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NONISOMORPHIC LIE ALGEBRAS ADMITTED BY GAS DYNAMICS MODELS

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Abstract. Group classification of gasdynamic equations with respect to the state equation consists of 13 types of finite-dimensional Lie algebras of different dimensions, from 11 to 14. Some types depend on a parameter. Five pairs of Lie algebras appear to be equivalent. The equivalent transformations for Lie algebras contain the equivalent transformations for gasdynamic equations. The equivalence test resulted in nine nonisomorphic Lie algebras with different structures. One type has 3 different structures for different parameters. Each of these Lie algebras is represented as a semidirect sum of a six-dimensional Abeilian ideal with a subalgebra, which is decomposed into a semidirect or direct sum in its turn. The optimal systems for subalgebras are constructed. The Abeilian ideal is added in 6 cases while constructing the optimal system. There remain 3 Lie algebras of the dimensions 12, 13, 15 for which the optimal systems are not constructed.

Keywords: gas dynamics, Lie algebra, equivalent transformation, optimal system

Equations of gas dynamics are as follows

$$\rho D\vec{u} + \nabla p = 0, \quad D\rho + \rho\nabla\cdot\vec{u} = 0, \quad Dp + \rho f_\rho \nabla\cdot\vec{u} = 0, \quad DS = 0 \tag{1}$$

with the state equation in the general form $p = f(\rho, S)$, $D = \partial_t + \vec{u}\cdot\nabla$, $\vec{u}$, p, ρ, S are velocity, pressure, density, entropy.

Equivalence transformations of the system (1) leave the system (1) unaltered, but change the state equation only:

$$p' = g(\rho, p, S), \quad \rho' = h(\rho, p, S), \quad S' = k(\rho, p, S), \quad p' = f'(\rho', S'). \tag{2}$$

Statement 1. *Equivalence transformations of the system* (1) *have the form*

$$p' = ap + b, \quad \rho' = a\rho, \quad S' = K(S), \quad f'(a\rho, K(S)) = af(\rho, S) + b, \tag{3}$$

where a, b are arbitrary constants, $K(S)$ is an arbitrary function.

Proof. Let us substitute the expressions (2) into the system (1) with the variables p', ρ', S'. By virtue of (1), one obtains the equalities

$$g_\rho = f_\rho\left(h\rho^{-1} - g_p\right), \quad g_S = f_S\left(h\rho^{-1} - g_p\right),$$

$$k_\rho + f_\rho k_p = 0, \quad h_\rho + f_\rho h_p = \rho^{-1}h, \quad g = f'(h, k).$$

Solution of the equations has the form

$$k = K(I, S), \quad g = pG_1(I) + G_2(I), \quad h = \rho G_1(I), \quad I = p - f(\rho, S),$$

$$f'(\rho G_1(I), K(I, S)) = (I + f(\rho, S))\, G_1(I) + G_2(I).$$

Assuming that $I = 0$, $a = G_1(0)$, $b = G_2(0)$, $K(0, S) = K(S)$, one obtains the formulae (3).

The system (1) with an arbitrary state equation admits an 11-dimensional Lie algebra L_{11}, with the basis given by the following operators in the Cartesian system of coordinates: $X_i = \partial_{x^i}$, $X_{3+i} =$

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$t\partial_{x^i} + \partial u^i$, $X_{6+i} = \varepsilon_{ij}^k \left(x^j_{x^k} + u^j\partial_{u^k}\right)$, $i = 1, 2, 3$, $X_{10} = \partial_t$, $X_{11} = t\partial_t + x^j\partial_{x^j}$. The algebra L_{11} is expanded for special state equations [1]. Extensions with the accuracy up to the transformations (3) are represented in the table, where $Y_0 = t\partial_t - u^i\partial_{u^i}$, $Y_{\varphi(p)} = \rho\varphi'(p)\partial_\rho + \varphi(p)\partial_p$, φ, f are arbitrary functions, γ, $\gamma_1 = 2\gamma(\gamma-1)^{-1}$ are parameters, k is the dimension of the algebra.

№	$p = f(\rho, S)$	k	Auxiliary operators
1	$f(\rho, S)$	11	—
2	$\rho^\gamma f(S\rho)$	12	$Y_0 - (\gamma_1 - 2)\rho\partial_\rho - \gamma_1 p\partial_p$, $\gamma_1 \neq 0, 2$
3	$\rho f(S\rho)$	12	Y_p
4	$f(S\rho)$	12	$Y_0 + 2\rho\partial_\rho = X_{12}$
5	$Sf(\rho)$	12	$Y_0 + 2p\partial_p$
6	$S\rho^\gamma$	13	Y_p, $Y_0 + 2\rho\partial_\rho$
7	$S\rho^{5/3}$	14	Y_p, $Y_0 + 2\rho\partial_\rho$, $x^iX_{3+i} + t(Y_0 - 3\rho\partial_\rho - 5p\partial_p) = Z$
8	$\ln\rho + f(\rho S)$	12	$Y_0 + 2\rho\partial_\rho + 2\partial_p$
9	$f(\rho) + S$	12	Y_1
10	$\rho^\gamma + S$	13	Y_1, $Y_0 - (\gamma_1 - 2)\rho\partial_\rho - \gamma_1 p\partial_p$, $\gamma \neq 0, \pm 1$
11	$\rho + S$	13	Y_1, Y_p
12	$\ln\rho + S$	13	Y_1, $Y_0 + 2\rho\partial_\rho$
13	S	∞	$Y_{\varphi(p)}$, $Y_0 + 2\rho\partial_\rho$

