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ABOUT THE CAMASSA-HOLM EQUATION WITH A SELF-CONSISTENT SOURCE

I.I. BALTAEVA, G.U. URAZBOEV

Abstract. The paper is devoted to solving the Camassa-Holm equation with a self-consistent source of a special type by the inverse scattering method. The main result consists in determining the evolution of the scattering data for the spectral problem associated with the Camassa-Holm equation with a self-consistent source of a special type. In contrast to the classical Camassa-Holm equation, the eigenvalues of the spectral problem are moving in the problem under consideration. The resulting equalities determine the evolution of the scattering data completely; this fact allows us to apply the inverse scattering method for solving the considered problem.

Keywords: the Camassa-Holm equation, inverse scattering problem, scattering data, Lax pair, eigenvalue, eigenfunction.

1. Introduction

The system of equations

$$\begin{cases} u_t - u_{xxt} + 2\omega u_x + 3uu_x - 2u_x u_{xx} - uu_{xxx} = \\ \quad = \sum\limits_{k=1}^{N} (m_x g_k f_k + 2(m+\omega)(g_k f_k)'_x), \\ g_{kxx} = \left(\frac{1}{4} + \lambda_k(m+\omega)\right) g_k, \\ f_{kxx} = \left(\frac{1}{4} + \lambda_k(m+\omega)\right) f_k, \quad k = 1, 2, ..., N, \ x \in R, \end{cases} \tag{1}$$

where $u = u(x,t)$, $m = u - u_{xx}$, $\omega = const \in R$, is considered in the present paper.

Let us assume that the function $u = u(x,t)$ is sufficiently smooth and is tending to its limits rapidly enough when $x \longrightarrow \pm\infty$, and

$$\int\limits_{-\infty}^{\infty} (1+|x|) \left(|u(x,t)| + \sum_{k=1}^{3} \left| \frac{\partial^k u(x,t)}{\partial x^k} \right| \right) dx < \infty. \tag{2}$$

In the problem under consideration, $g_k = g_k(x,t)$ is an eigenfunction of the equation $y_{xx} = (\frac{1}{4} + \lambda(m+\omega))y$, and corresponds to the eigenvalue λ_k, and $f_k = f_k(x,t)$ is a solution to the equation $f_{kxx} = (\frac{1}{4} + \lambda_k(m+\omega))f_k$ linearly independent of g_k, and

$$W\{g_k, f_k\} \equiv g_k f'_{kx} - g'_{kx} f_k = \omega_k(t), \quad k = 1, 2, ..., N, \tag{3}$$

where $\omega_k(t)$ are given functions of t.

Representations for solutions $u(x,t)$, $g_k(x,t)$, $f_k(x,t)$, $k = 1, ..., N$ of the problem (1) are obtained in the present paper by means of the inverse problem method for the equation $y_{xx} = (\frac{1}{4} + \lambda(m+\omega)y)$.

Note that the Camassa-Holm equation without a source was most completely solved by the inverse scattering method in [1]. The works [2-3] illustrate that the Korteweg-de Vries (KdV) equation with a self-consistent source can be solved by means of the inverse scattering method

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for the Sturm-Liouville operator. The sine-Gordon equation is solved with a self-consistent source corresponding to moving eigenvalues of the Dirac operator in [4].

2. Scattering problem

Consider the equation

$$\psi_{xx} = \left(\frac{1}{4} + \lambda(m+\omega)\right)\psi, \tag{4}$$

where $m = u - u_{xx}$, $\lambda(k) = -\frac{1}{\omega}(k^2 + \frac{1}{4})$, with the function $u(x)$, satisfying the condition

$$\int\limits_{-\infty}^{\infty} ((1+|x|)(|u(x)| + |u_{xx}(x)|)dx < \infty. \tag{5}$$

