# STABILITY OF BASIS PROPERTY OF A TYPE OF PROBLEMS ON EIGENVALUES WITH NONLOCAL PERTURBATION OF BOUNDARY CONDITIONS 

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#### Abstract

The article is devoted to a spectral problem for a multiple differentiation operator with an integral perturbation of boundary conditions of one type which are regular, but not strongly regular. The unperturbed problem has an asymptotically simple spectrum, and its system of normalized eigenfunctions creates the Riesz basis. We construct the characteristic determinant of the spectral problem with an integral perturbation of the boundary conditions. The perturbed problem can have any finite number of multiple eigenvalues. Therefore, its root subspaces consist of its eigen and (maybe) adjoint functions. It is shown that the Riesz basis property of a system of eigen and adjoint functions is stable with respect to integral perturbations of the boundary condition.


Keywords: Riesz basis, regular boundary conditions, eigenvalues, root functions, spectral problem, integral perturbation of boundary condition, characteristic determinant

## 1. Problem statement

It is well known that a system of eigenfunctions of the operator, given by a formally self-adjoint differential expression with arbitrary self-adjoint boundary conditions providing a discrete spectrum, form an orthonormal basis of the space $L_{2}$. The problem of preserving the basis properties with a (weak, in a sense ) perturbation of the initial operator was investigated in many works. For example, the similar question for the case of a self-adjoint, and nonself-adjoint initial operator was investigated in $[1-3]$, and in $[4-6]$, respectively.

The present paper is devoted to the spectral problem

$$
\begin{gather*}
l(u) \equiv-u^{\prime \prime}(x)=\lambda u(x), 0<x<1  \tag{1}\\
U_{1}(u) \equiv u^{\prime}(0)-u^{\prime}(1)+\alpha u(0)=0  \tag{2}\\
U_{2}(u) \equiv u(0)-u(1)=\int_{0}^{1} \overline{p(x)} u(x) d x, p(x) \in L_{2}(0,1), \tag{3}
\end{gather*}
$$

which is close to investigations [3, 5].
Here $\alpha \neq 0$ is an arbitrary complex number. In [3], the stability of basis properties of a periodic problem (the case $\alpha=0$ ) for Equation (1) was studied with an integral perturbation of the boundary condition. It was proved that the set $P$ of functions $p(x)$, that provide the problem by the basis property of eigenfunctions, is dense in $L_{1}(0,1)$, the set $L_{1}(0,1) \backslash P$ is also dense in $L_{1}(0,1)$.

The problem on basis property of the root functions of the operator with more general integral boundary conditions is solved positively in [5], where the Riesz basis property with brackets is proved under the Birkhoff regularity condition [7, p. 66-67] of the boundary conditions of the unperturbed problem; and the Riesz basis property is proved under the auxiliary assumption of strengthened regularity. In our case, the unperturbed boundary conditions (2), (3) (when

[^0]$p(x) \equiv 0$ ) are regular, but not strongly regular boundary conditions. Therefore, in this case, the results [5] are not applicable to investigate the Riesz basis and further investigation is necessary.

It follows from [5] that the system of eigen and adjoint functions of the problem (1)-(3) is complete and minimal in $L_{2}(0,1)$. In the present paper, a characteristic determinant for the spectral problem (1)-(3) is constructed. On the basis of the resulting formula, stability of the Riesz basis property for the system of eigen and adjoint functions of the problem with integral perturbation of the boundary condition is proved.

## 2. UnPERTURBED PROBLEM

In this section $p(x) \equiv 0$. As it follows from [8], for any $\alpha \neq 0$ the unperturbed problem

$$
\begin{equation*}
l(u)=\lambda u, U_{1}(u)=0, U_{2}(u)=0 \tag{4}
\end{equation*}
$$

has an asymptotically simple spectrum, and the system of its normalized eigenfunctions generates the Riesz basis in $L_{2}(0,1)$. One can readily verify by straightforward calculation that the spectral problem (4) has two series of eigenfunctions

$$
\lambda_{1 k}^{0}=(2 \pi k)^{2}, k=1,2, \ldots ; \lambda_{2 k}^{0}=(2 \omega k)^{2}, k=0,1,2, \ldots,
$$

to which almost normalized eigenfunctions

$$
\begin{equation*}
u_{1 k}^{0}(x)=\sqrt{2} \sin 2 \pi k x, u_{2 k}^{0}(x)=\sqrt{2} \cos \left(2 \omega_{k} x\right)-\frac{\sqrt{2 \alpha}}{4 \omega k} \sin \left(2 \omega_{k} x\right) \tag{5}
\end{equation*}
$$