In order to construct different submodels of the system (3), it is necessary to enumerate nonsimilar subalgebras of the Lie algebras from the table of expansions (optimal system). Subalgebras of isomorphic Lie algebras are isomorphic. Therefore, first let us determine nonisomorphic algebras from the tale of expansions.

Statement 2. [2]. *Finite-dimensional subalgebras of the algebra $L_\infty = \{X_{\varphi(p)}\}$ are similar to the following $\{Y_1\}$, $\{Y_1, Y_p\}$, $\{Y_1, Y_p, Y_{p^2}\}$.*

Statement 3. *Table of nonisomorphic finite-dimensional Lie algebras is as follows Table 1.*

$f(\rho, S)$	L_{11}
$f(S\rho)$	$\{Y_0 + 2\rho\partial_\rho\}\dot{\oplus}L_{11}$
$f(\rho) + S$	$\{Y_1\} \oplus L_{11}$
$S\rho^\gamma$	$\{Y_p, Y_0 + 2\rho\partial_\rho\}\dot{\oplus}L_{11}$
$\rho^\gamma + S$	$\{Y_1, Y_0 - (\gamma_1 - 2)\rho\partial_\rho - \gamma_1 p\partial_p\}\dot{\oplus}L_{11} = M_{\gamma_1}$, $\gamma_1 = 0, \pm 1$
$\rho^{-1} + S$	$\{Y_1, Y_0 + \rho\partial_\rho - p\partial_p\}\dot{\oplus}L_{11} = M_1$
$\rho^{1/3} + S$	$\{Y_1, Y_0 + 3\rho\partial_\rho + p\partial_p\}\dot{\oplus}L_{11} = M_{-1}$
$\rho + S$	$\{Y_1, Y_p\} \oplus L_{11}$
$S\rho^{5/3}$	$\{X_{10}, X_{11}, Y_p, Y_0 + 2\rho\partial_\rho, Z\}\dot{\oplus}L_9 = L_{14}$
S	$\{Y_1, Y_p, Y_{p^2}\} \oplus L_{12}$

Here L_9, L_{11} are ideals, $\dot{\oplus}$ is a semi-direct sum, $\oplus$ is a direct sum of ideals.

Proof. The Lie algebra $N = 3'$ with the auxiliary operator $Y_{p'} = \rho'\partial_{\rho'} + p'\partial_{p'}$ for the state equation $p' = \rho' f'(S'\rho')$ is equivalent to the Lie algebra $N = 9$ with the auxiliary operator $Y_1 = \partial_p$ for the state equation with the pressure $p = f(\rho) + S$ decomposed into the sum. This can be easily verified by means of the substitution $\rho = \dfrac{p'}{\rho'}$, $p = \ln p'$, $S = -\ln S'$, $f(f'(\tau)) = \ln(\tau f'(\tau))$. The substitution is consistent with the state equation, but it changes the second equation of the system (1):

$$D\ln\rho = D\ln f'(\tau) = \tau f'_\tau \left(f'(\tau)\right)^{-1} D\ln\rho', \quad \tau = S'\rho'.$$

The Lie algebra $N = 2'$ with the auxiliary operator $Y_0 - (\gamma_1 - 2)\rho'\partial_{\rho'} - \gamma_1 p'\partial_{p'}$ for the state equation $p' = \rho'^\gamma f'(S'\rho')$ is equivalent to the Lie algebra $N = 4$ with the auxiliary operator $Y_0 + 2\rho\partial_\rho$ for the state equation with the density $p = f(S\rho)$ decomposed into a product. This can be verified by making the substitution $\rho = \rho'^{\frac{1-\gamma}{2}}$, $p = p'\rho'^{-\gamma}$, $S = S'^{\frac{1-\gamma}{2}}$, $f'(\tau) = f\left(\tau^{\frac{1-\gamma}{2}}\right)$. The substitution is consistent with the state equation, but changes the second equation of the system (1) $D\ln\rho = \frac{1-\gamma}{2}D\ln\rho'$.

