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CERTAIN GENERATING FUNCTIONS OF HERMITE-BERNOULLI-LEGENDRE POLYNOMIALS

N.U. KHAN, T. USMAN

Abstract. The special polynomials of more than one variable provide new means of analysis for the solutions of a wide class of partial differential equations often encountered in physical problems. Most of the special function of mathematical physics and their generalization have been suggested by physical problems. It turns out very often that the solution of a given problem in physics or applied mathematics requires the evaluation of an infinite sum involving special functions. Problems of this type arise, e.g., in the computation of the higher-order moments of a distribution or while calculating transition matrix elements in quantum mechanics. Motivated by their importance and potential for applications in a variety of research fields, recently, numerous polynomials and their extensions have been introduced and studied. In this paper, we introduce a new class of generating functions for Hermite-Bernoulli-Legendre polynomials and study certain implicit summation formulas by using different analytical means and applying generating function. We also introduce bilateral series associated with a newly-introduced generating function by appropriately specializing a number of known or new partly unilateral and partly bilateral generating functions. The results presented here, being very general, are pointed out to be specialized to yield a number of known and new identities involving relatively simpler and familiar polynomials.

Keywords: 2-variable Hermite polynomials, Generalized Bernoulli numbers and polynomials, 2-variable Legendre polynomials, 3-variable Hermite-Bernoulli-Legendre polynomials, summation formulae, generating functions.

Mathematics Subject Classification: 33B10, 33C45, 33C47, 33C90

1. Introduction

Generalized and multivariable forms of the special functions of mathematical physics have witnessed a significant evolution during the recent years. In particular, the special polynomials of two variables provided new means of analysis for the solutions to large classes of partial differential equations often encountered in physical problems. Most of the special function of mathematical physics and their generalization have been suggested by physical problems.

To give an example, we recall that the 2-variable Hermite Kampé de Fériet polynomials $H_n(x, y)$ [3] defined by the generating function

$$\exp(xt + yt^2) = \sum_{n=0}^{\infty} H_n(x, y)\frac{t^n}{n!}, \tag{1.1}$$

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are the solution of the heat equation

$$\frac{\partial}{\partial y} f(x,y) = \frac{\partial^2}{\partial x^2} f(x,y), \qquad f(x,0) = x^n.$$

The higher order Hermite polynomials sometimes called the Kampé de Fériet polynomials of order m or the Gould-Hopper polynomials $H_n^{(m)}(x,y)$ are defined by ([11, Eq. (6.3)])

$$H_n^{(m)}(x,y) = n! \sum_{k=0}^{[\frac{n}{m}]} \frac{y^k x^{n-mk}}{k!(n-mk)!},$$

where m is the positive integer. These polynomials are specified by the generating function

$$\exp(xt + yt^m) = \sum_{n=0}^{\infty} H_n^{(m)}(x,y) \frac{t^n}{n!} \tag{1.2}$$

are the solution of the generalized heat equation [7]

$$\frac{\partial}{\partial y} f(x,y) = \frac{\partial^m}{\partial x^m} f(x,y), \qquad f(x,0) = x^n.$$

We also note that

$$H_n^{(1)}(x,y) = (x+y)^n, \qquad H_n^{(2)}(x,y) = H_n(x,y),$$

where $H_n(x,y)$ denotes the 2-variable Hermite Kampé de Fériet polynomials defined by (1.1).

We recall that the 2-variable Legendre polynomials $S_n(x,y)$ and $R_n(x,y)$ are given by Dattoli et al. [8]

$$S_n(x,y) = n! \sum_{k=0}^{[\frac{n}{2}]} \frac{x^k y^{n-2k}}{[(n-2k)!(k!)^2]}, \tag{1.3}$$

and

$$R_n(x,y) = (n!)^2 \sum_{k=0}^{\infty} \frac{(-1)^{n-k} x^{n-k} y^k}{[(n-2k)!]^2 (k!)^2}, \tag{1.4}$$

respectively, and are related with the ordinary Legendre polynomials $P_n(x)$ [14] as

$$P_n(x) = S_n\left(-\frac{1-x^2}{4}, x\right) = R_n\left(\frac{1-x}{2}, \frac{1+x}{2}\right).$$

