, \ j = 1, \ldots, m, \ n \in \mathbb{N}. \tag{21}$$

Since $H(\Pi_{L_j}(\lambda_n^j), M) = H(\lambda_n^j, M)$ evidently, all the series converge uniformly on the set M.

Upon cancelling the terms, the sum of the series (21) is written in the form

$$\sum_{n=1}^{\infty} b_n e^{\gamma_n x}. \tag{22}$$

Let us assume that terms of the series (22) are bounded at the point $x \in \mathbb{E}$. Since terms of the series (21) are positive, they are bounded as well by a number $c > 0$ at this point:

$$a_n^j e^{\lambda_n^j x} \leqslant c, \ j = 1, \ldots, m, \ n \in \mathbb{N}.$$

For any point $s \in \widetilde{S}_1$, there is a sequence $\left\{\lambda_{n_k}^1 : k \in \mathbb{N}\right\}$, such that

$$\lim_{k \to \infty} |\lambda_{n_k}^1| = \infty, \ \lim_{k \to \infty} \frac{\lambda_{n_k}^1}{|\lambda_{n_k}^1|} = s.$$

Therefore,

$$-\ln|\lambda_{n_k}^1| - H(\lambda_{n_k}^1, M) + \lambda_{n_k}^1 x \leqslant \ln c, \ k \in \mathbb{N}.$$

Whence, one readily obtains the inequality

$$sx \leqslant H(s, M). \tag{23}$$

Proposition (9) entails the injection $x \in \overline{\mathrm{conv}_{\widetilde{S_1}} M}$ and hence, $x \in L_2$. Manifestly, in this case $\lambda_n^2 x = \Pi_{L_2}(\lambda_n^2)x, \ n \in \mathbb{N}$ and moreover, $H(\lambda_n^2, K) = H(\Pi_{L_2}(\lambda_n^2), K)$.

Repeating these considerations several times, one finally obtains the desired result.

Let us turn to the second main result of the present paper.

Theorem 2. *Let us assume that the set $M \subset \mathbb{E}$ is convex, the linear space $L \subset \mathbb{E}$ is parallel to the space* aff M*, and the set S equals to $T(L, \Lambda)$. Furthermore, let the set M be weakly S-convex and either closed or relatively open.*

Then, there is a series (16), with the sets of absolute convergence and boundedness of the common term coinciding with the set M.

Proof. As usually, one can consider that $0 \in \operatorname{ri} M$.

First, let us assume that the set M is closed.

The compacts $K_r = \{x \in M : |x| \leqslant r\}$, $r \in \mathbb{N}$, exhaust the set M, and their affine hull coincides obviously with the linear space L. On the other hand, the compacts

$$F_r = \{x \in \mathbb{E} \setminus M : |x| \leqslant r, \rho(x, M) \geqslant 1/r\}, \ r \in \mathbb{N},$$

exhaust the compliment of the set M.

Evidently, the set $M_r = \overline{\operatorname{conv}_S K_r}$ lies inside the set M. Therefore, it does not intersect with the set F_r, $r \in \mathbb{N}$. In this case, applying the statement 19 and Lemma 5 sequentially, one obtains the natural numbers $p_1, q_1, p_2, q_2, \ldots$ with the condition $p_r \leqslant q_r < p_{r+1}$, and $a_{p_r}, \ldots, a_{q_r} \geqslant 0$, $r \in \mathbb{N}$ for which the inequalities

$$\max_{x \in K_r} \sum_{n=p_r}^{q_r} a_n e^{\lambda_n x} \leqslant 2^{-r}, \ \min_{x \in F_r} \max_{p_r \leqslant n \leqslant q_r} a_n e^{\lambda_n x} \geqslant 2^r$$

hold.

Assuming that $a_n = 0$ for indices $n = 1, \ldots, p_1 - 1$ and $q_r < n < p_{r+1}$, $r \in \mathbb{N}$, one obtains the series (16) with the required properties.

Let us consider the case with respect to the open set M.

The set $\partial M = \overline{M} \setminus M$ is closed, because it is the boundary of the set M in the space L. Therefore, adding the set $\{x \in \partial M : |x| \leqslant r\}$, $r \in \mathbb{N}$ to the set F_r, one obtains the compacts that exhaust the compliment of the set M.

The compacts

$$K_r = \left\{x \in \frac{r}{r+1}\overline{M} : |x| \leqslant r\right\}, \ r \in \mathbb{N},$$

obviously exhaust the set M.

Invoking the statement 1, one can construct the required function similarly to the above.

The theorem is proved.