The Lie algebra $N = 5'$ with the auxiliary operator $Y_0 + 2p'\partial p'$ for the state equation $p' = S'f'(\rho')$ is equivalent to the Lie algebra $N = 4$ with the auxiliary operator $Y_0 + 2\rho\partial_\rho$ for the state equation with the separated density $p = f(S\rho)$. The substitution $p = \rho'$, $\rho = p'$, $S = S'^{-1}$, $f'(f(\tau)) = \tau$ is consistent with the state equations, but it changes the second equation of the system (1) $D\ln\rho = \tau f'_\tau(f(\tau))^{-1} D\ln\rho'$, $\tau = \rho'$.

The Lie algebra $N = 8'$ with the auxiliary operator $Y_0 + 2\rho'\partial_{\rho'} + 2\partial_{p'}$ for the state equation $p' = \ln\rho' + f'(\rho' S')$ is equivalent to the Lie algebra $N = 4$. The equivalence transformations are as follows: $p = p' - \ln\rho'$, $\rho = \rho'$, $S = S'$, $f(\tau) = f'(\tau)$. The substitution is consistent with the state equations, but it alters the first equation of the system $\rho^{-1}\bigtriangledown p = \rho'^{-1}\bigtriangledown p' - \rho'^{-2}\bigtriangledown \rho'$.

The Lie algebra $N = 12'$ with the auxiliary operators $Y_1 = \partial_{p'}$, $Y_0 + 2\rho'\partial_{\rho'}$ for the state equation $p' = \ln\rho' + S'$ is equivalent to the Lie algebra $N = 6$ with the auxiliary operators $Y_p = \rho\partial_\rho + p\partial_p$, $Y_0 + 2\rho\partial_\rho$ for the state equation of a polytropic gas $p = S\rho^\gamma$. The equivalence transformations have the form $p = e^{p'}$, $\rho = \rho' e^{p'}$, $S = e^{(1-\gamma)S'}$. The substitution is consistent with the state equations only if $\gamma = \frac{1}{2}$ and it alters only the second equation of the system (1) $D\ln\rho = 2D\ln\rho'$.

Other pairs of Lie algebras from the table of expansions are nonequivalent and nonisomorphic, as one can see from the expansions of these algebras into semidirect and direct sums.

Systems of subalgebras for the algebras M_{γ_1}, $\gamma_1 \neq \pm 1$, M_1, M_{-1} differ from each other [3].

In order to construct optimal systems of nonisomorphic finite-dimensional Lie algebras it is convenient to expand them into semidirect sums, where one of the addends is the Abel ideal $J_6 = \{X_1, \dots, X_6\}$ $(J_3 = \{X_7, X_8, X_9\})$:

$$L_{11} = (J_3 \oplus \{X_{10}, X_{11}\}) \dot{\oplus} J_6 \quad [1],$$

$$\{X_{12}\}\dot{\oplus}L_{11} = \left(J_3 \oplus \left(\{X_{10}\}\dot{\oplus}\{X_{11}, X_{12}\}\right)\right)\dot{\oplus}J_6 \quad [4],$$

$$\{Y_1\} \oplus L_{11} = \left(J_3 \oplus \left(\{X_{10}, X_{11}\} \oplus \{Y_1\}\right)\right)\dot{\oplus}J_6,$$

$$\{Y_p, X_{12}\}\dot{\oplus}L_{11} = \left(\left(J_3 \oplus \left(\{X_{10}\}\dot{\oplus}\{X_{11}, X_{12}\}\right)\right) \oplus \{Y_p\}\right)\dot{\oplus}J_6 \quad [5],$$

$$M_{\gamma_1} = \left(\left(J_3 \oplus \{Y_1, Y_0 - (\gamma_1 - 2)\rho\partial_\rho - \gamma_1 p\partial_p\}\right)\dot{\oplus}\{X_{10}, X_{11}\}\right)\dot{\oplus}J_6 \quad [3],$$

$$\{Y_1, Y_p\} \oplus L_{11} = \left(J_3 \oplus \{X_{10}, X_{11}\} \oplus \{Y_1, Y_p\}\right)\dot{\oplus}J_6,$$

$$L_{14} = \left(J_3 \oplus \{X_{10}, X_{11}, X_{12}, Z\} \oplus \{Y_p\}\right)\dot{\oplus}J_6 \quad [6],$$

$$\{Y_1, Y_p, Y_{p^2}\} \oplus L_{12} = \left(J_3 \oplus \left(\{X_{10}\}\dot{\oplus}\{X_{11}, X_{12}\}\right) \oplus \{Y_1, Y_p, Y_{p^2}\}\right)\dot{\oplus}J_6.$$

Optimal systems are constructed for the Lie algebras with references. Optimal systems of subalgebras that are complements to the Abel ideal J_6 are presented in [2].

Thus, it remains to finish the construction of optimal systems for three expansions that are not found in scientific literature.

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