The present section contains information on the direct and inverse scattering problem for the problem (4-5) necessary for our further exposition. Provided that the condition (5) is met, Equation (4) possesses the Jost solutions with the following asymptotics:

$$\begin{aligned} \psi_1(x,k) &= e^{-ikx} + o(1), \quad x \longrightarrow +\infty, \\ \psi_2(x,k) &= e^{ikx} + o(1), \quad x \longrightarrow +\infty, \end{aligned} \tag{6}$$

$$\begin{aligned} \varphi_1(x,k) &= e^{-ikx} + o(1), \quad x \longrightarrow -\infty, \\ \varphi_2(x,k) &= e^{ikx} + o(1), \quad x \longrightarrow -\infty. \end{aligned} \tag{7}$$

When k are real, the pairs $\{\varphi_1, \varphi_2\}$ and $\{\psi_1, \psi_2\}$ are pairs of linearly independent solutions for Equation (4). Therefore,

$$\varphi_1(x,k) = a(k)\psi_1(x,k) + b(k)\psi_2(x,k). \tag{8}$$

One can readily see that

$$a(k) = -\frac{1}{2ik} W\{\psi_2(x,k), \varphi_1(x,k)\}.$$

The function $a(k)$ admits an analytic continuation into the upper half-plane and has a finite number of zeroes $k = ik_n, k_n > 0$. Meanwhile,

$$\lambda_n = -\frac{1}{\omega}\left(-k_n^2 + \frac{1}{4}\right), \quad n = 1, 2, ..., N$$

is an eigenvalue of Equation (4) so that

$$\varphi_1(x, ik_n) = b_n \psi_2(x, ik_n), \quad n = 1, 2, ..., N. \tag{9}$$

Moreover, the following expansion on the half-plane $Imk > 0$ takes place for the coefficient $a(k)$:

$$lna(k) = -i\alpha k + \sum_{n=1}^{N} \ln \frac{k - ik_n}{k + ik_n} - \frac{1}{2i\pi} \int\limits_{-\infty}^{\infty} \frac{\ln(1 - |R(k')|^2)}{k' - k} dk',$$

where

$$\alpha = \int\limits_{-\infty}^{\infty} \left(\sqrt{1 + \frac{m(x)}{\omega}} - 1\right) dx, \quad R(k) = \frac{b(k)}{a(k}.$$

The set $\{R(k),\ k \in R,\ k_n,\ b_n,\ n = 1, 2, ..., N\}$ is called the scattering data for Equation (4). The inverse scattering problem consists in recovering the function $m(x)$, and consequently $u(x)$, of Equation (4) by the scattering data.

The inverse problem of recovering the function $u(x)$ by the scattering data is solved by means of the following equations [1]:

$$\overline{\psi_1}(x,k)=\left(\frac{\xi(x)}{\xi'(x)}\right)^{\frac{1}{2}}+\int\limits_{-\infty}^{\infty}R(k')\overline{\psi_2}(x,k')[\xi(x)]^{2ik'}\frac{dk'}{k'-k}+\sum_{n=1}^{N}\frac{b_n[\xi(x)]^{-2k_n}\overline{\psi_1}(x,-ik_n)}{\dot{a}(ik_n)(k-ik_n)},$$

$p=1,2,...,N,$

$$\overline{\psi_1}(x,-ik_p)=\left(\frac{\xi(x)}{\xi'(x)}\right)^{\frac{1}{2}}+\int\limits_{-\infty}^{\infty}R(k')\overline{\psi_2}(x,k')[\xi(x)]^{2ik'}\frac{dk'}{k'+ik_p}+i\sum_{n=1}^{N}\frac{b_n[\xi(x)]^{-2k_n}\overline{\psi_1}(x,-ik_n)}{\dot{a}(ik_n)(k_p+k_n)},$$

$$e^{-\frac{x}{2}}[\xi(x)]^{\frac{1}{2}}=\left(\frac{\xi(x)}{\xi'(x)}\right)^{\frac{1}{2}}+\int\limits_{-\infty}^{\infty}R(k')\overline{\psi_2}(x,k')[\xi(x)]^{2ik'}\frac{dk'}{k'+i/2}+i\sum_{n=1}^{N}\frac{b_n[\xi(x)]^{-2k_n}\overline{\psi_1}(x,-ik_n)}{\dot{a}(ik_n)(k_n+1/2)},$$