From equation (1.3) and (1.4) we have

$$S_n(x,0) = n! \frac{x^{[\frac{n}{2}]}}{[(\frac{n}{2})!]^2}, \qquad S_n(0,y) = y^n, \qquad R_n(x,0) = (-x)^n, \qquad R_n(0,y) = y^n.$$

The generating functions for two variable Legendre polynomials $S_n(x,y)$ and $R_n(x,y)$ are given by [8]

$$e^{yt} C_0(-xt^2) = \sum_{n=0}^{\infty} S_n(x,y) \frac{t^n}{n!}, \qquad C_0(xt) C_0(-yt) = \sum_{n=0}^{\infty} R_n(x,y) \frac{t^n}{(n!)^2},$$

where $C_0(x)$ is the 0-th order Tricomi function [14]

$$C_0(x) = \sum_{r=0}^{\infty} \frac{(-1)^r x^r}{(r!)^2}.$$

We note the following link between the Hermite Kampé de Fériet polynomials $H_n(x,y)$ and the 2-variable Legendre polynomial $S_n(x,y)$ (see [6]):

$$H_n(y, -D_x^{-1}) = S_n(x,y),$$

where D_x^{-1} denotes the inverse of derivative operator $D_x := \frac{\partial}{\partial x}$ and is defined in such a way that

$$D_x^{-n}\{f(x)\} = \frac{1}{(n-1)!}\int_0^x (x-\xi)^{n-1} f(\xi)d\xi,$$

so that for $f(x) = 1$, we have

$$D_x^{-n}\{1\} = \frac{x^n}{n!}.$$

In view of the equations (1.1) and (1.2), we note the following relation:

$$H_n^{(2)}\left(x, -\frac{1}{2}\right) = H_n\left(x, -\frac{1}{2}\right) = He_n(x),$$

where $He_n(x)$ denotes the classical Hermite polynomials [1].

We recall that the 3-variable Legendre-Hermite polynomials ${}_SH_n(x, y, z)$ are defined by the series [10]

$${}_SH_n(x, y, z) = n! \sum_{k=0}^{[\frac{n}{2}]} \frac{z^k L_{n-2k}(x, y)}{k!(n-2k)!}, \tag{1.5}$$

and are specified by the following generating function:

$$\exp(yt + zt^2)C_0(-xt^2) = \sum_{n=0}^{\infty} {}_SH_n(x, y, z)\frac{t^n}{n!}.$$

The classical Bernoulli number B_n, Bernoulli polynomials $B_n(x)$ and their generalization $B_n^{(\alpha)}(x)$ (of real or complex) of order α are usually defined by means of the following generating functions ([1], [2], [4], see also [15])

$$\left(\frac{t}{e^t - 1}\right) = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}, \qquad \left(\frac{t}{e^t - 1}\right) e^{xt} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!},$$

and

$$\left(\frac{t}{e^t - 1}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} B_n^{(\alpha)}(x)\frac{t^n}{n!}, \qquad |t| < 2\pi; \quad 1^{\alpha} = 1.$$

The B_n are rational numbers and in particular $B_n^1(0) = B_n(0) = B_n$.

The generalized Hermite-Bernoulli polynomials of two variables ${}_HB_n^{(\alpha)}(x, y)$ are introduced by Pathan [12] in the form

$$\left(\frac{t}{e^t - 1}\right)^{\alpha} e^{xt+yt^2} = \sum_{n=0}^{\infty} {}_HB_n^{(\alpha)}(x, y)\frac{t^n}{n!}, \tag{1.6}$$

which are essentially generalization of Bernoulli numbers, Bernoulli polynomials, Hermite polynomials and Hermite-Bernoulli polynomials ${}_HB_n(x, y)$ introduced by Dattoli et al. ([9], p.386 (1.6)) in the form

$$\left(\frac{t}{e^t - 1}\right) e^{xt+yt^2} = \sum_{n=0}^{\infty} {}_HB_n(x, y)\frac{t^n}{n!}.$$

The special polynomials of more than one variable provide new means of analysis for the solutions of a wide class of partial differential equations often encountered in physical problems. It turns out very often that the solution of a given problem in physics or applied mathematics requires calculating of an infinite sum involving special functions. Problem of this type arise, for example, in the computation of the higher-order moments of a distribution or while calculating the transition matrix elements in quantum mechanics. In [5], Dattoli showed that the summation formulae of special functions often encountered in applications ranging from electromagnetic process to combinatorics can be written in terms of Hermite polynomials of more than one variable.