Corollary 1. *Let us assume that the convex set $M \subset \mathbb{E}$ is closed, or relatively open, and it is not a set of Λ-integrability.*

Then, there is a point $x \in \mathbb{E} \setminus M$, where all integrals determined on the set M of the form (15) *are defined.*

Indeed, let the linear space $L \subset \mathbb{E}$ be parallel to the space $\operatorname{aff} M$, and the set S be equal to $T(L, \Lambda)$.

According to Theorem 1, any integral of the form (15), determined on the set M, is determined on the set $\operatorname{ri}\operatorname{conv}_S M$. If the statement is not true, the injection $\operatorname{ri}\operatorname{conv}_S M \subset M$ should hold. In this case, Corollary 2 of Proposition 5 entails that the set $\operatorname{ri} M$, and hence $\overline{M}$ as well, is weakly Λ-convex. This leads to a contradiction and the theorem is proved.

Corollary 2. *Let M be a subset of $\mathbb{E}$ and $S = P(\Lambda)$.*

In order for any set of Λ-integrability, containing the set M, to have a nonempty interior, it is necessary and sufficient that the condition $\operatorname{int}\operatorname{conv}_S M \neq \varnothing$ *hold.*

Necessity. Let the linear space $L \subset \mathbb{E}$ be parallel to the space $\operatorname{aff} M$, and the set $\widetilde{S}$ be equal to $T(L, \Lambda)$. Applying the corollary of Theorem 1 and Theorem 2, one obtains that the set $D = \overline{\operatorname{conv}_{\widetilde{S}}} M$ is a set of Λ-integrability, obviously containing the set M. In this case, the set D is full-dimensional, and one can readily deduce from the definition of the set $T(L, \Lambda)$ that in this case, the equality $T(L, \Lambda) = S$ holds. By the known properties of convex sets, the set under consideration is nonempty.

Sufficiency follows from Proposition 16.

Let us prove the strengthened Theorem 2 for relatively open convex sets of a complex space.

Let $\mathbb{H}$ be a finite-dimensional Hilbert space with the scalar product $zw,\ z, w \in \mathbb{H}$. As it has been mentioned, the space can be considered as a Euclidean space with the scalar product $\operatorname{Re} zw,\ z, w \in \mathbb{H}$. Therefore, all the above results hold for it.

Let Λ be a closed subset of the space $\mathbb{H}$. Consider series of the form

$$\sum_{n=1}^{\infty} a_n e^{\lambda_n z}, \tag{24}$$

where $z,\ \lambda_n \in \Lambda,\ a_n \in \mathbb{C},\ n \in \mathbb{N}$.

Theorem 3. *Let the set $U \subset \mathbb{H}$ be convex and relatively open, the real linear space $L \subset \mathbb{H}$ be parallel to the space* $\operatorname{aff} U$ *and the set S be equal to $T(L, \Lambda)$. Furthermore, let the set U be weakly S-convex.*

Then, there is a series (24) *with nonnegative coefficients whose sets of absolute convergence and boundedness of the common term coincide with the set U, and the sum of the series $f(z)$ is unbounded at points of the relative boundary ∂U of the set U:*

$$\varlimsup_{z \to z_0, z \in U} |f(z)| = \infty,\ z_0 \in \partial U.$$

Proof. One can consider, that $0 \in \operatorname{ri} U$.

Let $K_n,\ n \in \mathbb{N}$ be a sequence of compacts of the set U, the relative interior of which exhausts the set, and the set of points

$$\{s_n \in \mathbb{S} : n \in \mathbb{N}\}$$

be dense everywhere on the set $\{z/|z| : z \in \partial U\}$.

Suppose that $V_1 = K_1,\ M_1 = \overline{\operatorname{conv}_S V_1}$. The closed set M_1 lies inside the set U as stated above. Therefore, there is a number $t_1 > 0$, such that the point $z_1 = t_1 s_1$ belongs to the set $U \setminus M_1$. Let us take the first set of $K_j,\ j \in \mathbb{N}$, containing the point z_1, as the set V_2.

Likewise, find the sequence of compacts V_j and numbers $t_j > 0$ with the property

$$z_j \in V_{j+1} \setminus \overline{\operatorname{conv}_S V_j},$$

where $z_j = t_j s_j,\ j \in \mathbb{N}$. Manifestly, these compacts exhaust the set U.

In the previous theorem, we constructed a series with nonnegative coefficients

$$\sum_{n=1}^{\infty} \alpha_n e^{\lambda_n z},\ \lambda_n \in \Lambda, \tag{25}$$

with sets of absolute convergence and boundedness of the common term coinciding with the set U. Let us denote the sum of the series by $h(z)$.