where

$$\xi(x)=\exp\{x+\int\limits_{\infty}^{x}\left(\sqrt{\frac{m(y)+\omega}{\omega}}-1\right)dy\},$$

$$\overline{\psi_1}(x,k)\equiv\psi_1(x,k)[\xi(x)]^{ik},$$

$$\overline{\varphi_1}(x,k)\equiv\varphi_1(x,k)\exp\{ik(x+\int\limits_{-\infty}^{x}\left(\sqrt{\frac{m(y)+\omega}{\omega}}-1\right)dy)\},$$

$$\frac{\overline{\varphi_1}(x,k)}{e^{i\alpha k}a(k)}=\overline{\psi_1}(x,k)+R(k)\overline{\psi_2}(x,k)[\xi(x)]^{2ik}.$$

Note that the functions h_n, defined by the equalities

$$h_n(x)=\frac{\frac{d}{dk}(\varphi_1-b_n\psi_2)|_{k=ik_n}}{\dot{a}(ik_n)},\quad n=1,2,...,N, \tag{10}$$

where $\varphi_{1n}=\varphi_1(x,ik_n),\psi_{2n}=\psi_2(x,ik_n),n=1,2,...,N$ are solutions to the equations $h_{nxx}=(\frac{1}{4}+\lambda_n(m+\omega))h_n$, and the asymptotics

$$\begin{aligned}&h_n\sim -b_ne^{-k_nx}\text{ when } x\longrightarrow-\infty,\\&h_n\sim e^{-k_nx}\text{ when } x\longrightarrow+\infty\end{aligned} \tag{11}$$

are true for them. According to (9), (6), (11), the equalities

$$W\{\varphi_{1n},h_n\}\equiv\varphi_{1n}h_n'-\varphi_{1n}'h_n=2k_nb_n,\quad n=1,2,...,N \tag{12}$$

hold. In what follows, we will need the following lemma.

Lemma 1. If functions f and g are solutions to equations

$$f_{xx}=\left(\frac{1}{4}+\lambda_1(m+\omega)\right)f,$$

$$g_{xx}=\left(\frac{1}{4}+\lambda_2(m+\omega)\right)g,$$

the following equality holds for them:

$$(m+\omega)fg=\frac{1}{\lambda_1-\lambda_2}\frac{d}{dx}W\{g,f\}.$$

The lemma is proved by direct verification.

Lemma 2. The following equality holds:

$$\dot{a}(ik_n) = \frac{1}{i\omega} \int\limits_{-\infty}^{\infty} (m+\omega)\varphi_{1n}\psi_{2n}dx, \tag{13}$$

where $\dot{a}(ik_n) = \frac{da(k)}{dk}|_{k=ik_n}$, $n = 1, 2, ..., N$.

Proof. Differentiating the equations

$$\varphi_{1xx}(x,k) = \left(\frac{1}{4} + \lambda(k)(m(x)+\omega)\right)\varphi_1(x,k),$$

$$\psi_{2xx}(x,k) = \left(\frac{1}{4} + \lambda(k)(m(x)+\omega)\right)\psi_2(x,k)$$

with respect to k, one obtains

$$\dot{\varphi}_{1xx}(x,k) = \left(\frac{1}{4} + \lambda(k)(m(x)+\omega)\right)\dot{\varphi}_1(x,k) + \dot{\lambda}(k)(m(x)+\omega)\varphi_1(x,k),$$

$$\dot{\psi}_{2xx}(x,k) = \left(\frac{1}{4} + \lambda(k)(m(x)+\omega)\right)\dot{\psi}_2(x,k) + \dot{\lambda}(k)(m(x)+\omega)\psi_2(x,k).$$

Then, one can readily conclude that

$$\dot{\psi}_2\varphi_{1xx} - \dot{\psi}_{2xx}\varphi_1 = -\dot{\lambda}(k)(m(x)+\omega)\varphi_1\psi_2,$$

$$\psi_2\dot{\varphi}_{1xx} - \psi_{2xx}\dot{\varphi}_1 = \dot{\lambda}(k)(m(x)+\omega)\varphi_1\psi_2.$$