In this paper, we derive the generating function for the Hermite-Bernoulli-Legendre polynomials ${}_SB_n^{(\alpha)}(x,y,z)$ in terms of certain multi-variable special polynomials. Also we find some implicit summation formulae by using different analytical means and applying generating functions. We also introduce bilateral series associated with the newly-introduced generating function by appropriately specializing a number of known or new partly unilateral and partly bilateral generating functions.

2. A new class of Hermite-Bernoulli-Legendre polynomials

We define the Hermite-Bernoulli-Legendre polynomials ${}_SB_n^{(\alpha)}(x,y,z)$ by means of generating function as follows:

$$\left(\frac{t}{e^t-1}\right)^{\alpha} e^{yt+zt^2}C_0(-xt^2)=\sum_{n=0}^{\infty}{}_SB_n^{(\alpha)}(x,y,z)\frac{t^n}{n!} \tag{2.1}$$

or equivalently, by the series

$${}_SB_n^{(\alpha)}(x,y,z)=\sum_{m=0}^{n}\sum_{k=0}^{[\frac{m}{2}]}\frac{B_{n-m}^{(\alpha)}\ S_{m-2k}(x,y)\ z^k\ n!}{(m-2k)!\ k!\ (n-m)!}\ ,$$

where $S_n(x,y)$ are the 2-variable Legendre polynomials in (1.3).

As $x=0$, (2.1) reduces to

$$\left(\frac{t}{e^t-1}\right)^{\alpha} e^{yt+zt^2}=\sum_{n=0}^{\infty}{}_HB_n^{(\alpha)}(y,z)\frac{t^n}{n!} \tag{2.2}$$

or equivalently (see [13], p.682),

$${}_SB_n^{(\alpha)}(0,y,z)={}_HB_n^{(\alpha)}(y,z)=\sum_{n=0}^{m}\binom{n}{m}B_{n-m}^{(\alpha)}\ H_m(y,z)\ ,$$

where ${}_HB_n^{(\alpha)}(y,z)$ are the 2-variable generalized Hermite-Bernoulli polynomials in (1.6).

Letting $\alpha=0$, (2.1) yields

$$e^{yt+zt^2}C_0(-xt^2)=\sum_{n=0}^{\infty}{}_SH_n(x,y,z)\frac{t^n}{n!} \tag{2.3}$$

or equivalently,

$${}_SB_n^{(\alpha)}(x,y,z)={}_SH_n(x,y,z)=n!\sum_{r=0}^{n}\sum_{k=0}^{[\frac{r}{2}]}\frac{(-1)^{r-2k}(x)^{r-2k}(y)^{n-r}z^k}{(n-r)!k!(r-2k)!},$$

where ${}_SH_n(x,y,z)$ are the 3-variable Hermite-Legendre polynomials in (1.5).

3. Implicit summation formulae involving Hermite-Bernoulli-Legendre polynomials

In order to find implicit summation formulae involving Hermite-Bernoulli-Legendre polynomials ${}_SB_n^{(\alpha)}(x,y,z)$, the same consideration as developed for the Hermite-Bernoulli polynomials in Pathan et al. [13] holds as well. First we prove the following results involving Hermite-Bernoulli-Legendre polynomials ${}_SB_n^{(\alpha)}(x,y,z)$.