correspond. Here $\omega_{k}=\pi k+\varepsilon_{k}$ are roots of the equation $\operatorname{tg} \omega=-\alpha / 4 \omega$. One of important properties of the problem is the asymptotic separateness of eigenvalues. As it follows from [9], there is an estimate

$$
\begin{equation*}
\frac{C_{1}}{k}<\left|\sqrt{\lambda_{1 k}^{0}}-\sqrt{\lambda_{2 k}^{0}}\right|<\frac{C_{2}}{k}, \tag{6}
\end{equation*}
$$

homogeneous in $k$. The system is biorthogonal to the system (5), and is determined by eigenfunctions of the problem conjugate to (4):

$$
l^{*}(v)=\bar{\lambda} v, v^{\prime}(0)-v^{\prime}(1)+\bar{\alpha} v(0)=0, v(0)-v(1)=0 .
$$

The eigenfunctions have the form

$$
\begin{equation*}
v_{1 k}^{0}(x)=\sqrt{2} \sin 2 \pi k x, v_{2 k}^{0}(x)=\beta_{k}\left\{\sqrt{2} \cos \left(2 \overline{\omega_{k}} x\right)-\frac{\sqrt{2 \bar{\alpha}}}{4 \overline{\omega_{k}}} \sin \left(2 \overline{\omega_{k}} x\right)\right\} \tag{7}
\end{equation*}
$$

The coefficient $\beta_{k}$ is determined from the biorthogonality relation $\left(u_{2 k}^{0}, v_{2 k}^{0}\right)=1$ and has the asymptotics $\beta_{k}=1+O\left(k^{-1}\right)$. The systems (5) and (7) form the Riesz basis in $L_{2}(0,1)$ and are mutually biorthogonal.

## 3. Characteristic determinant of the problem (1)-(3)

Satisfying the conditions (2), (3) by the general solution $u(x, \lambda)=C_{1} \cos \sqrt{\lambda} x+C_{2} \sin \sqrt{\lambda} x$ of Equation (1), one obtains a linear system with respect to the coefficients $C_{k}$ :

$$
\left\{\begin{array}{l}
C_{1}[\alpha+\sqrt{\lambda} \sin \sqrt{\lambda}]+C_{2}[\sqrt{\lambda}-\sqrt{\lambda} \cos \sqrt{\lambda}]=0  \tag{8}\\
C_{1}\left[1-\cos \sqrt{\lambda}-\int_{0}^{1} \overline{p(x)} \cos \sqrt{\lambda} x d x\right]+C_{2}\left[-\sin \sqrt{\lambda}-\int_{0}^{1} \overline{p(x)} \sin \sqrt{\lambda} x d x\right]=0
\end{array}\right.
$$

Its determinant is a characteristic determinant of the problem (1)-(3):

$$
\Delta_{1}(\lambda)=\left|\begin{array}{rr}
\alpha+\sqrt{\lambda} \sin \sqrt{\lambda} & 1-\cos \sqrt{\lambda}-\int_{0}^{1} \overline{p(x)} \cos \sqrt{\lambda} x d x  \tag{9}\\
\sqrt{\lambda}(1-\cos \sqrt{\lambda}) & -\sin \sqrt{\lambda}-\int_{0}^{1} \frac{1}{p(x)} \sin \sqrt{\lambda} x d x
\end{array}\right| .
$$

One can readily see that the characteristic determinant of the unperturbed problem (1)-(3) is derived when $p(x) \equiv 0$. Let us denote it by $\Delta_{0}(\lambda)=-2 \sqrt{\lambda}(1-\cos \sqrt{\lambda})-\alpha \sin \sqrt{\lambda}$.