These equalities provide

$$W\{\dot{\psi}_{2n}, \varphi_{1n}\} + W\{\psi_{2n}, \dot{\varphi}_{1n}\} = -\frac{2ik_n}{\omega} \int\limits_{-\infty}^{\infty} (m+\omega)\varphi_{1n}\psi_{2n}dx.$$

On the other hand, differentiating the equality

$$a(k) = -\frac{1}{2ik} W\{\psi_2(x,k), \varphi_1(x,k)\},$$

with respect to k, and substituting it instead of $k = ik_n$, one has

$$2k_n\dot{a}(ik_n) = W\{\dot{\psi}_{2n}, \varphi_{1n}\} + W\{\psi_{2n}, \dot{\varphi}_{1n}\}.$$

Hence,

$$\dot{a}(ik_n) = \frac{1}{i\omega} \int\limits_{-\infty}^{\infty} (m+\omega)\varphi_{1n}\psi_{2n}dx.$$

Lemma 2 is proved.

3. Evolution of the scattering data

Let the function $u(x,t)$ in (4) be a solution to the equation

$$u_t - u_{xxt} + 2\omega u_x + 3uu_x - 2u_x u_{xx} - uu_{xxx} = G, \tag{14}$$

where the function $G = G(x,t)$ is sufficiently smooth and $G(x,t) = o(1)$ when $x \to \pm\infty, t \geq 0$.

Lemma 3. If the function $u(x,t)$ is a solution to Equation (14) in the class of functions (2) then, the scattering data of the problem (4) with the function $u(x,t)$ depend on t as follows:

$$\frac{dR}{dt} = -\frac{4ik\omega}{4k^2+1}R - \frac{4k^2+1}{8ik\omega a^2(k)} \int\limits_{-\infty}^{\infty} G\varphi_1^2 dx, \ \ (Imk = 0),$$

$$\frac{db_n}{dt} = \frac{4\omega k_n}{1-4k_n^2} b_n + \frac{1-4k_n^2}{8\omega k_n} \int_{-\infty}^{\infty} G\varphi_{1n} h_n dx,$$

$$\frac{dk_n}{dt} = i \frac{4k_n^2 - 1}{8\omega k_n b_n \dot{a}(ik_n)} \int_{-\infty}^{\infty} G\varphi_{1n}^2 dx.$$

Proof. When k are real, we seek the Lax pair for Equation (14) in the form:

$$\varphi_{1xx} = (1 + \lambda(m+\omega))\varphi_1, \tag{15}$$

$$\varphi_{1t} = (\frac{1}{2\lambda} - u)\varphi_{1x} + \frac{u_x}{2}\varphi_1 + \gamma\varphi_1 + F(x,k,t), \tag{16}$$

where $m(x) = u - u_{xx}$,a $\varphi_1(x,k,t)$ are the Jost solutions of the equation $\varphi_{1xx} = (\frac{1}{4} + \lambda(m+\omega))\varphi_1$ with the asymptotics (7). Using the equality $\varphi_{1xxt} = \varphi_{1txx}$, on the basis of the equalities (14), (15), and (16), we obtain

$$F_{xx} - \left(\frac{1}{4} + \lambda(m+\omega)\right) F = \lambda G\varphi_1. \tag{17}$$

Let us find solution to this equation in the form

$$F(x,k,t) = A(x)\varphi_1(x,k,t) + B(x)\varphi_2(x,k,t).$$

Then, derive the system of equations

$$\begin{cases} A_x\varphi_1 + B_x\varphi_2 = 0, \\ A_x\varphi_{1x} + B_x\varphi_{2x} = \lambda G\varphi_1 \end{cases} \tag{18}$$

to determine $A(x)$ and $B(x)$. Using the asymptotics of the function $\varphi_1(x,k,t)$ and (2), let us pass to the limit in the equality (16) when $x \longrightarrow -\infty$. The passage to the limit results in