Theorem 3.1. *The following implicit summation formulae for Hermite-Bernoulli-Legendre polynomials* ${}_SB_n^{(\alpha)}(x,y,z)$ *holds true:*

$$ {}_SB_{k+l}^{(\alpha)}(x,v,z) = \sum_{n,p=0}^{k,l} \binom{k}{n}\binom{l}{p} (v-y)^{n+p}\ {}_SB_{k+l-p-n}^{(\alpha)}(x,y,z) \tag{3.1} $$

Proof. We replace t by $t+u$ and rewrite the generating function (2.1) as

$$ \left(\frac{t+u}{e^{t+u}-1}\right)^{\alpha} e^{z(t+u)^2} C_0(-x(t+u)^2) = e^{-y(t+u)} \sum_{k,l=0}^{\infty} {}_SB_{k+l}^{(\alpha)}(x,v,z)\ \frac{t^k}{k!}\ \frac{u^l}{l!} $$

Replacing y by v in the above equation and equating the resulting equation to the above equation, we get

$$ e^{(v-y)(t+u)} \sum_{k,l=0}^{\infty} {}_SB_{k+l}^{(\alpha)}(x,y,z)\frac{t^k}{k!}\frac{u^l}{l!} = \sum_{k,l=0}^{\infty} {}_SB_{k+l}^{(\alpha)}(x,v,z)\frac{t^k}{k!}\frac{u^l}{l!} \tag{3.2} $$

By expanding exponential function (3.2) we arrive at

$$ \sum_{N=0}^{\infty} \frac{[(v-y)(t+u)]^N}{N!} \sum_{k,l=0}^{\infty} {}_SB_{k+l}^{(\alpha)}(x,y,z)\frac{t^k}{k!}\frac{u^l}{l!} = \sum_{k,l=0}^{\infty} {}_SB_{k+l}^{(\alpha)}(x,v,z)\frac{t^k}{k!}\frac{u^l}{l!}, $$

which by using formula ([15, Eq. (2)])

$$ \sum_{N=0}^{\infty} f(N)\frac{(x+y)^N}{N!} = \sum_{n,m=0}^{\infty} f(m+n)\frac{x^n}{n!}\frac{y^m}{m!} $$

in the left hand side becomes

$$ \sum_{n,p=0}^{\infty} \frac{(v-y)^{n+p}}{n!p!} \sum_{k,l=0}^{\infty} {}_SB_{k+l}^{(\alpha)}(x,y,z)\frac{t^k}{k!}\frac{u^l}{l!} = \sum_{k,l=0}^{\infty} {}_SB_{k+l}^{(\alpha)}(x,v,z)\frac{t^k}{k!}\frac{u^l}{l!} \tag{3.3} $$

Now replacing k by $k-n$, l by $l-p$ and using Lemma 1 in [15] in the left hand side of (3.3), we get

$$ \sum_{n,p=0}^{\infty}\sum_{k,l=0}^{\infty} \frac{(v-y)^{n+p}}{n!p!}\ {}_SB_{k+l-n-p}^{(\alpha)}(x,y,z)\frac{t^k}{(k-n)!}\frac{u^l}{(l-p)!} = {}_SB_{k+l}^{(\alpha)}(x,v,z)\frac{t^k}{k!}\frac{u^l}{l!} $$

Finally, by equating the coefficients of the like powers of t and u in the above equation, we get the required result. □

By taking $l=0$ in Eq. (3.1), we immediately deduce the following result.

Corollary 3.1. *The following implicit summation formula for Hermite-Bernoulli-Legendre polynomials* ${}_SB_n^{(\alpha)}(x,y,z)$ *holds true:*

$$ {}_SB_k^{(\alpha)}(x,v,z) = \sum_{n=0}^{k} \binom{k}{n} (v-y)^n {}_SB_{k-n}^{(\alpha)}(x,y,z). $$

Replacing v by $v+y$ and setting $x=z=0$ in Theorem 3.1, we get the following result involving Hermite-Bernoulli-Legendre polynomial of one variable:

$$ {}_SB_{k+l}^{(\alpha)}(v+y) = \sum_{n,p=0}^{k,l} \binom{k}{n}\binom{l}{p} (v)^{m+n}\ {}_SB_{k+l-p-n}^{(\alpha)}(y), $$

whereas by setting $v = 0$ in Theorem 3.1, we get another result involving Hermite-Bernoulli-Legendre polynomial of one and two variable

$$ {}_SB_{k+l}^{(\alpha)}(x,z) = \sum_{n,p=0}^{k,l} \binom{k}{n} \binom{l}{p} (-y)^{n+m} {}_SB_{k+l-p-n}^{(\alpha)}(x,y,z). $$