Applying the statement 19 and Lemma 5, one sequentially finds the numbers $\beta_j > 0$ and points $\mu_j \in \Lambda$, satisfying the inequalities

$$\max_{z \in V_j} |\beta_j e^{\mu_j z}| \leqslant 2^{-j},\ |\beta_j e^{\mu_j z_j}| \geqslant 2 + 2^j + \left| h(z_j) + \sum_{k=1}^{j-1} \beta_k e^{\mu_k z_j} \right|,$$

$j \in \mathbb{N}$.

The series

$$\sum_{n=1}^{\infty} \beta_n e^{\mu_n z}, \tag{26}$$

converges absolutely on the set U. Let us demonstrate that the series obtained as a sum of the latter series and the series (25) satisfies the required conditions upon cancelling the terms.

Indeed, coefficients of the series are nonnegative. Therefore, the common term of the resulting series is unbounded outside the set U, and the inequalities $|f(z_j)| \geqslant 2^j,\ j \in \mathbb{N}$ hold for its sum $f(z)$.

Let z_0 be an arbitrary point of the set ∂U and $s = z_0/|z_0|$. In this case, there is a sequence of points $\{s_{n_k} : k \in \mathbb{N}\}$ converging to the point s and it remains only to prove that the relation

$$\lim_{k\to\infty} t_{n_k} s_{n_k} = z_0$$

holds.

Suppose this is not true. Then, there is a number $\varepsilon > 0$ and a subsequence $\{u_p : p \in \mathbb{N}\}$ of the sequence $\{t_{n_k} s_{n_k} : k \in \mathbb{N}\}$, such that either the inequalities $|u_p| \geqslant |z_0| + \varepsilon,\ p \in \mathbb{N}$, or the inequalities $|u_p| \leqslant |z_0| - \varepsilon,\ p \in \mathbb{N}$ hold.

Let us consider the first case. The points

$$\frac{u_p}{|u_p|}\left(|z_0| + \varepsilon\right),\ p \in \mathbb{N},$$

obviously, belong to the set U and converge to the point $z_0 + \varepsilon s$. Thus, the point z_0 lies on the interval $(0, z_0 + \varepsilon s)$, and the point $z_0 + \varepsilon s$ lies inside the set $\overline{U}$, which contradicts the relation $z_0 \notin U$ (see [7], p. 9).

In the second case, one can consider that the sequence $\{|u_p| : p \in \mathbb{N}\}$ converges to a number t_0 and therefore, the sequence $\{u_p : p \in \mathbb{N}\}$ converges to the point $t_0 s$. This point belongs to the set U as stated above and hence, the set $\{u_p : p \in \mathbb{N}\} \cup \{t_0 s\}$ lies compactly in U. However, any compact of the set U falls within a certain compact $V_j,\ j \in \mathbb{N}$, containing only a finite number of points of the sequence $\{u_p : p \in \mathbb{N}\}$.

The resulting contradiction proves the theorem.

5. Applications

The given section contains some properties of sets of Λ-integrability.

proposition 20. *The following conditions are equivalent :*

1. *The set $\{0\}$ is not a set of Λ-integrability.*
2. *The set Λ lies inside a half-space of the space $\mathbb{E}$.*
3. *Any set of Λ-integrability is unbounded.*

Proof. $1 \Rightarrow 2$. The corollary of Theorem 2 entails that there is a vector $x_0 \in \mathbb{E},\ x_0 \neq 0$ such that any integral of the form (15), defined at zero, is defined at the point x_0 as well. Let us prove, that there is a number $c \in \mathbb{R}$ with the property

$$\lambda x_0 \leqslant c,\ \lambda \in \Lambda. \tag{27}$$

Indeed, otherwise, there is a sequence

$$\{\lambda_n \in \Lambda : n \in \mathbb{N}\},$$

such that $\lambda_n x_0 \geqslant n^2,\ n \in \mathbb{N}$. The series

$$\sum_{n=1}^{\infty} \frac{e^{\lambda_n x}}{\lambda_n x_0}$$

converges at zero but, evidently, diverges at the point x_0, which proves the desired implication.

$2 \Rightarrow 3$. By condition, there is a vector $x_0 \in \mathbb{E},\ x_0 \neq 0$ and a number $c \in \mathbb{R}$ such that the inequality (27) holds for them. If an integral of the form (15) is determined at a point $x_1 \in \mathbb{E}$, then the inequality

$$\int_\Lambda e^{\lambda(x_1 + t x_0)}\, d\mu(\lambda) \leqslant e^{tc} \int_\Lambda e^{\lambda x_1}\, d\mu(\lambda),\ t \geqslant 0,$$

provides the validity of the second implication.