$$F(x,t) \longrightarrow 0, \quad \text{when} \quad x \longrightarrow -\infty.$$

Hence, solution of the system of equations (18) has the form:

$$A(x) = -\frac{\lambda}{2ik} \int_{-\infty}^{x} G\varphi_1\varphi_2 dx + (\frac{ik}{2\lambda} - \gamma),$$

$$B(x) = \frac{\lambda}{2ik} \int_{-\infty}^{x} G\varphi_1^2 dx.$$

In this case, the second equation of the Lax pair has the form

$$\varphi_{1t} = (\frac{1}{2\lambda} - u)\varphi_{1x} + \frac{u_x}{2}\varphi_1 + \gamma\varphi_1 + \left(-\frac{\lambda}{2ik} \int_{-\infty}^{x} G\varphi_1\varphi_2 dx + (\frac{ik}{2\lambda} - \gamma)\right)\varphi_1 + \frac{\lambda}{2ik} \int_{-\infty}^{x} G\varphi_1^2 dx \varphi_2. \tag{19}$$

Passing to the limit $x \longrightarrow \infty$ in the equality (19) by virtue of (2), (6), (8), and substituting $\gamma = \frac{ik}{2\lambda}$, one obtains

$$a_t = -\frac{\lambda}{2ik} \int_{-\infty}^{\infty} G\varphi_1\varphi_2 dx a(k,t) + \frac{\lambda}{2ik} \int_{-\infty}^{\infty} G\varphi_1^2 dx \overline{b}(k,t), \tag{20}$$

$$b_t = \frac{ik}{\lambda} b(k,t) - \frac{\lambda}{2ik} \int_{-\infty}^{\infty} G\varphi_1\varphi_2 dx b(k,t) + \frac{\lambda}{2ik} \int_{-\infty}^{\infty} G\varphi_1^2 dx \overline{a}(k,t). \tag{21}$$

Multiply (21) by a and subtract from it the quality (20) multiplied by b. Then, using the definition of the function $R(k)$ and substituting $\lambda = -\frac{1}{\omega}(k^2+\frac{1}{4})$, we obtain

$$\frac{dR}{dt} = -\frac{4ik\omega}{4k^2+1}R - \frac{4k^2+1}{8ik\omega a^2(k)}\int\limits_{-\infty}^{\infty} G\varphi_1^2 dx.$$

In the general case, eigenvalues of the equation $y_{xx} = (\frac{1}{4}+\lambda(m+\omega))y$ depend on time. Therefore, differentiating the equalities

$$\varphi_1(x, ik_n, t) = b_n(t)\psi_2(x, ik_n, t), \;\; n = 1, ..., N, \tag{22}$$

with respect to t, we obtain

$$\frac{\partial\varphi_{1n}}{\partial t} + \frac{\partial\varphi_1}{\partial k}|_{k=ik_n}\frac{d(ik_n)}{dt} = \frac{db_n}{dt}\psi_{2n} + b_n\left(\frac{\partial\psi_{2n}}{\partial t} + \frac{\partial\psi_2}{\partial k}|_{k=ik_n}\frac{d(ik_n)}{dt}\right),$$

i.e.

$$\frac{\partial\varphi_{1n}}{\partial t} = \frac{b_n}{dt}\psi_{2n} - \dot{a}(ik_n)h_n\frac{d(ik_n)}{dt} + b_n\frac{\partial\psi_{2n}}{\partial t} \tag{23}$$

according to the notation (10). Similarly to the case of the continuous spectrum, we seek the Lax pair in case of the discreet spectrum in the following form:

$$\varphi_{1nxx} = (1+\lambda_n(m+\omega))\varphi_{1n}, \tag{24}$$

$$\varphi_{1nt} = (\frac{1}{2\lambda} - u)\varphi_{1nx} + \frac{u_x}{2}\varphi_{1n} + \gamma\varphi_{1n} + F_n. \tag{25}$$