Along with the above result, we will exploit extended forms of Hermite-Bernoulli-Legendre polynomial ${}_SB_n^{(\alpha)}(x,y)$ by setting $z = 0$ in the Theorem 3.1 to get

$$ {}_SB_{k+l}^{(\alpha)}(x,v) = \sum_{n,p=0}^{k,l} \binom{k}{n} \binom{l}{p} (v-y)^{n+m} {}_SB_{k+l-p-n}^{(\alpha)}(x,y). $$

A straightforward expression of the ${}_HB_n^{(\alpha)}(y,z)$ is suggested by a special case of the Theorem 3.1 for $x = 0$ in the following form

$$ {}_HB_{k+l}^{(\alpha)}(v,z) = \sum_{n,p=0}^{k,l} \binom{k}{n} \binom{l}{p} (v-y)^{n+m} {}_HB_{k+l-p-n}^{(\alpha)}(y,z), $$

where ${}_HB_{k+l}^{(\alpha)}(y,z)$ are the 2-variable generalized Hermite-Bernoulli polynomials in (1.6).

Similarly, a straightforward expression of the ${}_SH_n(x,y,z)$ is suggested by a special case of the Theorem 3.1 for $\alpha = 0$ in the following form

$$ {}_SH_{k+l}(x,v,z) = \sum_{n,p=0}^{k,l} \binom{k}{n} \binom{l}{p} (v-y)^{n+m} {}_SH_{k+l-p-n}(x,y,z), $$

where ${}_SH_{k+l}(x,y,z)$ are the 3-variable Hermite-Legendre polynomials in (1.5).

Theorem 3.2. *The following implicit summation formula for Hermite-Bernoulli-Legendre polynomials ${}_SB_n^{(\alpha)}(x,y,z)$ holds true:*

$$ {}_SB_n^{(\alpha)}(x,y+v,z+u) = \sum_{m=0}^{n} \binom{n}{m} {}_SB_{n-m}^{(\alpha)}(x,y,z) H_m(v,u), \tag{3.4} $$

where $H_m(v,u)$ are the 2-variable Hermite polynomials in (1.1).

Proof. We replace y by $y+v$ and z by $z+u$ in (2.1), use (1.1) and rewrite the generating function as

$$ \sum_{n=0}^{\infty} {}_SB_n^{(\alpha)}(x,y+v,z+u)\frac{t^n}{n!} = \left\{ \left(\frac{t}{e^t-1}\right)^{\alpha} e^{yt+zt^2} C_0(-xt^2) \right\} e^{vt+ut^2}, $$

$$ \sum_{n=0}^{\infty} {}_SB_n^{(\alpha)}(x,y+v,z+u)\frac{t^n}{n!} = \sum_{n=0}^{\infty} {}_SB_n^{(\alpha)}(x,y,z)\frac{t^n}{n!} \sum_{m=0}^{\infty} H_m(v,u)\frac{t^m}{m!}. $$

Now, replacing n by $n-m$ and comparing the coefficients of t^n, we arrive at (3.4). □

Theorem 3.3. *The following implicit summation formula for Hermite-Bernoulli-Legendre polynomials ${}_SB_n^{(\alpha)}(x,y,z)$ holds true:*

$$ {}_SB_n^{(\alpha)}(x,y,z) = \sum_{m=0}^{[\frac{n}{2}]} \frac{(-1)^m(-x)^m {}_HB_{n-2m}^{(\alpha)}(y,z)\ n!}{(n-2m)!(m!)^2}. \tag{3.5} $$