Let us represent the function $p(x)$ in the form of the Fourier series in terms of the Riesz basis (7):

$$
\begin{equation*}
p(x)=\sum_{k=1}^{\infty} a_{1 k} v_{1 k}^{0}(x)+\sum_{k=0}^{\infty} a_{2 k} v_{2 k}^{0}(x) . \tag{10}
\end{equation*}
$$

Using (10), find a more convenient representation of the determinant $\Delta_{1}(\lambda)$. To this end, first, calculate integrals involved in (9). Easy calculations show that

$$
\begin{gathered}
\left(\lambda-\lambda_{j k}^{0}\right) \int_{0}^{1} \overline{v_{j k}^{0}(x)} \cos \sqrt{\lambda} x d x=\overline{v_{j k}^{0}(0)}[\alpha \cos \sqrt{\lambda}+\sqrt{\lambda} \sin \sqrt{\lambda}]+\overline{v_{j k}^{0}(0)}[\cos \sqrt{\lambda}-1], \\
\left(\lambda-\lambda_{j k}^{0}\right) \int_{0}^{1} \overline{v_{j k}^{0}(x)} \sin \sqrt{\lambda} x d x=\overline{v_{j k}^{0}(0)}[\sqrt{\lambda}+\alpha \sin \sqrt{\lambda}-\sqrt{\lambda} \cos \sqrt{\lambda}]+\overline{v_{j k}^{0}(0)}[\sin \sqrt{\lambda}], \\
j=1,2 .
\end{gathered}
$$

Therefore, invoking (7), one obtains

$$
\begin{gathered}
\int_{0}^{1} \overline{p(x)} \cos \sqrt{\lambda} x d x=A(\lambda)[\cos \sqrt{\lambda}-1]+B(\lambda)[\alpha+\sqrt{\lambda} \sin \sqrt{\lambda}] \\
\int_{0}^{1} \overline{p(x)} \sin \sqrt{\lambda} x d x=A(\lambda)[\sin \sqrt{\lambda}]+B(\lambda) \sqrt{\lambda}[1-\cos \sqrt{\lambda}]
\end{gathered}
$$

where the following notation is introduced:

$$
A(\lambda)=2 \sqrt{2} \pi \sum_{k=1}^{\infty} \frac{k \overline{a_{1 k}}}{\lambda-\lambda_{1 k}^{0}}, B(\lambda)=\sqrt{2} \sum_{k=0}^{\infty} \frac{\overline{a_{2 k} \beta_{k}}}{\lambda-\lambda_{2 k}^{0}}
$$

Using the result, one reduces the determinant (9) to the form

$$
\begin{equation*}
\Delta_{1}(\lambda)=\Delta_{0}(\lambda)[1+A(\lambda)] \equiv \Delta_{0}(\lambda)\left[1+2 \sqrt{2} \pi \sum_{k=1}^{\infty} \frac{k \overline{a_{1 k}}}{\lambda-\lambda_{1 k}^{0}}\right] \tag{11}
\end{equation*}
$$

by means of standard transformations.
Let us formulate the obtained result in the following Lemma.
Lemma 1. Characteristic determinant of the problem (1)-(3) with perturbed boundary conditions can be represented in the form (11), where $\Delta_{0}(\lambda)$ is a characteristic determinant of the unperturbed problem, and $a_{1 k}$ are coefficients of expansion (10) of the function $p(x)$ into a biorthogonal series in terms of the Riesz basis (7).

The function $A(\lambda)$ in representation (11) has poles at the points $\lambda=\lambda_{1 k}^{0}$ of the first order. However, the function $\Delta_{0}(\lambda)$ has zeroes of the first order at the same points. Therefore, the function $\Delta_{1}(\lambda)$, represented according to the formula (11), is an entire analytical function of the variable $\lambda$.

## 4. Eigenvalues of the problem (1)-(3)

Analyzing the formula (11), one can readily see that $\Delta_{1}\left(\lambda_{2 k}^{0}\right)=0$ for all $k \geq 0$. It means that all eigenvalues of the second series $\lambda_{2 k}^{0}$ of the unperturbed problem are eigenvalues of the perturbed problem (1)-(3): $\lambda_{2 k}^{1}=\lambda_{2 k}^{0}$. Moreover, one can easily observe that eigenvalues of the first series $\lambda_{1 j}^{0}$ of the unperturbed problem are eigenvalues of the perturbed problem (1)-(3), provided that the Fourier coefficient corresponding to the number $j$ in the expansion (10) equals to zero: $a_{1 j}=0$.

The case of a simple form of the characteristic determinant (11) takes place when $p(x)$ is represented in the form (10) with the finite first sum. That is when there is a number $N$ such that $a_{1 k}=0$ for all $k>N$. In this case, the formula (11) takes the form

$$
\Delta_{1}(\lambda)=\Delta_{0}(\lambda)\left(1+2 \sqrt{2} \pi \sum_{k=0}^{N} \frac{k \overline{a_{1 k}}}{\lambda-\lambda_{1 k}^{0}}\right) .
$$

On the basis of this particular case of the formula (11), one can readily prove the following lemma.