Then, we obtain the equation

$$F_{nxx} - (\frac{1}{4}+\lambda_n(m+\omega))F_n = \lambda G\varphi_{1n} \tag{26}$$

to determine $F_n(x,t)$. Let us solve (26) in the form

$$F_n(x,t) = A_n(x,t)\varphi_{1n} + B_n(x)h_n.$$

Likewise, in order to find $A_n(x,t)$ and $B_n(x,t)$, we obtain a system of equations resulting in

$$A_n(x,t) = -\left(\frac{\lambda_n}{2k_nb_n}\int\limits_{-\infty}^{x} G\varphi_{1n}h_n dx + (\frac{k_n}{2\lambda_n}+\gamma)\right),$$

$$B_n(x) = \frac{\lambda_n}{2k_nb_n}\int\limits_{-\infty}^{x} G\varphi_{1n}^2 dx$$

in case of the continuous spectrum as well. Thus, on the basis of (23), the second equation of the Lax pair in this case has the form :

$$\varphi_{1nt} = (\tfrac{1}{2\lambda_n} - u)\varphi_{1nx} + \tfrac{u_x}{2}\varphi_{1n} + \gamma\varphi_{1n} - \left(\tfrac{\lambda_n}{2k_nb_n}\int\limits_{-\infty}^{x} G\varphi_{1n}h_n dx + (\tfrac{k_n}{2\lambda_n}+\gamma)\right)\varphi_{1n} + \tfrac{\lambda_n}{2k_nb_n}\int\limits_{-\infty}^{x} G\varphi_{1n}^2 dx h_n. \tag{27}$$

Passing to the limit in this equality when $x \longrightarrow \infty$ and using the asymptotics (2), (11), (22) and (7), we obtain

$$-\frac{k_n}{2\lambda_n}b_ne^{-k_nx} - \left(\frac{\lambda_n}{2k_nb_n}\int\limits_{-\infty}^{\infty} G\varphi_{1n}h_n dx + \frac{k_n}{2\lambda_n}\right)b_ne^{-k_nx} + \frac{\lambda_n}{2k_nb_n}\int\limits_{-\infty}^{\infty} G\varphi_{1n}^2 dx e^{k_nx} =$$

$$= \frac{db_n}{dt} e^{-k_n x} - \dot{a}(ik_n) \frac{d(ik_n)}{dt} e^{k_n x}.$$

Substituting $\lambda_n = -\frac{1}{\omega}(-k_n^2 + \frac{1}{4})$ and comparing coefficients of the exponents, we have

$$\frac{db_n}{dt} = \frac{4\omega k_n}{1-4k_n^2} b_n + \frac{1-4k_n^2}{8\omega k_n} \int_{-\infty}^{\infty} G\varphi_{1n} h_n dx,$$

$$\frac{dk_n}{dt} = i \frac{4k_n^2 - 1}{8\omega k_n b_n \dot{a}(ik_n)} \int_{-\infty}^{\infty} G\varphi_{1n}^2 dx.$$

Lemma 3 is proved.

Since the function h_n is the solution to the equation $h_{nxx} = (\frac{1}{4} + \lambda_n(m+\omega))h_n$, the representation

$$h_n = \frac{\beta_n}{\dot{a}(ik_n)} \varphi_{1n} + \alpha_n f_n, \quad n = 1, 2, ..., N$$

holds for it. According to (11), we have $\alpha_n = \frac{2k_n b_n d_n}{\omega_n}$, where d_n is determined from the equality $g_n = d_n \varphi_{1n}$.

Moreover, (3) provides that

$$W\{h_n, f_n\} = \frac{\beta_n \omega_n}{\dot{a}(ik_n) d_n}, \quad n = 1, 2, ..., N. \tag{28}$$