The last implication is manifest.

proposition 21. *Any set of Λ-integrability has a nonempty interior if and only if the set Λ lies inside a convex acute cone.*

Proof. Let

$$S = P(\Lambda),\ K_\varepsilon = \mathrm{conv} S_\varepsilon,\ U_\varepsilon = \{tx : t \geqslant 0,\ x \in K_\varepsilon\},\ \varepsilon > 0.$$

Manifestly, the set U_ε is a closed convex cone with the vertex at zero.

Suppose that all sets of Λ-integrability have a nonempty interior. Then, one can readily deduce the inequality $\mathrm{int}\,\mathrm{conv}_S\{0\} \neq \varnothing$ form Corollary 2 of Theorem 2.

Let us prove existence of the number $\varepsilon > 0$, for which the cone U_ε is accute.

Indeed, otherwise, for any number $\varepsilon > 0$ there are various points $\alpha_\varepsilon,\ \beta_\varepsilon \in \mathbb{E}$, such that the straight line $\{t\alpha_\varepsilon + (1-t)\beta_\varepsilon : t \in \mathbb{R}\}$ lies inside the cone U_ε. In this case, as one can readily demonstrate, the vectors $\pm s_\varepsilon$, where $s_\varepsilon = (\alpha_\varepsilon - \beta_\varepsilon)/|\alpha_\varepsilon - \beta_\varepsilon|$, belong to the cone U_ε.

There are a sequence $\{\varepsilon_n\}$ tending to zero and a vector $s_0 \in \mathbb{S}$ such that $s_{\varepsilon_n} \to s_0$. Manifestly, $\pm s_0 \in U_0$.

One can readily demonstrate that the inequalities

$$sx \leqslant 0,\ s \in V_0$$

hold for the points x of the set $\mathrm{conv}_S\{0\}$. Therefore, $s_0 x = 0$, which contradicts the nonempty interior of the set.

Thus, there is a number $\varepsilon > 0$ with the required property. According to the statement 15, there is a number $R \geqslant 0$ such that the set Λ is involved in the set $\{x \in \mathbb{E} : |x| < R\} \cup U_\varepsilon$. However, the cone U_ε has a nonempty interior and if $x_0 \in \mathrm{int}\, U_\varepsilon$, then for some number $t \geqslant 0$ the cone $-tx_0 + U_\varepsilon$ contains a solid sphere $\{x \in \mathbb{E} : |x| < R\}$ and the cone U_ε. The first half of the proposition is proved.

Conversely, let us assume that the injection $\Lambda \subset \alpha + V$ holds for a convex acute closed cone $V \subset \mathbb{E}$ with the vertex at zero and a vector $\alpha \in \mathbb{E}$. In this case, the set S, obviously lies in the cone V.

Manifestly, it is sufficient to prove that the set $\mathrm{conv}_S\{0\}$ has a nonempty interior.

Suppose this is not true. Then, obviously, the latter set lies in a hyperplane passing through the origin of coordinates, i.e. there is a vector $\alpha \in \mathbb{S}$ such that the implication

$$sx \leqslant 0,\ s \in S \Rightarrow \alpha x = 0$$

holds. Using the Khan-Banach theorem, one can readily deduce the injection $\pm\alpha \in V$. Therefore, the cone V is not acute.

The statement is proved.

proposition 22. *Let $L \subset \mathbb{E}$ be a linear subspace, $\{0\} \neq L \neq \mathbb{E}$.*

In order for any set of Λ-integrability, containing the vicinity of zero of the space L, to contain the vicinity of zero for the space $\mathbb{E}$, it is necessary and sufficient that the relations $0 \notin \Pi_L(P(\Lambda))$ *hold.*

Proof. Necessity. Suppose that $M = \mathbb{B} \cap L,\ S = T(L, \Lambda)$. According to Theorem 1, the set $\overline{\mathrm{conv}_S M}$ is a set of Λ-integrability. Hence, it contains the solid sphere $\delta\mathbb{B}$ for a certain number $\delta > 0$. On the basis of definition of the set $T(L,\Lambda)$, one concludes that the equality $T(L,\Lambda) = P(\Lambda)$ holds.

Manifestly, the relations

$$H(s, M) = H(\Pi_L(s), M) = |\Pi_L(s)|,\ s \in \mathbb{S}$$

hold. Therefore, Proposition 9 provides

$$sx \leqslant |\Pi_L(s)|,\ |x| \leqslant \delta,\ s \in S,$$

and, with the assumption that $x = \delta s$, the first half of the proposition is proved.