Proof. Using the generating function (2.1), we have

$$ \sum_{n=0}^{\infty} {}_SB_n^{(\alpha)}(x,y,z)\frac{t^n}{n!} = \left(\frac{t}{e^t-1}\right)^{\alpha} e^{yt+zt^2} C_0(-xt^2), $$

$$\sum_{n=0}^{\infty} {}_SB_n^{(\alpha)}(x,y,z)\frac{t^n}{n!} = \left\{\sum_{n=0}^{\infty} {}_HB_n^{(\alpha)}(y,z)\frac{t^n}{n!}\right\}\left\{\sum_{m=0}^{\infty} \frac{(-1)^m(-xt^2)^m}{(m!)^2}\right\}.$$

Replacing n by $n-2m$ and comparing the coefficients of t^n, we obtain (3.5). □

Theorem 3.4. *The following implicit summation formula for Hermite-Bernoulli-Legendre polynomials* ${}_SB_n^{(\alpha)}(x,y,z)$ *holds true:*

$${}_SB_n^{(\alpha)}(x,y,z) = \sum_{m=0}^{n}\sum_{k=0}^{[\frac{m}{2}]} \frac{(-1)^k(x)^k B_{n-m}^{(\alpha)}(y-u)H_{m-2k}(u,z)n!}{(m-2k)!(n-m)!(k!)^2} \tag{3.6}$$

Proof. By exploiting the generating function (1.1), we can write Eq. (2.1) as

$$\left\{\left(\frac{t}{e^t-1}\right)^{\alpha} e^{yt-ut}\right\} e^{ut+zt^2}C_0(-xt^2) = \sum_{n=0}^{\infty} B_n^{(\alpha)}(y-u)\frac{t^n}{n!}\sum_{m=0}^{\infty} H_m(u,z)\frac{t^m}{m!}\sum_{k=0}^{\infty}\frac{(-1)^k(-xt^2)^k}{(k!)^2}$$

Replacing m by $m-2k$, we get

$${}_SB_n^{(\alpha)}(x,y,z) = \sum_{n=0}^{\infty} B_n^{(\alpha)}(y-u)\frac{t^n}{n!} \sum_{m=0}^{\infty}\left\{\sum_{k=0}^{[\frac{m}{2}]}\frac{H_{m-2k}(u,z)(-1)^k(-x)^k}{(m-2k)!(k!)^2}\right\}t^m$$

Now we replace n by $n-m$:

$$\sum_{n=0}^{\infty} {}_SB_n^{(\alpha)}(x,y,z)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\left\{\sum_{m=0}^{n}\sum_{k=0}^{[\frac{m}{2}]}\frac{(-1)^k(-x)^k B_{n-m}^{(\alpha)}(y-u)H_{m-2k}(u,z)}{(n-m)!(m-2k)!(k!)^2}\right\}t^n$$

We equate the coefficients of the like powers of t^n and this leads us to (3.6). □

By employing the same technique used in the proof of above theorem, from (1.6) and (2.1), we can obtain semi-addition formula for ${}_SB_n^{(\alpha)}(x,y,z)$ and the known addition formula for $B_n^{(\alpha)}(x)$ (see, e.g., [[16], p. 84, Eq. (25)]) which are given in the following theorem.

Theorem 3.5. *The following implicit summation formula for Hermite-Bernoulli-Legendre polynomials* ${}_SB_n^{(\alpha)}(x,y,z)$ *holds true:*

$${}_SB_n^{(\alpha+\beta)}(x,u+v,z) = \sum_{k=0}^{n}\binom{n}{k} {}_SB_{n-k}^{(\alpha)}(x,u,z)B_k^{(\beta)}(v) \tag{3.7}$$

and

$$B_n^{(\alpha+\beta)}(u+v) = \sum_{k=0}^{n}\binom{n}{k} B_{n-k}^{(\alpha)}(u)B_k^{(\beta)}(v) \tag{3.8}$$

Proof. In view of the relation

$${}_SB_n^{(\alpha)}(0,y,0) = B_n^{(\alpha)}(y),$$

(3.8) is just a special case of (3.7). □

4. Generating functions of the Hermite-Bernoulli-Legendre polynomials involving bilateral series

We consider the function

$$V^{(\alpha)} = V^{(\alpha)}(x, y, w, z; s, t) := \left(\frac{t}{e^t - 1}\right)^{\alpha} e^{s - wt/s + yt + zt^2}\ C_0(-xt^2)$$