Let us apply the result of Lemma 3 when

$$G = \sum_{k=1}^{N} (m_x g_k f_k + 2(m+\omega)(g_k f_k)'_x).$$

Using Lemma 1 for $k \neq n$, one obtains

$$\int_{-\infty}^{\infty} (2((m+\omega) f_k g_k)'_x - m_x g_k f_k) \varphi_{1n}^2 dx = \int_{-\infty}^{\infty} (((m+\omega) f_k g_k)'_x - m_x g_k f_k) \varphi_{1n}^2 dx +$$

$$+ (m+\omega) f_k g_k \varphi_{1n}^2 |_{-\infty}^{\infty} - \int_{-\infty}^{\infty} (m+\omega) f_k g_k (\varphi_{1n}^2)'_x dx = \int_{-\infty}^{\infty} (m+\omega)(f'_{kx} g_k \varphi_{1n}^2 + f_k g'_{kx} \varphi_{1n}^2 -$$

$$-2 f_k g_k \varphi_{1n} \varphi'_{1nx}) dx = \int_{-\infty}^{\infty} (m+\omega)(g_k \varphi_{1n}(f'_{kx} \varphi_{1n} - \varphi'_{1nx} f_k) + f_k \varphi_{1n}(g'_{kx} \varphi_{1n} - \varphi'_{1nx} g_k)) dx =$$

$$= \frac{1}{\lambda_k - \lambda_n} \int_{-\infty}^{\infty} \left(\frac{d}{dx} W\{\varphi_{1n}, g_k\} W\{\varphi_{1n}, f_k\} + \frac{d}{dx} W\{\varphi_{1n}, f_k\} W\{\varphi_{1n}, g_k\} \right) dx =$$

$$= \frac{1}{\lambda_k - \lambda_n} W\{\varphi_{1n}, g_k\} W\{\varphi_{1n}, f_k\} |_{-\infty}^{\infty} = 0.$$

According to (3), we have

$$\int_{-\infty}^{\infty} (2((m+\omega) g_n f_n)'_x - m_x g_n f_n) \varphi_{1n}^2 dx = \int_{-\infty}^{\infty} (m+\omega)(g_n \varphi_{1n}(\varphi_{1n} f'_{nx} - f_n \varphi'_{1nx}) +$$

$$+ f_n \varphi_{1n}(\varphi_{1n} g'_{nx} - g_n \varphi'_{1nx})) dx = \int_{-\infty}^{\infty} (m+\omega) g_n \varphi_{1n} W\{\varphi_{1n}, f_n\} dx =$$

$$= \int_{-\infty}^{\infty} (m+\omega)\varphi_{1n}^2 W\{g_n, f_n\}dx = \omega_n \int_{-\infty}^{\infty} (m+\omega)\varphi_{1n}^2 dx.$$

By virtue of (13), the equation for k_n in Lemma 3 can be written in the form

$$\frac{dk_n}{dt} = \frac{1-4k_n^2}{8k_n}\omega_n. \tag{29}$$

According to (3), the asymptotics

$$f_n \sim \frac{\omega_n}{2c_n k_n} e^{k_n x}, \text{ when, } x \longrightarrow \infty, \tag{30}$$

$$f_n \sim -\frac{\omega_n}{2d_n k_n} e^{-k_n x}, \text{ when, } x \longrightarrow -\infty, \tag{31}$$

hold for the functions $f_n(x,t)$. Here c_n are determined from the equality $g_n = c_n\psi_{2n}$. Using Lemma 1 and the asymptotics (11), (30), (31) for $k \neq n$, we have

$$\int_{-\infty}^{\infty} (2((m+\omega)f_k g_k)'_x - m_x g_k f_k)\varphi_{1n} h_n dx = \int_{-\infty}^{\infty} (((m+\omega)f_k g_k)'_x - m_x g_k f_k)\varphi_{1n} h_n dx +$$

$$+(m+\omega)f_k g_k \varphi_{1n} h_n|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} (m+\omega)f_k g_k(\varphi_{1n}h_n)'_x dx = \int_{-\infty}^{\infty} (m+\omega)(f'_{kx} g_k \varphi_{1n} h_n + f_k g'_{kx}\varphi_{1n} h_n -$$

$$-f_k g_k \varphi_{1n} h'_{nx} - f_k g_k \varphi'_{1nx} h_n)dx = \int_{-\infty}^{\infty} (m+\omega)(f_k\varphi_{1n}(g'_{kx}h_n - h'_{nx}g_k) + g_k h_n(f'_{kx}\varphi_{1n} - \varphi'_{1nx}f_k))dx =$$