Lemma 2. For any prescribed numbers, a complex number $\widehat{\lambda}$ and a natural one $\widehat{m}$, there is always a function $p(x)$ such that $\hat{\lambda}$ is an eigenvalue of the problem (1)-(3) of the multiplicity $\widehat{m}$.

The perturbed problem can have any finite number of multiple eigenvalues by virtue of the latter lemma. Therefore, its root subspaces consist of eigen and (maybe) adjoint functions.

Let us find the asymptotics of eigenvalues of the first series in the general case. Standard reasoning, connected to the application of the Rouche theorem, leads to the conclusion that for numbers $j$ large enough, zeroes of the first series of the characteristic determinant (11) have the form $\sqrt{\lambda_{1 j}^{1}}=2 \pi j+\delta_{j},\left|\delta_{j}\right|<1$. Let us specify the asymptotics $\delta_{j}$. One can readily see that

$$
1+2 \sqrt{2} \frac{\pi j}{4 \pi j+\delta_{j}} \frac{\overline{a_{1 j}}}{\delta_{j}}+\sqrt{2} \sum_{k=0, k \neq j}^{\infty} \frac{2 \pi k}{\left[2 \pi k+2 \pi j+\delta_{j}\right]} \frac{\overline{a_{1 k}}}{\left[2 \pi j-2 \pi k+\delta_{j}\right]}=0
$$

whence the following asymptotics is obtained:

$$
\begin{equation*}
\left|\delta_{j}\right|=(\sqrt{2})^{-1}\left|a_{1 j}\right| \cdot|1+o(1)| \tag{12}
\end{equation*}
$$

Since the Fourier coefficients with respect to the Riesz basis belong to the space of sequences $l_{2}$, then it follows that the numerical sequence $\left\{\delta_{j}\right\} \in l_{2}$. This yields the unknown asymptotics of eigenvalues of the first series. Thus, the theorem is proved.

Lemma 3. Eigenvalues of the problem (1)-(3) form two series: $\lambda_{1 k}^{1}=\left(2 \pi k+\delta_{k}\right)^{2}, \lambda_{2 k}^{1}=\lambda_{2 k}^{0}$, where $\lambda_{2 k}^{0}$ are eigenvalues of the second series of the unperturbed problem, and $\left\{\delta_{k}\right\} \in l_{2}$. Meanwhile, $\delta_{j}=0$ when $a_{1 j}=0$. The values of the coefficients $a_{2 k}$ do not influence eigenvalues of the problem (1)-(3).

For smoother functions $p(x)$, the asymptotics of the first series of eigenvalues can be specified.
Lemma 4. If $p(x) \in W_{2}^{1}(0,1)$, then eigenvalues of the first series of the problem (1)-(3) have the asymptotics $\lambda_{1 k}^{1}=\left(2 \pi k+\delta_{k}\right)^{2}$, where $\left\{k \delta_{k}\right\} \in l_{2}$. Eigenvalues of the problem (1)-(3) are asymptotically simple and separated.

Proof. Straightforward calculation readily verifies that the system of functions

$$
\begin{gathered}
v_{1 k}^{0}(x)(2 \pi k)^{-1}=\sqrt{2} \cos 2 \pi k x, k=1,2, \ldots \\
v_{2 k}^{0}{ }^{\prime}(x)\left(2 \bar{\omega}_{k}\right)^{-1}=\beta_{k}\left\{-\sqrt{2} \sin \left(2 \bar{\omega}_{k} x\right)-\frac{\sqrt{2} \bar{\alpha}}{4 \bar{\omega}_{k}} \cos \left(2 \bar{\omega}_{k} x\right)\right\}, k=0,1,2, \ldots
\end{gathered}
$$

is quadratically close to the trigonometric system $\{1, \sqrt{2} \cos 2 \pi k x,-\sqrt{2} \sin (2 \pi k x)\}$ and hence, forms the Riesz basis in $L_{2}(0,1)$. Therefore, one has $\left\{k a_{1 k}\right\} \in l_{2}$ for $p(x) \in W_{2}^{1}(0,1)$. The asymptotics (12) provides the first part of the lemma. The second part of the lemma is obtained by comparing with the asymptotics (6) of eigenvalues of the unperturbed problem. The lemma is proved