Since projection of a compact is a compact, the sufficiency is readily deduced from Proposition 6.

Corollary *Let us assume that F is an irreducible polynomial of n variables over a field of complex numbers, and its main part does not vanish on a real sphere.*

Then, any function $u \in C^\infty(\{x \in \mathbb{R}^n : |x| < 1\})$, satisfying the equation

$$F\left(\frac{d}{dx}\right) u = 0,$$

is real analytic.

If we assume that $\Lambda = \{\lambda \in \mathbb{C}^n : F(\lambda) = 0\}$, the set $P(\Lambda)$ coincides with the set of zeroes of the main part of the polynomial F on a complex unit sphere and the desired result is readily deducible from results of the paper [8].

Let us consider now the Taylor type series in a finite-dimensional Hilbert space $\mathbb{H}$.

proposition 23. *Let us assume for the vectors $\alpha_1, \dots, \alpha_l \in \mathbb{H}$ that*

$$V = \{z \in \mathbb{H} : \operatorname{Re} \alpha_j z \leqslant 0,\ j = 1, \dots, l\},$$

and denote by Λ the set

$$\left\{\sum_{j=1}^{l} n_j \alpha_j : n_j \in \mathbb{N}_0,\ j = 1, \dots, l\right\}$$

numbered somehow.

Then, series of the form (24) *have the following properties.*

1. *The series* (24)*, converging absolutely at the points $z_1, \dots, z_k \in \mathbb{H}$, converges absolutely and uniformly on the set* $\operatorname{conv}\{z_1, \dots, z_k\} + V$.
2. *Any set of absolute Λ-integrability $M \subset \mathbb{H}$ satisfies the equality $M + V = M$.*
3. *For a convex closed set $M \subset \mathbb{H}$, $M + V = M$, there is a series of the from* (24)*, with thesets of absolute convergence and boundedness of the common term coinciding with the set M.*
4. *For a convex relatively open set $D \subset \mathbb{H}$, $D + V = D$, there is a series of the form* (24)*, absolutely converging on the set U, whose sum is unbounded at every point of the relative boundary of the set.*

Proof. Let us designate by S the set

$$\mathbb{S} \cap \left\{\sum_{j=1}^{l} t_j \alpha_j : t_j \geqslant 0,\ j = 1, \dots, l\right\}.$$

Manifestly, This is a closed set satisfying the injection $P(\Lambda) \subset S$. Let us demonstrate that the reverse injection holds true as well.

Indeed, let the vector $s = \sum_{j=1}^{l} t_j \alpha_j,\ t_j \geqslant 0$ have a unit norm and, to be specific, $t_1 > 0$. There are sequences of natural number $\left\{m_j^n : n \in \mathbb{N}\right\}$ $j = 1, \dots, l$, satisfying the relations

$$\lim_{n\to\infty} m_1^n = \infty,\ \lim_{n\to\infty} \frac{m_j^n}{m_1^n} = \frac{t_j}{t_1},\ j = 2, \dots, l.$$

In this case,

$$\lim_{n\to\infty} \frac{\sum_{j=1}^{l} m_j^n \alpha_j}{\left|\sum_{j=1}^{l} m_j^n \alpha_j\right|} = s,$$

and the equality

$$\lim_{n\to\infty} \frac{\left|\sum_{j=1}^{l} m_j^n \alpha_j\right|}{m_1^n} = \frac{1}{t_1}$$

provides that

$$\lim_{n\to\infty}\left|\sum_{j=1}^{l} m_j^n \alpha_j\right| = \infty,$$

and hence, $s \in P(\Lambda)$.

Note that the set S is spherically convex and the equality $\mathrm{conv}_S\,\{0\} = V$ holds. Therefore, Proposition 12 entails the relation $\mathrm{conv}_S M = M + V$ for any convex set $M \subset \mathbb{H}$.

The injections

$$\frac{\lambda_n}{|\lambda_n|} \in S,\ n \in \mathbb{N}$$

are obvious and the first item follows from Lemma 4 and thus, entails the second item in its turn.

Let us assume now that the condition $M + V = M$ is satisfied for the convex set $M \subset \mathbb{H}$, and the linear space $L \subset \mathbb{H}$ is parallel to the space $\mathrm{aff}\, M$. Since $T(L,\Lambda) \supset P(\Lambda) = S$, the relation $\mathrm{conv}_{T(L,\Lambda)} M \subset \mathrm{conv}_S M = M$ holds. Therefore, obviously, $\mathrm{conv}_{T(L,\Lambda)} M = M$ and the remaining items follow from Theorems 2 and 3.

The statement is proved.

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