$$= \frac{1}{\lambda_k - \lambda_n}\int_{-\infty}^{\infty} \left(\frac{d}{dx}W\{\varphi_{1n}, f_k\}W\{h_n, g_k\} + \frac{d}{dx}W\{h_n, g_k\}W\{\varphi_{1n}, f_k\}\right)dx =$$

$$= \frac{1}{\lambda_k - \lambda_n}W\{\varphi_{1n}, f_k\}W\{h_n, g_k\}|_{-\infty}^{\infty} = 0.$$

According to (28),

$$\int_{-\infty}^{\infty} (2((m+\omega)g_n f_n)'_x - m_x g_n f_n)\varphi_{1n} h_n dx = \int_{-\infty}^{\infty} (m+\omega)(g_n\varphi_{1n}(h_n f'_{nx} - f_n h'_{nx}) +$$

$$+f_n h_n(\varphi_{1n} g'_{nx} - g_n \varphi'_{1nx}))dx = \int_{-\infty}^{\infty} (m+\omega) g_n \varphi_{1n} W\{h_n, f_n\}dx =$$

$$= \frac{\beta_n \omega_n}{\dot{a}(ik_n)}\int_{-\infty}^{\infty} (m+\omega)\varphi_{1n}^2 dx.$$

The last two equalities and the formula (13) provide

$$\int_{-\infty}^{\infty} G\varphi_{1n} h_n dx = i\omega\beta_n b_n \omega_n.$$

Hence,

$$\frac{db_n}{dt} = \frac{4\omega k_n}{1-4k_n^2} b_n + i\frac{1-4k_n^2}{8k_n}\beta_n b_n \omega_n, \; n = 1, 2, ..., N. \tag{32}$$

Likewise, using the definition of the Jost solutions, Lemma 1 and asymptotics (11), (30), (31), one can demonstrate that

$$\int_{-\infty}^{\infty} G\varphi_1^2 dx = ab\omega \sum_{n=1}^{N} \frac{\omega_n}{k_n}\left(1 - \frac{k_n^2 - k^2}{k_n^2 + k^2}\right).$$

Therefore,

$$\frac{dR}{dt} = -i\left(\frac{4k\omega}{4k^2+1} - \frac{4k^2+1}{8k}\sum_{n=1}^{N}\frac{\omega_n}{k_n}\left(1 - \frac{k_n^2-k^2}{k_n^2+k^2}\right)\right)R. \tag{33}$$

Let us join (29),(32) and (33) into the following statement.

Theorem. If the functions $u(x,t), g_k(x,t), f_k(x,t), k = 1,2,...,N$ are the solution to the problem (1-3) then, the scattering data for Equation (4) with the function $u(x,t)$ vary in t as follows:

$$\frac{dR}{dt} = -i\left(\frac{4k\omega}{4k^2+1} - \frac{4k^2+1}{8k}\sum_{n=1}^{N}\frac{\omega_n}{k_n}\left(1 - \frac{k_n^2-k^2}{k_n^2+k^2}\right)\right)R, \quad (Imk = 0),$$

$$\frac{dk_n}{dt} = \frac{1-4k_n^2}{8k_n}\omega_n,$$

$$\frac{db_n}{dt} = \frac{4\omega k_n}{1-4k_n^2}b_n + i\frac{1-4k_n^2}{8k_n}\beta_n b_n \omega_n, \ n = 1,2,...,N.$$

The resulting equalities determine the evolution of the scattering data completely. This allows us to apply the method of the inverse scattering problem to solve the problem $(1-3)$.

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Iroda Ismailovna Baltaeva,
Urgench State University,
Hamid Olimjon Str., 14,
220100, Urgench, Uzbekistan
E-mail: iroda-b@mail.ru

Gairat Urazalievich Urazboev,
Urgench State University,
Hamid Olimjon Str., 14,
220100, Urgench, Uzbekistan
E-mail: gayrat71@mail.ru

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