## 5. Eigenfunctions of the problem (1)-(3)

In what follows, we assume that that $p(x) \in W_{2}^{1}(0,1)$. Using the notation of Section 3, we can rewrite the system (8) in the form

$$
\left\{\begin{array}{l}
C_{1}[\alpha+\sqrt{\lambda} \sin \sqrt{\lambda}]+C_{2} \sqrt{\lambda}[1-\cos \sqrt{\lambda}]=0  \tag{13}\\
\left\{C_{1} \sqrt{\lambda}[1-\cos \sqrt{\lambda}]+C_{2}[-\sin \sqrt{\lambda}]\right\}[1+A(\lambda)]=0
\end{array}\right.
$$

When $\lambda=\lambda_{2 k}^{1}=\lambda_{2 k}^{0}$, one has $C_{2}=-\frac{\alpha}{4 \omega_{k}} C_{1}$. Therefore, eigenvalues of the second series of the perturbed and unperturbed problems (1) - (3) coincide: $u_{2 k}^{1}(x)=u_{2 k}^{0}(x)$.

When $\lambda=\lambda_{1 k}^{1}$, one has $1+A\left(\lambda_{1 k}^{1}\right)=0$. Therefore, the second equation in the system (13) turns into an identity. By virtue of Lemma 4

$$
\lim _{k \rightarrow \infty}\left[\alpha+\sqrt{\lambda_{1 k}^{1}} \sin \sqrt{\lambda_{1 k}^{1}}\right]=\lim _{k \rightarrow \infty}\left[\alpha+\left(2 \pi k+\delta_{k}\right) \sin \delta_{k}\right]=\alpha \neq 0
$$

Hence, for numbers $k$ large enough, the coefficient of $C_{1}$ in the first equation of the system is other than zero. For such numbers $k$, we find $C_{1}=-\gamma_{k} C_{2}$, where

$$
\begin{equation*}
\gamma_{k}=\sqrt{\lambda_{1 k}^{1}}\left[1-\cos \sqrt{\lambda_{1 k}^{1}}\right] /\left[\alpha+\sqrt{\lambda_{1 k}^{1}} \sin \sqrt{\lambda_{1 k}^{1}}\right] . \tag{14}
\end{equation*}
$$

Meanwhile, due to the asymptotics of $\lambda_{1 k}^{1}$ from Lemma 4, we have $\left\{k \gamma_{k}\right\} \in l_{2}$. This proves the following lemma.

Lemma 5. If $p(x) \in W_{2}^{1}(0,1)$, then for numbers $k$ large enough, eigenfunctions of the problem (1)-(3) construct two series:

$$
\begin{align*}
& u_{1 k}^{0}(x)=\sqrt{2} \sin 2 \pi k x-\gamma_{k} \sqrt{2} \cos 2 \pi k x \\
& u_{2 k}^{0}(x)=\sqrt{2} \cos \left(2 \omega_{k} x\right)-\frac{\sqrt{2} \alpha}{4 \omega_{k}} \sin \left(2 \omega_{k} x\right) \tag{15}
\end{align*}
$$

where the sequence $\gamma_{k}$ is determined by (14) and $\left\{k \gamma_{k}\right\} \in l_{2}$.

## 6. Main Result

As it follows from Lemma 2, the problem (1)-(3) can have multiple eigenvalues, but according to Lemma 4 they can be of a finite number only. Therefore, the root subspaces of the problem (1)-(3) consist of eigen and (maybe) of a finite number of adjoint functions.

The main result of the work is the following.
Theorem. For any function $p(x) \in W_{2}^{1}(0,1)$, the system of eigen and adjoint functions of the problem (1)-(3) forms the Riesz basis $L_{2}(0,1)$.

Proof. Due to [5], the system of eigen and adjoint functions of the problem (1)-(3) forms the Riesz basis with brackets. The affiliation $\left\{k \gamma_{k}\right\} \in l_{2}$ allows us to justify easily that the system of eigenfunctions (15) is quadratically close to the system (5) of eigenfunctions of the unperturbed problem that forms the Riesz basis in $L_{2}(0,1)$. Hence, the system of eigen and adjoint functions of the problem (1)-(3) forms the Riesz basis in $L_{2}(0,1)$ as well. The theorem is proved.

The results of the present paper, unlike [3], demonstrate the stability of basis properties of the root functions under integral perturbation of the boundary conditions of one type of problems that are regular, but not strongly regular.

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