Expanding $e^{s-wt/s}$ and using (2.1), we get

$$V^{(\alpha)} = \sum_{M=0}^{\infty} \frac{s^M}{M!} \sum_{K=0}^{\infty} \left(\frac{-wt}{s}\right)^K \frac{1}{K!} \sum_{N=0}^{\infty} {}_S B_n^{(\alpha)}(x, y, z) \frac{t^N}{N!} \tag{4.1}$$

We replace the summation indices M and N in (4.1) by $K + N = n$ and $M - K = m$, respectively and we rearrange the triple series:

$$V^{(\alpha)} = \sum_{m=-\infty}^{\infty} \sum_{n=m^*}^{\infty} s^m t^n \sum_{K=0}^{n} \frac{(-w)^K}{K!(m+K)!(n-K)!}\ {}_S B_{n-K}^{(\alpha)}(x, y, z);$$

hereinafter $m^* = \max\{0, -m\}$, $m \in \mathbb{Z}$. Thanks to the absolute convergence of the series involved, we are led to the generating function

$$\begin{aligned} &\left(\frac{t}{e^t - 1}\right)^{\alpha} e^{s - wt/s + yt + zt^2}\ C_0(-xt^2) \\ &\quad = \sum_{m=-\infty}^{\infty} \sum_{n=m^*}^{\infty} s^m t^n \sum_{K=0}^{n} \frac{(-w)^K}{K!(m+K)!(n-K)!}\ {}_S B_{n-K}^{(\alpha)}(x, y, z). \end{aligned} \tag{4.2}$$

Some special cases of (4.2) are demonstrated in the following corollaries.

Corollary 4.1. *Setting $\alpha = 1$ in (4.2), we get*

$$\left(\frac{t}{e^t - 1}\right) e^{s - wt/s + yt + zt^2}\ C_0(-xt^2) = \sum_{m=-\infty}^{\infty} \sum_{n=m^*}^{\infty} s^m t^n \sum_{K=0}^{n} \frac{(-w)^K}{K!(m+K)!(n-K)!}\ {}_S B_{n-K}^{(1)}(x, y, z).$$

Corollary 4.2. *Setting $\alpha = 0$ and using ${}_S B_n^{(0)}(x, y, z) = {}_S H_n(x, y, z)$ in (4.2), we get*

$$e^{s - wt/s + yt + zt^2}\ C_0(-xt^2) = \sum_{m=-\infty}^{\infty} \sum_{n=m^*}^{\infty} s^m t^n \sum_{K=0}^{n} \frac{(-w)^K}{K!(m+K)!(n-K)!}\ {}_S H_{n-K}(x, y, z),$$

where ${}_S H_{n-K}(x, y, z)$ are the 3-variable Hermite-Legendre polynomials given in (1.5).

Corollary 4.3. *Setting $x = 0$ in (4.2) and using (1.6), we get*

$$\left(\frac{t}{e^t - 1}\right)^{\alpha} e^{s - wt/s + yt + zt^2} = \sum_{m=-\infty}^{\infty} \sum_{n=m^*}^{\infty} s^m t^n \sum_{K=0}^{n} \frac{(-w)^K}{K!(m+K)!(n-K)!}\ {}_H B_{n-K}^{(\alpha)}(y, z),$$

where ${}_H B_{n-K}^{(\alpha)}(y, z)$ are the 2-variable generalized Hermite-Bernoulli polynomials in (1.6).

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Nabiullah Khan,
Department of Applied Mathematics,
Faculty of Engineering and Technology,
Aligarh Muslim University, Aligarh-202002, India.
E-mail: nukhanmath@gmail.com

Talha Usman,
Department of Applied Mathematics,
Faculty of Engineering and Technology,
Aligarh Muslim University, Aligarh-202002, India.
E-mail: talhausman.maths@gmail.com